RICH PHENOMENA IN A NETWORK OF TWO RINGS COUPLED THROUGH A 'BUFFER' CELL

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We study curious dynamical features that appear in a network of two unidirectional rings coupled asymmetrically through a 'buffer' cell. One example of such phenomena was presented by Golubitsky, Nicol and Stewart (Some curious phenomena in coupled cell systems, *J. Nonlinear Sci.* **14** (2) (2004) 207–236), where they have shown simulation results in which two rings of cells, coupled asymmetrically through a 'buffer' cell, appear to exhibit rotating wave states with incommensurate frequencies. In this paper, we propose a bifurcation scenario where the phenomena presented by Golubitsky *et. al.* is obtained through a sequence of Hopf bifurcations starting at an equilibrium. We use XPPAUT and MATLAB to compute numerically the relevant states.

Keywords: Coupled cell systems, Hopf bifurcation, secondary Hopf bifurcation

1. Introduction

We study a network of two unidirectional rings of cells coupled through a 'buffer' cell. The first ring is built with three cells and the second one with five cells, see Figure 1.

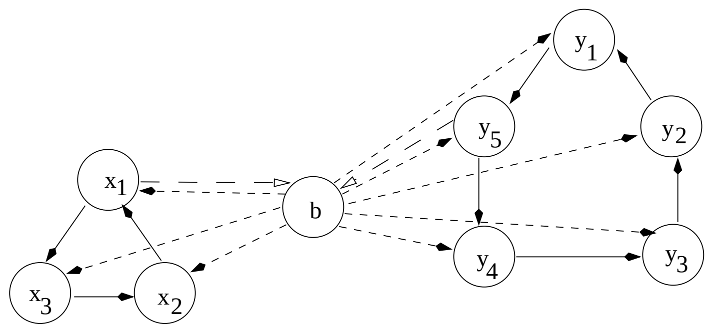

Figure 1. Network of two coupled unidirectional rings, one with three cells and the other with five, connected through a 'buffer' cell b.

Golubitsky *et. al.* [3], present one simulation of this network where cells in each ring appear to exhibit rotating wave states with well defined frequencies, and the 'buffer' cell appears to be rotating at incommensurate frequencies. They suggest that these states lie on thin tori, not closed loops, and their conjecture is that as so they are presumably quasiperiodic.

In this paper we present numerical simulations showing a sequence of three Hopf bifurcations which preceed the appearance of such phenomena. Specifically, we consider a one-parameter family of coupled cell systems with structure consistent with the network in Figure 1 which has a stable equilibrium for certain values of the parameter. By variation of the parameter we numerically compute a cascade of three Hopf bifurcations: the first from the equilibrium, the second from a periodic solution and the third from a quasi-periodic solution. In particular, the solution shown by Golubitsky *et. al.* [3] arises by further variation of the parameter.

2. Coupled Cell Networks Formalism

Coupled cell networks can be represented by directed graphs whose nodes are identified with dynamical systems or 'cells' and whose edges ('arrows') represent the couplings between them. Identical cells/couplings are represented with the same node/arrow type. For a survey, overview and examples, see Golubitsky and Stewart [5].

In Figure 1, the nodes in the two networks are the circles and represent cells. All cells have identical internal dynamics and are represented by circles. There are three different types of edges, drawn as different styles of arrows and different tips.

2.1. *Symmetries in a coupled cell system*

Coupled dynamical systems often possess a group of symmetries Γ. A symmetry is a transformation that sends solutions to solutions. A symmetry of a network is a pair of permutations, one on the cells and one on the arrows, that preserves the network structure. The permutation action of the symmetry group of the network on the cells induces an action of the symmetry group Γ on the phase space X of the coupled cell system by permuting the coordinates.

A coupled cell system

$$\frac{dx}{dt} = f(x),$$

where $x \in X$ and f is smooth, has symmetry γ if and only if satisfies the equivariance condition:

$$f(\gamma x) = \gamma f(x), \qquad \forall x \in X, \gamma \in \Gamma.$$

Symmetry is an important concept in the study of networks.

2.2. *'Interior symmetry' in a coupled cell system*

The concept of 'interior symmetry' was introduced in Golubitsky, Pivato and Stewart [4]. It represents an intermediate class between the class of general networks and the class of symmetric networks. In this case, there is a group of permutations that acts in a subset of cells but not on the entire set of cells, that partially preserves the network structure (cell- and edges-types) and that permutes cell coordinates.

