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An Alternative Point Process Framework for Modeling Multivariate Extreme Values

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An alternative limiting point process to that of de Haan (1985) is studied that holds regardless of whether the underlying data generation mechanism is asymptotically dependent or asymptotically independent. We characterize its intensity function in terms of the coefficient of tail dependence and an angular measure which satisfies a normalisation condition. We use this point process to derive a generalisation of standard componentwise maxima results that holds for both asymptotic dependence and asymptotic independence. We illustrate our results using a flexible parametric example and provide methods for simulating from both the limiting point process and the limiting componentwise maxima distribution.

Keywords Asymptotic dependence; Asymptotic independence; Coefficient of tail dependence; Componentwise maxima; Extreme value theory; Joint survivor function; Multivariate extreme values; Point processes; Simulation.

Mathematics Subject Classification 60G70; 62E20.

1. Introduction

1.1. Classical Background

The classical results of multivariate extremes can often be framed in terms of the point process representation of de Haan (1985); for example, see Coles and Tawn (1991, 1994). To illustrate this, in the following we concentrate on the two-dimensional case and consider independent and identically distributed (i.i.d.) bivariate random variables $(X_1, Y_1), \dots, (X_n, Y_n)$ with unit Fréchet distributed margins, so that $\Pr(X \leq x) = \Pr(Y \leq x) = \exp(-1/x)$ on $x > 0$, and joint distribution function $F(x, y)$ where F is assumed to be in the domain of attraction of a bivariate extreme value (BEV) distribution (Resnick, 1987). The choice of unit

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Fréchet distributed margins admits no loss of generality here, as probability integral transformations may be used to derive corresponding results for arbitrary marginal distributions. Defining the componentwise maxima $M_{X,n} = \max(X_1, \dots, X_n)$ and $M_{Y,n} = \max(Y_1, \dots, Y_n)$, it is clear $\Pr(M_{X,n}/n \leq x) = \Pr(M_{Y,n}/n \leq x) = \exp(-1/x)$ so that $M_{X,n}/n$ and $M_{Y,n}/n$ are both exactly unit Fréchet distributed, or more loosely, a normalisation of n^{-1} is required in order to stabilize the componentwise maxima.

Following de Haan (1985), it can be shown that the point process \mathcal{P}_n defined by

$$\mathcal{P}_n = \left\{ \left(\frac{X_i}{n}, \frac{Y_i}{n} \right); i = 1, \dots, n \right\}$$

converges weakly in the limit as $n \rightarrow \infty$ to a non homogeneous Poisson process \mathcal{P} on $[0, \infty) \times [0, \infty) \setminus (0, 0)$. Furthermore, by changing variables to the pseudo-radial and angular coordinates $R = X + Y$ and $W = X/R$, it can be shown that \mathcal{P} has a point intensity μ that factorizes as

$$\mu(dr \times dw) = r^{-2} dr dH(w),$$

where H is a non negative measure on $[0, 1]$ that is arbitrary apart from having to satisfy

$$\int_0^1 w dH(w) = \int_0^1 (1-w) dH(w) = 1. \quad (1.1)$$

An equivalent, but more straightforward, way to express the conditions (1.1) is that $H/2$ is the cumulative probability distribution function of any random variable on $[0, 1]$ that has mean equal to $1/2$.

As an illustration of how \mathcal{P} can be used to derive classical multivariate extreme value results, we use it here to derive the general characterisation of a BEV distribution with unit Fréchet distributed margins. Following the set up above, let G denote any non degenerate limiting bivariate distribution function of $(M_{X,n}/n, M_{Y,n}/n)$ and define the set $A_{xy} = \{(0, x] \times (0, y]\}$ for fixed positive x and y . Then

$$\begin{aligned} G(x, y) &= \lim_{n \rightarrow \infty} \Pr\left(\frac{M_{X,n}}{n} \leq x, \frac{M_{Y,n}}{n} \leq y\right) \\ &= \lim_{n \rightarrow \infty} \Pr(\text{all points of } \mathcal{P}_n \text{ are in } A_{xy}) \\ &= \lim_{n \rightarrow \infty} \Pr(\text{no points of } \mathcal{P}_n \text{ are outside } A_{xy}) \\ &= \Pr(\text{no points of } \mathcal{P} \text{ are outside } A_{xy}) \\ &= \exp(-1 \times \text{Expected number of points of } \mathcal{P} \text{ outside } A_{xy}) \\ &= \exp\left\{-\int_0^1 \int_{r_{xy}(w)}^{\infty} r^{-2} dr dH(w)\right\}, \end{aligned} \quad (1.2)$$

where $r_{xy}(w) = \min\{x/w, y/(1-w)\}$. On performing the r -integration in Eq. (1.2), we obtain $G(x, y) = \exp\{-V(x, y)\}$ where V is a function that determines the

dependence structure of $G(x, y)$ and is given by

$$V(x, y) = \int_0^1 \max\left(\frac{w}{x}, \frac{1-w}{y}\right) dH(w) \quad (1.3)$$

for H satisfying the regularity conditions (1.1). This is the classical representation of a BEV distribution with unit Fréchet distributed margins; see de Haan and Resnick (1977) and Pickands (1981). It is easy to see from equation (1.3) that V is a homogeneous function of order -1 , that is $V(nx, ny) = n^{-1}V(x, y)$. Hence, we have that G is max-stable since $G^n(nx, ny) = G(x, y)$, and G is also in its own domain of attraction since if each $(X_i, Y_i) \sim G$ in the above then $(M_{X,n}/n, M_{Y,n}/n) \sim G$ for each $n \geq 1$.

1.2. Limitations of the Classical Theory

The notions of asymptotic dependence and asymptotic independence are important for describing the limitations of the classical theory and so are now defined. Letting κ denote the limiting conditional probability in

$$\lim_{u \rightarrow \infty} \Pr(X > u \mid Y > u) = \kappa$$

then X and Y are said to be asymptotically dependent if $\kappa > 0$ and asymptotically independent if $\kappa = 0$ (Sibuya, 1960). Asymptotic independence is an important feature that arises in both practical data modelling applications and the study of the dependence features of most classical families of distributions; see de Haan and de Ronde (1998), Ledford and Tawn (1997), Capéraà et al. (2000), and Heffernan (2000).