The network in Figure 1 is an example of $\mathbf{Z}_3 \times \mathbf{Z}_5$ 'interior symmetry'. This means that if consider the subnetwork formed by ignoring the couplings from cell x_1 to the 'buffer' cell and from cell y_5 to the 'buffer' cell, then the resulting network is $\mathbf{Z}_3 \times \mathbf{Z}_5$-symmetric. This is a consequence of a general characterization of a coupled cell network with 'interior symmetry' (see Proposition 3.3 in Antoneli, Dias and Paiva [1]).

3. Numerical Simulations

The simulation is performed using Matlab [8] and XPPAUT [2]. We consider as internal dynamics of the nine cells the function given by:

$$g(u) = \mu\, u - \frac{1}{10}u^2 - u^3$$

and we use linear coupling.

The simulated coupled cell system is thus:

$$\begin{array}{ll} \dot{x_j} &= g(x_j) + c\,(x_j - x_{j+1}) + d\,b \quad j = 1, \ldots, 3 \\ \dot{b} &= g(b) + \lambda\,(x_1 + y_5) \\ \dot{y_j} &= g(y_j) + c\,(y_j - y_{j+1}) + d\,b \quad j = 1, \ldots, 5 \end{array} \tag{1}$$

where c, d, λ are parameters, and the indexing assumes $x_4 = x_1$ and $y_6 = y_1$. Note that if $\lambda = 0$, then the coupled cell system (1) has global symmetry $\mathbf{Z}_3 \times \mathbf{Z}_5$.

We simulate solutions of this system in the two cases, $\lambda = 0$ (symmetric) and $\lambda \neq 0$ ('interior symmetric') and compare the two situations.

We use XPPAUT to compute some bifurcation branches for the coupled cell system (1). We set $c = 0.75$, $d = 0.2$, and we vary the parameter μ. In Figure 2 we show a sequence of three Hopf bifurcations (valid for $\lambda = 0$ and $\lambda \neq 0$) given by XPPAUT and in what follows we will present the corresponding state for each branch and comment on the properties of the solutions obtained for each branch.

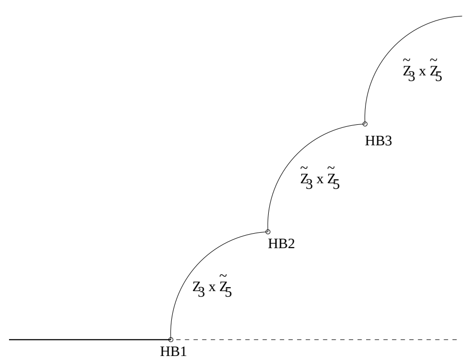

Figure 2. Schematic (partial) bifurcation diagram for the coupled cell system (1), around the equilibrium $(0, 0, 0, 0, 0, 0, 0, 0, 0)$, varying the bifurcation parameter μ, setting $c = 0.75$ and $d = 0.2$. Solid lines represent a stable equilibrium, dashed lines correspond to unstable equilibrium. Solid curves represent stable periodic orbits. See text for details.

The solutions corresponding to the first branch are explained using the Equivariant Hopf Theorem for coupled cell systems in the symmetric case, and the analog for coupled cell systems with ‘interior symmetry’.

The Equivariant Hopf Theorem provides states with spatio-temporal symmetries corresponding to subgroups of the symmetry group of the network, which have two dimensional fixed point subspaces (see Golubitsky and Stewart [6]). The interior symmetry-breaking Hopf bifurcation Theorem (see Antoneli, Dias and Paiva [1]) provides states whose linearization on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having spatio-temporal symmetries, corresponding to subgroups of the ‘interior symmetry’ group of the network, which have two dimensional fixed point subspaces. In Figure 3 (left), it is shown a time series solution of the coupled cell system (1) for $\lambda = 0$, representing a rotating wave state in the five-cells ring, that bifurcated at a Hopf point, HB1, from the trivial equilibrium branch. In Figure 3 (right), it is shown a time series solution of the coupled cell system (1) for $\lambda = 0.1$, representing the superposition of a rotating wave state in the five-cells ring and a synchronous periodic state in the three-cells ring, bifurcating at a Hopf point, HB1, from the trivial equilibrium branch. In the schematic bifurcation diagram of Figure 2, the above solutions for $\lambda = 0$ and $\lambda = 0.1$ of the coupled cell system (1), correspond to the first Hopf bifurcating branch, with ‘symmetry type’ $\mathbf{Z}_3 \times \tilde{\mathbf{Z}}_5$.