When the components of the underlying (X_i, Y_i) pairs in Sec. 1.1 are asymptotically dependent, then the de Haan (1985) limiting point process representation provides a sound foundation for modeling the observed joint tail dependence structure, and consequently constructions derived from this representation such as the class of BEV distributions are well motivated and typically useful in practice. This is not the case when the (X_i, Y_i) components are asymptotically independent because then the limiting point process \mathcal{P} is degenerate as the measure $H(w)$ is identically zero on the interior of $[0, 1]$ and places unit masses at $w = 0$ and $w = 1$. In this case, Eqs. (1.2) and (1.3) give that the resulting BEV distribution has exactly independent margins since $G(x, y) = \exp\{-(1/x + 1/y)\}$. The fact that the limiting point process, and hence constructions derived from it such as the class of BEV distributions, all correspond either to exact independence or asymptotic dependence leads to non regular inference issues; see Tawn (1988), Ledford and Tawn (1996), and Ramos and Ledford (2005). Considerable difficulties also arrive in situations where asymptotic independence is apparent, as neither exact independence nor asymptotic dependence may be adequate for describing the observed joint tail dependence structure; see Ledford and Tawn (1996, 1997) and Coles et al. (1999). The crux of the problem is that the currently available underlying limit theory is degenerate for cases of asymptotic independence. It is our view that an alternative limit theory is required that provides useful results for both asymptotic dependence and asymptotic independence. This is what is provided in this article.

1.3. *Alternatives to the Classical Approach*

Over the last decade or so there has been progress on developing extremal dependence models that accommodate both asymptotic dependence and asymptotic independence. Ledford and Tawn (1997) proposed a method based on joint survivor functions that could be applied to both asymptotically dependent and asymptotically independent data but overlooked some necessary regularity conditions and used a parametric form that was not guaranteed to correspond to a valid probability model. More recently, Ramos and Ledford (2009) extended the Ledford and Tawn (1997) results by establishing the required regularity conditions, developing a flexible parametric model that satisfied these and deriving a censored likelihood approach for estimation. Coles and Pauli (2002) developed a different approach based on a parametric copula that describes dependence throughout the entire domain rather than just the extremes, and Heffernan and Tawn (2004) derived a method with a more empirical flavour based on combining normalised limiting conditional distributions. However, none of these developments provided a modeling framework with generality and depth comparable to those of the de Haan (1985) point process and its encapsulation of other classical extreme results such as the characterization (1.2). Resnick (2002) and Maulik and Resnick (2004) also gave a characterization and representation of distributions possessing a property they called hidden regular variation, which is a property of the subfamily of distributions possessing both multivariate regular variation and asymptotic independence, and is based on an elaboration of the coefficient of tail dependence of Ledford and Tawn (1996, 1997). In these two works, hidden regular variation is represented in terms of a non negative measure called the hidden angular (or spectral) measure, which was estimated by Heffernan and Resnick (2005) using the rank transform. We construct a similar representation via our point process approach and thereby obtain a simple normalization condition that the hidden angular measure must satisfy. This condition is missing in the hidden regular variation papers cited above.

The starting point for this article is the limiting Poisson point process result that was outlined by Ledford and Tawn (1997). Focusing on the bivariate case, we derive regularity conditions on the intensity function of this point process in Sec. 2. In Sec. 3, we use our point process results to derive the analogue of the classical componentwise maxima results given in Sec. 1.1. A flexible parametric example is introduced in Sec. 4 that is used to illustrate the new point process and componentwise maxima results, and also to demonstrate how the joint tail modeling approach of Ramos and Ledford (2009) can be derived from the point process results developed in this paper. Methods for simulating from both the underlying point process and the limiting distribution of componentwise maxima are given in Sec. 5. In Sec. 6, we provide some concluding remarks and show how the bivariate results discussed in this article can be extended to the multivariate case. In the Appendices, we provide some proofs.

2. A Unified Point Process Framework

In this section, we build on the limiting Poisson process results of Ledford and Tawn (1997) and show that the intensity function which arises can be written in terms of a non negative measure that is arbitrary apart from having to satisfy some regularity conditions. Our treatment unifies the asymptotically dependent and asymptotically independent cases. The convergence of the relevant sequence of point processes

to a Poisson process on the cone $(0, \infty)^2$ and the convergence of the associated intensity measure can also be obtained from the results of Resnick (2002), Maulik and Resnick (2004), and Heffernan and Resnick (2005).

2.1. Derivation

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ denote independent and identically distributed bivariate random variables with unit Fréchet distributed margins and joint survivor function that is bivariate regularly varying with index $-1/\eta$ for some $\eta \in (0, 1]$, so that

$$\bar{F}(x, y) = \Pr(X > x, Y > y) = \frac{\mathcal{L}(x, y)}{(xy)^{1/(2\eta)}} \tag{2.1}$$

where \mathcal{L} is a bivariate slowly varying (BSV) function. Following Ledford and Tawn (1996, 1997), we refer to the parameter $\eta \in (0, 1]$ as the coefficient of tail dependence. The representation (2.1) accommodates a wide spectrum of dependence behaviour, including asymptotic independence, asymptotic dependence, complete dependence and exact independence. As a modeling framework, it also provides a smooth transition between complete dependence and exact independence, and allows negative dependence between the marginal extremes, yielding a very flexible and broadly applicable statistical model; see Ramos and Ledford (2009). Let g denote the limit function of \mathcal{L} , so that for all $(x, y) \in (0, \infty) \times (0, \infty)$ and $c > 0$ we have

$$g(x, y) = \lim_{r \rightarrow \infty} \frac{\mathcal{L}(rx, ry)}{\mathcal{L}(r, r)} \quad \text{and} \quad g(cx, cy) = g(x, y). \tag{2.2}$$

A consequence of Eq. (2.2) is that for any $x > 0$ and $y > 0$

$$g(x, y) = g\{x/(x + y), y/(x + y)\} = g\{w, 1 - w\} \equiv g^*(w), \tag{2.3}$$

where $w = x/(x + y) \in (0, 1)$, and thus it is clear that the bivariate function $g(x, y)$ depends only on the univariate quantity $w \in (0, 1)$.

Following Ledford and Tawn (1997), we consider the sequence of point processes

$$\mathcal{P}_n^* = \left\{ \left(\frac{X_i}{b_n}, \frac{Y_i}{b_n} \right) : i = 1, \dots, n \right\} \tag{2.4}$$

where the normalising constants b_n are defined implicitly as satisfying $n\bar{F}(b_n, b_n) = 1$, so that typically $b_n = O(n^\eta)$. Clearly, when $\eta < 1$ the normalising constants satisfy $b_n = o(n)$ and are therefore too light to stabilize the componentwise maxima $M_{X,n}$ and $M_{Y,n}$ within \mathcal{P}_n^* ; see Sec. 1. In contrast, when the componentwise maxima are asymptotically dependent then $b_n = O(n)$ and $M_{X,n}$ and $M_{Y,n}$ are stabilized. The impact of this distinction is that \mathcal{P}_n^* converges weakly to a non homogeneous Poisson process \mathcal{P}^* in the limit as $n \rightarrow \infty$ with domain $(0, \infty) \times (0, \infty)$ when $\eta < 1$ and domain $\{[0, \infty) \times [0, \infty)\} \setminus \{(0, 0)\}$ when the componentwise maxima are asymptotically dependent. The first case is discussed in more detail in Ledford and Tawn (1997) and the latter case in de Haan (1985), however for our purposes we will restrict attention to the domain $(0, \infty) \times (0, \infty)$ as then our results will hold

regardless of whether the componentwise maxima are asymptotically dependent or asymptotically independent.

We start by deriving the intensity function $\Lambda_\eta\{(x, \infty) \times (y, \infty)\}$ of \mathcal{P}^* for any $x > 0$ and $y > 0$, that is the expected number of points of \mathcal{P}^* within a set of the form $\{(x, \infty) \times (y, \infty)\}$. By definition, and substituting according to Eqs. (2.1) and (2.2), we have

$$\begin{aligned} \Lambda_\eta\{(x, \infty) \times (y, \infty)\} &= \lim_{n \rightarrow \infty} n\bar{F}(b_n x, b_n y) \\ &= \lim_{n \rightarrow \infty} n\bar{F}(b_n, b_n) \frac{\bar{F}(b_n x, b_n y)}{\bar{F}(b_n, b_n)} = \frac{g(x, y)}{(xy)^{1/(2\eta)}} \end{aligned} \tag{2.5}$$

since $n\bar{F}(b_n, b_n) = 1$. Substituting into Eq. (2.5) according to the pseudo-polar coordinates $r = x + y$ and $w = x/r$ and exploiting Eq. (2.3), it is clear that

$$\Lambda_\eta\{(x, \infty) \times (y, \infty)\} = r^{-1/\eta} g^*(w) / \{w(1 - w)\}^{1/(2\eta)}$$

and thus for any $x > 0$ and $y > 0$ we have $\Lambda_\eta\{(x, \infty) \times (y, \infty)\}$ factorizes into separate terms involving r and w . We now derive an alternative expression for $\Lambda_\eta\{(x, \infty) \times (y, \infty)\}$ by first expressing it as the integral of its point intensity μ_η over the set $\{(x, \infty) \times (y, \infty)\}$ and then changing variables to (r, w) . Thus, we obtain

$$\begin{aligned} \Lambda_\eta\{(x, \infty) \times (y, \infty)\} &= \int_{\{(x, \infty) \times (y, \infty)\}} \mu_\eta(dx \times dy) \\ &= \int_{w \in (0,1)} \int_{r=r'_{xy}(w)}^\infty r^{-(1+1/\eta)} dr dH_\eta(w) \end{aligned} \tag{2.6}$$

$$= \eta \int_{(0,1)} \left\{ \min\left(\frac{w}{x}, \frac{1-w}{y}\right) \right\}^{1/\eta} dH_\eta(w), \tag{2.7}$$

where $r'_{xy}(w) = \max\{x/w, y/(1 - w)\}$ and $H_\eta(w)$ is a non negative measure on $(0, 1)$; see Ramos and Ledford (2009) or Beirlant et al. (2004). The non negative measure $H_\eta(w)$ is arbitrary apart from having to satisfy a simple normalization condition that we now derive. Since $\lim_{n \rightarrow \infty} n\bar{F}(b_n, b_n) = 1$ implies that the expected number of points of \mathcal{P}^* within $(1, \infty) \times (1, \infty)$ is 1 also, then η and H_η are related via

$$1 = \Lambda_\eta\{(1, \infty) \times (1, \infty)\} = \eta \int_{(0,1)} \{\min(w, 1 - w)\}^{1/\eta} dH_\eta(w),$$

or equivalently, H_η must satisfy the following normalization condition:

$$\eta^{-1} = \int_0^{1/2} w^{1/\eta} dH_\eta(w) + \int_{1/2}^1 (1 - w)^{1/\eta} dH_\eta(w). \tag{2.8}$$

The measure H_η is a particular case of the hidden angular measure considered by Resnick (2002) and Maulik and Resnick (2004), however these articles omit the normalization condition (2.8).

The results obtained above add significant rigour to the treatment provided by Ledford and Tawn (1997) and extend the hidden regular variation results of Resnick and collaborators to include the integral constraint (2.8). The normalization

condition (2.8) is identical to that obtained in Ramos and Ledford (2009) but is different to that which arises in the de Haan (1985) classical point process treatment of multivariate extremes; see Eq. (1.1). Condition (2.8), although necessary, is not sufficient to ensure that an (η, H_η) -pair defines the intensity of a point process that can arise via construction (2.4) from a bivariate random variable (X, Y) with unit Frechét distributed marginals. As in Theorem 1 of Ramos and Ledford (2009), additional conditions are required for this so that the marginal tails of the point process are not heavier than those of a unit Frechét variable. See Appendix A for details.

In order to provide a brief illustration, here we examine a simple discrete model. Following Ramos and Ledford (2009), suppose initially that H_η has two atoms of equal mass a at positions w_1 and $(1 - w_1)$ in the interior of $(0, 1)$ so that $0 < w_1 < 1/2$. The normalization condition (2.8) implies $a = (2\eta)^{-1}w_1^{-1/\eta}$. Generalizing this to the case of an asymmetric discrete measure with positive masses $a_1, \dots, a_q, \dots, a_n$ at positions $0 < w_1 < \dots < w_q < \dots < w_n < 1$, where q is such that $w_{q-1} \leq 1/2 < w_q$, the normalization condition (2.8) is satisfied if and only if

$$\sum_{i=1}^{q-1} a_i w_i^{1/\eta} + \sum_{i=q}^n a_i (1 - w_i)^{1/\eta} = \eta^{-1}.$$

The additional conditions in Appendix A are satisfied for any $\eta \in (0, 1)$. We do not examine this model any further in this paper but instead concentrate later on a smooth and flexible parametric form for H_η that is more suited to applications.

3. Componentwise Maxima Revisited

In this section, we study an analogue of result (1.2) based on the sequence of point processes \mathcal{P}_n^* of Sec. 2. This topic was examined briefly by Ledford and Tawn (1997), however the more detailed treatment given here makes clear some issues that are not apparent within their treatment. Our approach is to obtain the limiting distribution of normalized componentwise maxima of points that occur within a region bounded away from the axes and then examine the behaviour of this distribution as the bounds approach the axes.

Under the same setup as Sec. 2, we additionally define $R_\varepsilon = \{(x, y) : x > \varepsilon, y > \varepsilon\}$ and $R_\varepsilon(x, y) = R_\varepsilon \setminus \{(\varepsilon, x) \times (\varepsilon, y)\}$, and let $M_{X,n,\varepsilon}$ and $M_{Y,n,\varepsilon}$ denote the componentwise maxima of those points out of $(X_1, Y_1), \dots, (X_n, Y_n)$ that occur within R_ε . Then for any $x > \varepsilon$ and $y > \varepsilon$ we have