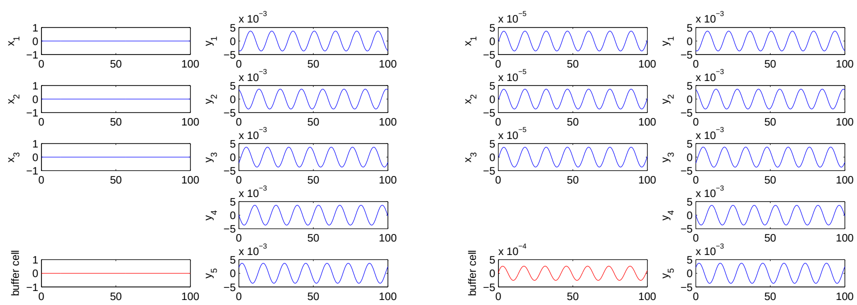

Figure 3. Simulation of the coupled system (1). Time series from the nine cells. (Left) Cells in the three cell ring are at equilibria and cells in the five cell ring show a rotating wave, for the symmetric case ($\lambda = 0$); (Right) Cells in the three cell ring are in synchrony with the ‘buffer‘ cell and cells in the five cell ring show a rotating wave, for the ‘interior symmetric’ case ($\lambda = 0.1$).

By further variation of the parameter μ, there is a secondary Hopf bifurcation point, labeled HB2, where the time series of the cells x_1, x_2, x_3, appear to show a $\mathbf{Z}_3$ rotating wave state. However, the full solution of the coupled cell system (1) is quasi-periodic. In Figure 4, we show, in the left panel, the individual time series of the nine cells, for the case $\lambda = 0$ and on the right panel we plot cell x_1 vs cells y_5, where the quasi-periodic behavior is indicated by the non-closed curve that fills the square densely. In Figure 5, we present, in the left panel, the individual time series of the nine cells, for the case $\lambda = 0.1$, note that the 'buffer' cell appears to exhibit a quasi-periodic state or a periodic orbit with very long period. On the right panel, we plot cell x_1 vs cells y_5, where the behavior is similar to the case $\lambda = 0$. These two figures represent, in the schematic bifurcation diagram of Figure 2, states corresponding to the the second Hopf bifurcating branch, with 'symmetry type' $\tilde{\mathbf{Z}}_3 \times \tilde{\mathbf{Z}}_5$. Continuing the variation of parameter μ,

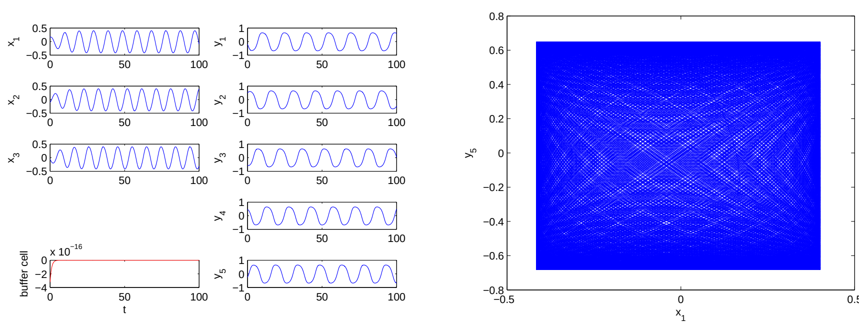

Figure 4. Simulation of the coupled system (1). (Left) Time series from the nine cells for $\lambda = 0$; (right) x_1 vs y_5 for $\lambda = 0$.

we find a third Hopf bifurcating point, HB3. Near this bifurcation point, we don't see any significant qualitative change on the dynamical features of the cells in comparison with the behavior plotted in Figures 4,5. The curious behavior exhibited by the coupled cell system studied in Golubitsky *et al* [3] (Figure 18), is obtained in this third bifurcating branch but further away from HB3. In Figures 6 and 7, we plot, on the left panel, the time series for the nine cells and on the right panel cell x_1 vs cell y_5, for the cases $\lambda = 0$ and $\lambda = 0.1$, respectively. The numerics strongly suggests that this dynamical feature needs a relaxation oscillation[a] phenomena to

[a]Solutions with long periods of quasi-static behavior interspersed with short periods of rapid transition.