$$\begin{aligned} & \Pr\left(\frac{M_{X,n,\varepsilon b_n}}{b_n} \leq x, \frac{M_{Y,n,\varepsilon b_n}}{b_n} \leq y\right) \\ &= \Pr\{\text{no}(X_i, Y_i) \text{ pair occurs within } R_{\varepsilon b_n}(xb_n, yb_n)\} \\ &= \Pr\{\text{every}(X_i, Y_i) \text{ pair occurs outside } R_{\varepsilon b_n}(xb_n, yb_n)\} \\ &= \{1 - \bar{F}(xb_n, \varepsilon b_n) - \bar{F}(\varepsilon b_n, yb_n) + \bar{F}(xb_n, yb_n)\}^n \\ &= \left[1 - \frac{1}{n} \left\{ \frac{\bar{F}(xb_n, \varepsilon b_n)}{\bar{F}(b_n, b_n)} + \frac{\bar{F}(\varepsilon b_n, yb_n)}{\bar{F}(b_n, b_n)} - \frac{\bar{F}(xb_n, yb_n)}{\bar{F}(b_n, b_n)} \right\}\right]^n \end{aligned}$$

since $n\bar{F}(b_n, b_n) = 1$. Substituting according to Eq. (2.5) and taking the limit as $n \rightarrow \infty$ gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr\left(\frac{M_{X,n,\varepsilon b_n}}{b_n} \leq x, \frac{M_{Y,n,\varepsilon b_n}}{b_n} \leq y\right) \\ = \exp\left[-\left\{\frac{g(x, \varepsilon)}{(x\varepsilon)^{1/(2\eta)}} + \frac{g(\varepsilon, y)}{(\varepsilon y)^{1/(2\eta)}} - \frac{g(x, y)}{(xy)^{1/(2\eta)}}\right\}\right] \end{aligned} \tag{3.1}$$

$$= \exp\{-1 \times \text{Expected number of points of } \mathcal{P}^* \text{ that occur in } R_\varepsilon(x, y)\}. \tag{3.2}$$

To examine the terms in Eq. (3.1) as $\varepsilon \rightarrow 0$, we recall Eqs. (2.5) and (2.7) and so obtain

$$\frac{g(x, y)}{(xy)^{1/(2\eta)}} = \eta \int_{(0,1)} \left\{ \min\left(\frac{w}{x}, \frac{1-w}{y}\right) \right\}^{1/\eta} dH_\eta(w). \tag{3.3}$$

From Eq. (3.3), we therefore have

$$\frac{g(x, \varepsilon)}{(x\varepsilon)^{1/(2\eta)}} = \eta \int_0^{x/(x+\varepsilon)} \left(\frac{w}{x}\right)^{1/\eta} dH_\eta(w) + \eta \int_{x/(x+\varepsilon)}^1 \left(\frac{1-w}{\varepsilon}\right)^{1/\eta} dH_\eta(w),$$

and hence

$$\lim_{\varepsilon \rightarrow 0} \frac{g(x, \varepsilon)}{(x\varepsilon)^{1/(2\eta)}} = \eta x^{-1/\eta} \int_{(0,1)} w^{1/\eta} dH_\eta(w) + \eta \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/\eta} \int_{x/(x+\varepsilon)}^1 (1-w)^{1/\eta} dH_\eta(w).$$

Similarly, Eq. (3.3) gives

$$\lim_{\varepsilon \rightarrow 0} \frac{g(\varepsilon, y)}{(\varepsilon y)^{1/(2\eta)}} = \eta \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/\eta} \int_0^{\varepsilon/(\varepsilon+y)} w^{1/\eta} dH_\eta(w) + \eta y^{-1/\eta} \int_{(0,1)} (1-w)^{1/\eta} dH_\eta(w).$$

Thus, provided the measure H_η satisfies the normalization condition (2.8) and is finite then both limits on the right-hand side of the above equations equal zero for any $x > 0$ and $y > 0$, respectively. Therefore, taking the limit of Eq. (3.1) as $\varepsilon \rightarrow 0$ gives a non degenerate result and

$$G_\eta(x, y) \equiv \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \Pr\left(\frac{M_{X,n,\varepsilon b_n}}{b_n} \leq x, \frac{M_{Y,n,\varepsilon b_n}}{b_n} \leq y\right) = \exp\{-V_\eta(x, y)\} \tag{3.4}$$

where

$$\begin{aligned} V_\eta(x, y) &= \text{Expected number of points of } \mathcal{P}^* \text{ that occur in } \lim_{\varepsilon \rightarrow 0} R_\varepsilon(x, y) \\ &= \eta \int_{(0,1)} \left\{ \max\left(\frac{w}{x}, \frac{1-w}{y}\right) \right\}^{1/\eta} dH_\eta(w). \end{aligned} \tag{3.5}$$

It is straightforward to see that the dependence function V_η defined in Eq. (3.5) is homogeneous of order $-1/\eta$, that is, $V_\eta(tx, ty) = t^{-1/\eta} V_\eta(x, y)$ for all $t > 0$. Thus, we

have $G_\eta^n(n^\eta x, n^\eta y) = G_\eta(x, y)$ and so G_η is max-stable. Examining now the marginal behavior implied by Eqs. (3.4) and (3.5), we obtain

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \Pr(M_{X,n,\varepsilon b_n}/b_n \leq x) = \exp\{-V_\eta(1, \infty)x^{-1/\eta}\} \quad \text{and}$$

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \Pr(M_{Y,n,\varepsilon b_n}/b_n \leq y) = \exp\{-V_\eta(\infty, 1)y^{-1/\eta}\},$$

that is, Fréchet distributions with shape parameter η and scale parameters equal to $V_\eta(1, \infty) = \eta \int_0^1 w^{1/\eta} h_\eta(w) dw$ and $V_\eta(\infty, 1) = \eta \int_0^1 (1-w)^{1/\eta} h_\eta(w) dw$, respectively. So, unlike for the classical results given in Sec. 1.1, the limiting marginal distributions here are not unit Fréchet. However, by adopting different normalizing conditions in the point process, it is possible to obtain a limiting distribution that has Fréchet margins with the same shape parameter but scale parameters equal to 1. Specifically, the point process

$$\mathcal{P}_n^\# = \left\{ \left(\frac{X_i}{V_\eta^\eta(1, \infty)b_n}, \frac{Y_i}{V_\eta^\eta(\infty, 1)b_n} \right) : i = 1, \dots, n \right\}$$

achieves this.

Equations (3.4) and (3.5) give the joint distribution of componentwise maxima for pairs of variables which are simultaneously large regardless of whether the underlying dependence structure is asymptotically dependent or asymptotically independent, and thus considerably extend the standard componentwise maxima result given in Eq. (1.2). It is hoped that this asymptotically motivated framework will provide versatile and improved modelling and extrapolation capabilities compared to the classical BEV paradigm, especially in cases where neither exact independence nor asymptotic dependence is appropriate. The above also clarifies the point process results of Ledford and Tawn (1997) and makes clear the convergence issues. We remark that the limiting point process \mathcal{P}^* of Sec. 2, in contrast to the de Haan (1985) approach, is based on a normalisation that stabilises the joint tail rather than the componentwise maxima and a domain that excludes the axes $x = 0$ and $y = 0$. If these key differences are omitted, then we are back in the classical BEV case and result (1.2) holds. This point seems to have been overlooked in the statement of the result given in Sec. 3.2.1 of Kotz and Nadarajah (2000).