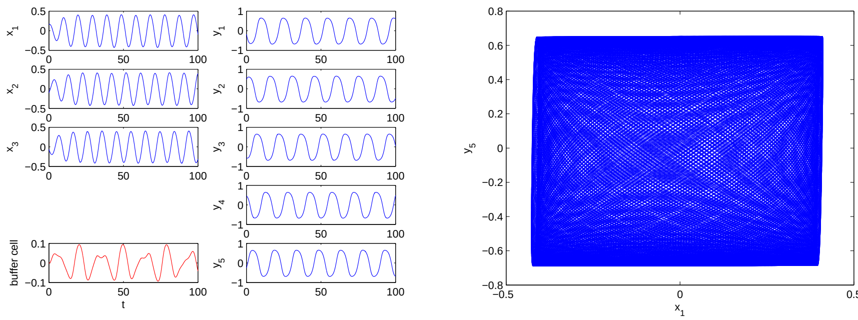

Figure 5. Simulation of the coupled system (1). (Left) Time series from the nine cells for $\lambda = 0.1$; (right) x_1 vs y_5 for $\lambda = 0.1$.

occur. Relaxation oscillations are studied in the context of the canard phenomenon [7]. Canard theory is usually applied in the study of ordinary differential equations where there are two time scales (usually denoted by fast-slow systems). In the example studied in this paper, it is not explicitly imposed a structure of two time scales.

In Table 1, we summarize the information, computed by XPPAUT, related to the schematic (partial) bifurcation diagram in Figure 2, and describe the dynamical behavior for the coupled system (1) for the different bifurcating branches.

Table 1. Description of the dynamical behavior of system (1) for variation of parameter μ. See text for more details.

Bifurcation Point	$\mu^\star$	Three cells ring	Five cells ring	Figure
Hopf Point (HB1)	1.35676275	equilibrium	rotating wave	3
Hopf Point (HB2)	1.12500000	rotating wave	rotating wave	4,5
Hopf Point (HB3)	0.51823726	rotating wave (relaxation oscillations)	rotating wave (relaxation oscillations)	6,7

4. Conclusion

In this paper we present numerical simulations showing dynamical features of a network of two unidirectional rings coupled through a ‘buffer’ cell.

We use XPPAUT and MATLAB to compute a partial bifurcation diagram and the corresponding dynamical states in each bifurcating branch.

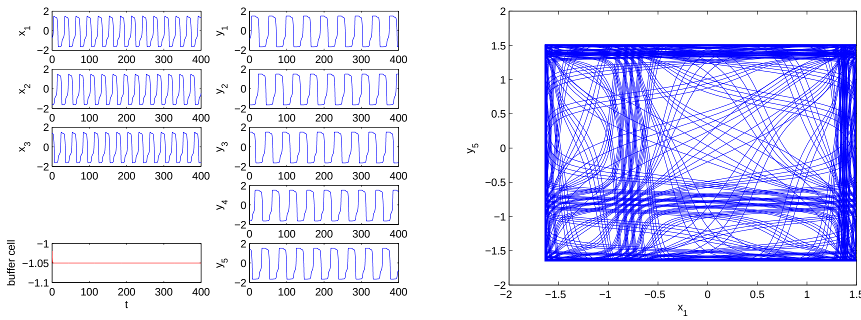

Figure 6. Simulation of the coupled system (1). (Left) time series from the nine cells; (right) x_1 vs cell y_5 for $\lambda = 0$.

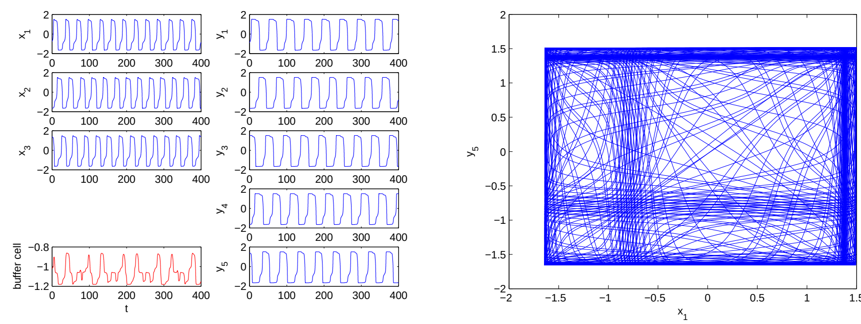

Figure 7. Simulation of the coupled system (1). (Left) time series from the nine cells; (right) x_1 vs cell y_5 for $\lambda = 0.1$.

We suggest a bifurcation scenario that explains the curious phenomena shown by Golubitsky *et al* [3] (Figure 18). More specifically, we present evidence that such type of solution can arise through a sequence of three Hopf bifurcations. In the last branch of solutions, we observe a transition from small amplitude solutions to large amplitude relaxation oscillations.

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