The constraint of H_η being finite causes some restriction since, as noted by Maulik and Resnick (2004), the class of regularly varying multivariate distributions with infinite hidden angular measure is at least as large as the class of distributions on $[1, \infty)$ which have infinite mean. Thus, an infinite measure H_η which is valid for the point process \mathcal{P}^* of Sec. 2 will yield the degenerate limit of zero as $\varepsilon \rightarrow 0$ for the componentwise maxima distribution in Eq. (3.4). We will see an explicit example of this later. This emphasizes that the point process representation in Sec. 2 accommodates a wider class of dependence behavior than that captured by the componentwise maxima results of this section. Geometrically, what is happening when the componentwise maxima results are degenerate is that $M_{X,n,\varepsilon b_n}$ and $M_{Y,n,\varepsilon b_n}$ grow too fast to be stabilized by the b_n normalization. We illustrate this later through simulation results for a particular parametric example.

Remark 3.1. In Eq. (3.5), the dependence function V_η is defined in terms of a given measure H_η . Restricting ourselves here to the class of models for which V_η is

differentiable, we can show that H_η , or more specifically its measure density h_η , can be recovered from a given V_η . Our approach is similar to that of Coles and Tawn (1991). Using, for example, Eqs. (2.7) and (3.5), it is clear that

$$\Lambda_\eta\{(x, \infty) \times (y, \infty)\} = V_\eta(x, \infty) + V_\eta(\infty, y) - V_\eta(x, y), \tag{3.6}$$

so taking the mixed derivative of Eq. (3.6) with respect to x and y we obtain

$$\frac{\partial^2 V_\eta}{\partial x \partial y} = -r^{-(2+1/\eta)} h_\eta(w), \tag{3.7}$$

where $r = x + y$ and $w = x/(x + y)$. Thus, provided the density h_η exists, Eq. (3.7) describes the relationship between the pseudo-angular measure H_η and the exponent measure V_η . This extends part of the Coles and Tawn (1991) result to the asymptotically independent case, however their result is more wide-ranging as it also covers the relationship between the lower dimensional pseudo-radial and exponent measures in higher dimensional cases.

4. A Parametric Example

In order to examine the properties of estimators and to undertake applied statistical modeling, it is useful to have a tractable parametric model that is flexible enough to exhibit a wide range of behaviours and also allows straightforward simulation. Here we introduce such a model via a particular parametric H_η family. This model will be the focus for much of the remainder of the article.

4.1. The η -Asymmetric Logistic Model

This is similar to the BEV-asymmetric logistic dependence structure of classical bivariate extremes; see Tawn (1988). As defined in Ramos and Ledford (2009), consider H_η with density h_η given by

$$h_\eta(w) = \frac{\eta - \alpha}{\alpha \eta^2 N_\varrho} \left\{ (\varrho w)^{-1/\alpha} + \left(\frac{1-w}{\varrho} \right)^{-1/\alpha} \right\}^{\alpha/\eta-2} \{w(1-w)\}^{-(1+1/\alpha)} \tag{4.1}$$

for $w \in (0, 1)$ where $N_\varrho = \varrho^{-1/\eta} + \varrho^{1/\eta} - (\varrho^{-1/\alpha} + \varrho^{1/\alpha})^{\alpha/\eta}$ and $\eta \in (0, 1]$, $\varrho > 0$ and $\alpha \in (0, 1]$. It is straightforward to show that h_η satisfies both the normalisation condition (2.8) and the conditions mentioned in Appendix A.

4.2. Point Process Intensity

For the η -asymmetric logistic model with measure density h_η satisfying Eq. (4.1), Eq. (2.7) gives that the intensity function of the limiting Poisson point process \mathcal{P}^* satisfies

$$\Lambda_\eta\{(x, \infty) \times (y, \infty)\} = N_\varrho^{-1} \left[(\varrho x)^{-1/\eta} + \left(\frac{y}{\varrho} \right)^{-1/\eta} - \left\{ (\varrho x)^{-1/\alpha} + \left(\frac{y}{\varrho} \right)^{-1/\alpha} \right\}^{\alpha/\eta} \right], \tag{4.2}$$

where $x > 0$ and $y > 0$.

4.3. Limiting Distribution of Componentwise Maxima

For the η -asymmetric logistic model defined by Eq. (4.1), the limiting componentwise maxima distribution given in Eq. (3.4) satisfies

$$G_\eta(x, y) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \Pr(M_{X,n,\varepsilon b_n} \leq x b_n, M_{Y,n,\varepsilon b_n} \leq y b_n) \\ = \begin{cases} \exp \left[-N_\varrho^{-1} \left\{ (\varrho x)^{-1/\alpha} + \left(\frac{y}{\varrho}\right)^{-1/\alpha} \right\}^{\alpha/\eta} \right] & \text{for } \alpha < \eta, \\ 0 & \text{for } \alpha \geq \eta. \end{cases} \tag{4.3}$$

The marginal distributions of G_η are easily obtained from Eq. (4.3) by substituting $x = \infty$ or $y = \infty$ and hence satisfy

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \Pr(M_{X,n,\varepsilon b_n} \leq x b_n) = \begin{cases} \exp(-N_\varrho^{-1} \varrho^{-1/\eta} x^{-1/\eta}) & \text{for } \alpha < \eta, \\ 0 & \text{for } \alpha \geq \eta, \end{cases} \text{ and} \\ \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \Pr(M_{Y,n,\varepsilon b_n} \leq y b_n) = \begin{cases} \exp(-N_\varrho^{-1} \varrho^{1/\eta} y^{-1/\eta}) & \text{for } \alpha < \eta, \\ 0 & \text{for } \alpha \geq \eta. \end{cases}$$

The degenerate $\alpha \geq \eta$ case in the above arises when the measure H_η is infinite, as discussed in Sec. 3. We return to this in Sec. 5.4.

4.4. Limiting Model for the Joint Tail

Let A_{st} denote a set of the form $\{(x', y') : x' > s, y' > t\}$ for some fixed $s > 0$ and $t > 0$. Following the notation of Sec. 2 and writing $b_n A_{st}$ for $\{(x', y') : x' > b_n s, y' > b_n t\}$, then on $x > s$ and $y > t$ we have that

$$\lim_{n \rightarrow \infty} \Pr\{(X, Y) \in b_n A_{xy} \mid (X, Y) \in b_n A_{st}\} = \lim_{n \rightarrow \infty} \frac{\Pr(X > b_n x, Y > b_n y)}{\Pr(X > b_n s, Y > b_n t)} \\ = \frac{\Lambda_\eta\{(x, \infty) \times (y, \infty)\}}{\Lambda_\eta\{(s, \infty) \times (t, \infty)\}}. \tag{4.4}$$

The normalization condition (2.8) and the conditions in Appendix A ensure the existence of a bivariate random variable (X, Y) with unit Fréchet distributed margins such that Eqs. (2.7) and (4.4) hold.

Substituting the η -asymmetric logistic intensity function given in Eq. (4.2) into Eq. (4.4), we obtain the limiting joint tail model

$$\lim_{n \rightarrow \infty} \Pr\{(X, Y) \in b_n A_{xy} \mid (X, Y) \in b_n A_{st}\} \\ = N_{st\varrho}^{-1} \left[(\varrho x)^{-1/\eta} + \left(\frac{y}{\varrho}\right)^{-1/\eta} - \left\{ (\varrho x)^{-1/\alpha} + \left(\frac{y}{\varrho}\right)^{-1/\alpha} \right\}^{\alpha/\eta} \right] \tag{4.5}$$

for $x > s$ and $y > t$, where $N_{st\varrho} = (\varrho s)^{-1/\eta} + (t/\varrho)^{-1/\eta} - \{(\varrho s)^{-1/\alpha} + (t/\varrho)^{-1/\alpha}\}^{\alpha/\eta}$. This is the basis of the modeling approach and likelihood methods adopted by Ramos and Ledford (2009) who chose the $s = t = 1$ special case of this more general result.

Clearly, since the $\alpha \geq \eta$ case gives rise to a non degenerate model in Eq. (4.5) and a degenerate limiting distribution in Eq. (4.3), the joint tail modeling approach of this section has more generality and wider applicability than an approach based on the componentwise maxima results.

5. Simulation

Methods for simulating from both the limiting point process and the limiting distribution of componentwise maxima are examined here. Most discussion focuses on the η -asymmetric logistic model introduced in Section 4.

5.1. The η -Asymmetric Logistic Limiting Point Process

The simulation scheme here is similar to that in Ramos and Ledford (2009). Differentiating Eq. (4.5) yields the joint density

$$\frac{\eta - \alpha}{\alpha \eta^2 N_{st\varrho}} \left\{ (\varrho x)^{-1/\alpha} + \left(\frac{y}{\varrho}\right)^{-1/\alpha} \right\}^{\alpha/\eta-2} (xy)^{-(1+1/\alpha)} \tag{5.1}$$

for $x > s, y > t, N_{st\varrho}$ as defined in Sec. 4.4 and fixed $s > 0$ and $t > 0$. Similarly to Shi et al. (1992), we construct (X, Y) pairs with joint density (5.1) by transforming a pair of easily simulated variables, V and Z say. This is achieved via $X^{-1} = \varrho Z^\eta \cos^{2\alpha} V$ and $\varrho Y^{-1} = Z^\eta \sin^{2\alpha} V$ where V and Z satisfy

$$V = \begin{cases} \arccos \left[\{1 - N_{st\varrho} (\varrho s)^{1/\eta} U_1\}^{\frac{\eta}{2(\eta-\alpha)}} \right] & \text{if } 0 < U_1 \leq u^* \\ \arcsin \left\{ \left(\left(\frac{t}{\varrho}\right)^{1/\eta} \left[N_{st\varrho} U_1 - (\varrho s)^{-1/\eta} + \left\{ (\varrho s)^{-1/\alpha} + \left(\frac{t}{\varrho}\right)^{-1/\alpha} \right\}^{\alpha/\eta} \right] \right)^{\frac{\eta}{2(\eta-\alpha)}} \right\} & \\ \text{if } u^* < U_1 < 1 & \end{cases} \tag{5.2}$$

and

$$Z = \begin{cases} U_2 (\varrho s)^{-1/\eta} \cos^{-2\alpha/\eta} V & \text{if } 0 < U_1 \leq u^* \\ U_2 \left(\frac{t}{\varrho}\right)^{-1/\eta} \sin^{-2\alpha/\eta} V & \text{if } u^* < U_1 < 1 \end{cases} \tag{5.3}$$

where U_1 and U_2 are independent uniformly distributed on $[0, 1]$ and

$$u^* = N_{st\varrho}^{-1} \left[(\varrho s)^{-1/\eta} - (\varrho s)^{-1/\alpha} \left\{ (\varrho s)^{-1/\alpha} + \left(\frac{t}{\varrho}\right)^{-1/\alpha} \right\}^{\alpha/\eta-1} \right].$$

A derivation is given in Appendix B. The above construction generalizes the Ramos and Ledford (2009) simulation method which deals with the $s = t = 1$ case only. Illustrative data sets generated using this scheme together with contours of the underlying joint density are given in Fig. 1. Note that $\alpha > 1$ in Eq. (5.1) is a valid density function however this is not a density that can arise for (X_i, Y_i) with unit

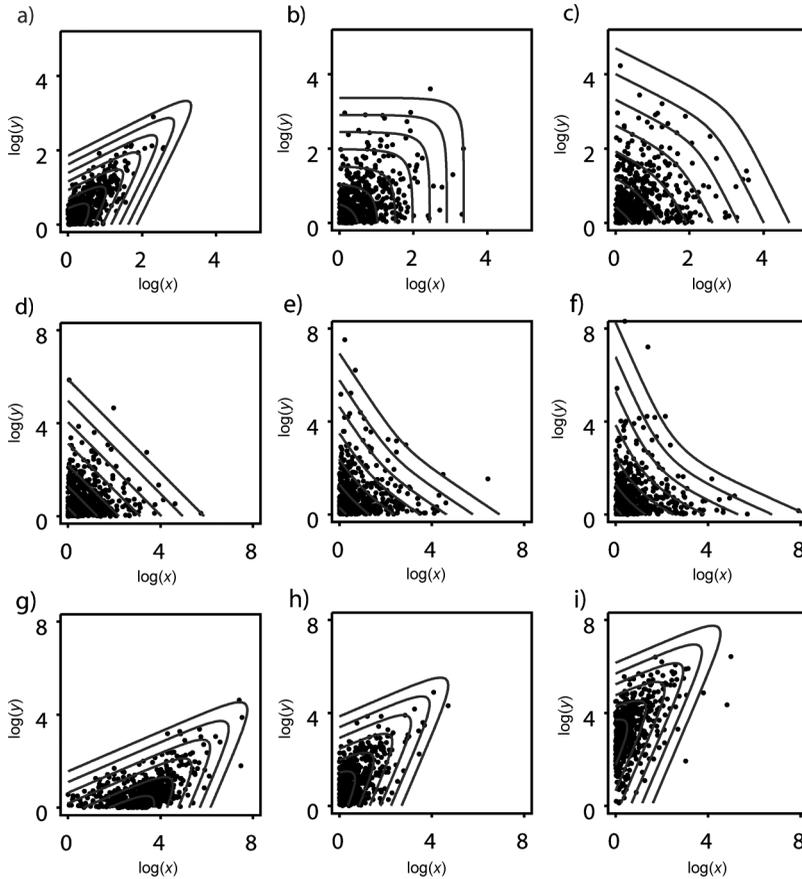


Figure 1. Simulated points from the η -asymmetric logistic point process obtained using the algorithm of Sec. 5.1 over region $\{(x, y) : x > 1, y > 1\}$. Density contours are also shown. Panels (a)–(f) all have $\eta = 0.4$ and $\varrho = 1$, corresponding to symmetric cases, with $\alpha = \{0.2, 0.4, 0.6, 0.8, 1, 1.3\}$ respectively. Panels (g)–(i) each have $\eta = 0.7$ and $\alpha = 0.4$, and show the effect of asymmetry with $\varrho = \{0.2, 1.5, 5\}$, respectively. Note that the axes are not all to the same scale. Panel f) illustrates an $\alpha > 1$ case of Eq. (5.1), see Sec. 5.1.

Fréchet marginals in definition (2.4) since the constraints in Appendix A are then violated.

5.2. More General Limiting Point Process Simulation

The approach described here is similar to that of Nadarajah (1999) and enables simulation of the limiting point process \mathcal{P}^* over a region bounded away from the axes and the origin. From Eq. (2.6), the conditional density of the pseudo-polar transforms (R, W) of (X, Y) in region $A = \{(r, w) : r > r_0, w \in (k, 1 - k)\}$ where $r_0 > 0$ and $0 < k < 1$ satisfies

$$f\{r, w \mid (R, W) \in A\} = \frac{r^{-(1+1/\eta)} dH_\eta(w)}{\int_k^{1-k} \int_{r_0}^\infty r^{-(1+1/\eta)} dr dH_\eta(w)} = \frac{r_0^{1/\eta} r^{-(1+1/\eta)} dH_\eta(w)}{\eta \int_k^{1-k} dH_\eta(w)} \quad (5.4)$$

for $r > r_0$ and $w \in (k, 1 - k)$. The factorization in Eq. (5.4) means that the R - and W -components can be simulated independently. Since the R -component has distribution function $1 - (r_0/r)^{1/\eta}$ it can be generated via $R = r_0 U^{-\eta}$ where U is uniformly distributed on $[0, 1]$. Simulation of the W -component may be undertaken by inverting its distribution function $\int_k^w dH_\eta(w') / \int_k^{1-k} dH_\eta(w')$, or alternatively via the rejection method (see Ripley, 1987, Sec. 3.2) when H_η has bounded measure density h_η over $(0, 1)$. In this latter case, the W -component has density $h_\eta^*(w) = h_\eta(w) / \int_k^{1-k} h_\eta(w') dw'$ for $w \in (k, 1 - k)$, so to apply the rejection method we require a density $g(w)$ from which it is straightforward to simulate and a constant M such that $h_\eta^*(w)/g(w) \leq M$ for $w \in (k, 1 - k)$. In practice, it is often convenient to take $g(w)$ as uniform on $(k, 1 - k)$ and $M = \sup\{(1 - 2k)h_\eta^*(w)\}$; see Nadarajah (1999). Simulation of (X, Y) over a region A_0 bounded away from the axes and the origin but not of the form of A may be undertaken by first applying the procedures above for a region A with $A_0 \subset A$ and then discarding those points falling outside A_0 .

5.3. The η -Asymmetric Logistic Limiting Componentwise Maxima Distribution

Our approach here is similar to the methods given by Shi et al. (1992) and Stephenson (2003) for generating from the logistic and asymmetric logistic multivariate extreme value distributions. Let (X, Y) denote a random variable with joint distribution function G_η as in Eq. (4.3) in the non degenerate case where $\alpha < \eta$. Then we may write $X^{-1} = \varrho(N_\varrho Z)^\eta \cos^{2\alpha} V$ and $\varrho Y^{-1} = (N_\varrho Z)^\eta \sin^{2\alpha} V$ where Z and V are independent with easily characterized distributions: V may be represented as $\arcsin(U^{1/2})$ where U is uniform on $[0, 1]$, while Z is a $(1 - \frac{\alpha}{\eta}) : \frac{\alpha}{\eta}$ mixture of a unit exponential random variable and the sum of two independent unit exponential random variables; see the derivation in Appendix C. Figure 2 shows examples of points generated using this approach with the corresponding densities superimposed.

5.4. Simulation Based Examination of the $\alpha \geq \eta$ Case

In Sec. 4.3, we saw that G_η for the η -asymmetric logistic model is degenerate for the case $\alpha \geq \eta$, i.e., when the parametric measure H_η is infinite, whereas the point process \mathcal{P}^* is non degenerate for all $\eta \in (0, 1]$ and $\alpha \in (0, 1]$. This suggests that when $\alpha \geq \eta$ the componentwise maxima of those points in region $R_{\varepsilon b_n}$ grow faster than b_n and therefore occur in separate observations. This effect can be seen in the simulation results in Fig. 1, since as α increases the points tend to concentrate towards the boundaries and furthermore tend to be larger in magnitude than the points towards the $w = 1/2$ region of the domain. This behaviour is also consistent with the results in Sec. 4.4 since by writing $x = s$ or $y = t$ in Eq. (4.5) it is straightforward to show that the marginal variables there have shape parameter α when $\alpha \geq \eta$, corresponding to marginal tails heavier than the joint tail which has shape parameter η .

6. Conclusions and Extension to the Multivariate Case

We derived non degenerate point process results for bivariate tails that hold under the mild regularity condition of bivariate regular variation and regardless of whether the underlying data generation mechanism exhibits asymptotic dependence or asymptotic independence. This is a considerable extension of the available point process theory of de Haan (1985) which is degenerate for asymptotically

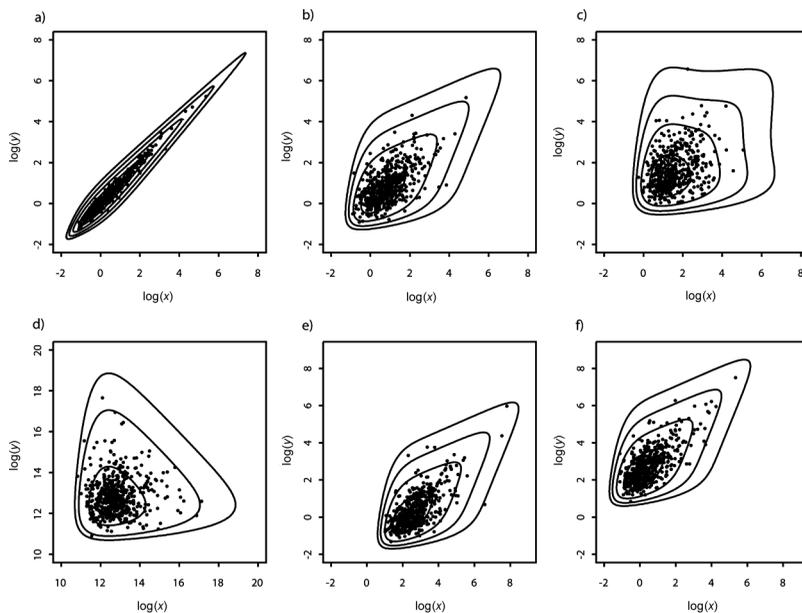


Figure 2. Simulated points from the η -asymmetric logistic componentwise maxima distribution obtained using the approach of Sec. 5.3. Density contours are also shown. All panels have $\eta = 0.7$. Panels (a)–(d) have $\rho = 1$, corresponding to symmetric cases, with $\alpha = \{0.1, 0.4, 0.6, 0.7 - 1e^{-8}\}$, respectively. Since G_η is degenerate when $\alpha \geq \eta$ ($= 0.7$ here), the biggest value used for α is $\eta - 1e^{-8}$. Panels (e) and (f) each have $\alpha = 0.4$, and show the effect of asymmetry with $\rho = \{0.3, 3\}$, respectively.

independent cases. We exploited our point process framework to derive non degenerate limit results for componentwise maxima that again hold for both asymptotically dependent and asymptotically independent cases. This provides a clear extension of the currently available models for componentwise maxima. The normalization condition (2.8) also extends existing hidden regular variation results.

The point process approach developed here has important consequences for statistical data analysis and provides an alternative theoretical underpinning of the modeling framework detailed by Ramos and Ledford (2009). In contrast to existing classical approaches, which tend to be based on weak convergence results concerning the limiting behaviour of normalized componentwise maxima and are non degenerate only for asymptotically dependent cases, our methods focus directly on the tail behavior of the joint survivor function of the underlying observations and capture within the same apparatus both the asymptotically dependent and asymptotically independent cases.

We illustrated our results using the flexible η -asymmetric logistic model of Ramos and Ledford (2009) and provided efficient methods for simulating from both the limiting point process and the limiting componentwise maxima distribution associated with this model. Likelihood based estimation is straightforward using this parameterization and is discussed in Ramos and Ledford (2009) where tests of asymptotic dependence vs. asymptotic independence ($H_0 : \eta = 1$ vs. $H_1 : \eta \leq 1$), tests of symmetry vs. asymmetry ($H_0 : \rho = 1$), and tests of ray-independence ($H_0 : \alpha = 2\eta$) are all examined. Deriving further parametric models and semi-parametric methods

that satisfy both the normalization condition (2.8) and the conditions in Appendix A remains an important open topic for further research.

6.1. Extension to the Multivariate Case

It is relatively straightforward to extend the previous results to deal with higher dimensions. Let \mathbf{X} denote a d -dimensional random variable with unit Fréchet marginal distributions and joint survivor function that is multivariate regularly varying in d -dimensions with index $-1/\eta$, so that for all $\mathbf{x} = (x_1, \dots, x_d)$ with each $x_i > 0$ we have

$$\lim_{u \rightarrow \infty} \frac{\Pr(\mathbf{X} > u\mathbf{x})}{\Pr(\mathbf{X} > u\mathbf{1})} = g(\mathbf{x}) \left(\prod_{i=1}^d x_i \right)^{-1/(d\eta)} \quad \text{where } g(c\mathbf{x}) = g(\mathbf{x}) \text{ for all } c > 0.$$

Then, for b_n satisfying $n\Pr(\mathbf{X} > b_n\mathbf{1}) = 1$, we have that the sequence of point processes

$$\mathcal{P}_n^\otimes = \left\{ \left(\frac{X_{1j}}{b_n}, \dots, \frac{X_{dj}}{b_n} \right) : j = 1, \dots, n \right\}$$

converges weakly in the limit as $n \rightarrow \infty$ to a non homogeneous Poisson process \mathcal{P}^\otimes on $(0, \infty)^d$. The limiting process \mathcal{P}^\otimes has intensity function

$$\Lambda_\eta^\otimes\{(x_1, \infty) \times \dots \times (x_d, \infty)\} = \eta \int_{S_d} \min_{1 \leq i \leq d} \left(\frac{w_i}{x_i} \right)^{1/\eta} dH_\eta(\mathbf{w}), \tag{6.1}$$

where $w_i = x_i / \sum_{j=1}^d x_j$ for $i = 1, \dots, d$ and $H_\eta(\mathbf{w})$ is a non negative measure on the open $(d - 1)$ -dimensional unit simplex

$$S_d = \{ \mathbf{w} \in \mathbb{R}^d : \sum w_i = 1, \text{ each } w_i > 0 \}.$$

As before, the non negative measure $H_\eta(\mathbf{w})$ is arbitrary apart from having to satisfy a normalization condition which in this multivariate case is given by

$$\eta^{-1} = \int_{S_d} \min_{1 \leq i \leq d} w_i^{1/\eta} dH_\eta(\mathbf{w}). \tag{6.2}$$

Defining in the obvious notation $\mathbf{R}_{\varepsilon b_n} = \{ \mathbf{x} : \mathbf{x} > \varepsilon b_n \mathbf{1} \}$ and denoting by $\mathbf{M}_{n, \varepsilon b_n}$ the d -vector of componentwise maxima of those points which occur within $\mathbf{R}_{\varepsilon b_n}$, we have

$$\begin{aligned} G_\eta(\mathbf{x}) &\equiv \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \Pr \left(\frac{\mathbf{M}_{n, \varepsilon b_n}}{b_n} \leq \mathbf{x} \right) \\ &= \exp \left[-\eta \int_{S_d} \left\{ \max_{1 \leq j \leq d} \left(\frac{w_j}{x_j} \right) \right\}^{1/\eta} dH_\eta(\mathbf{w}) \right] \\ &= \exp\{-V_\eta(\mathbf{x})\} \end{aligned} \tag{6.3}$$

which is non degenerate provided that H_η is finite.

The extension of result (3.7) to the multivariate case can be obtained from Eq. (6.1), the inclusion–exclusion principle and the definition of V_η in Eq. (6.3), giving

$$\begin{aligned} & \Lambda_\eta^\otimes \{(x_1, \infty) \times \cdots \times (x_d, \infty)\} \\ &= \sum_{i=1}^d \eta \int_{S_d} \left(\frac{w_i}{x_i}\right)^{1/\eta} dH_\eta(\mathbf{w}) - \sum_{i=1}^{d-1} \sum_{j=i+1}^d \eta \int_{S_d} \left\{ \max\left(\frac{w_i}{x_i}, \frac{w_j}{x_j}\right) \right\}^{1/\eta} dH_\eta(\mathbf{w}) \\ & \quad + \cdots + (-1)^{d+1} \eta \int_{S_d} \max_{1 \leq i \leq d} \left(\frac{w_i}{x_i}\right)^{1/\eta} dH_\eta(\mathbf{w}) \\ &= \sum_{i=1}^d V_\eta(\infty, \dots, \infty, x_i, \infty, \dots, \infty) \\ & \quad - \sum_{i=1}^{d-1} \sum_{j=i+1}^d V_\eta(\infty, \dots, \infty, x_i, \infty, \dots, \infty, x_j, \infty, \dots, \infty) \\ & \quad + \cdots + (-1)^{d+1} V_\eta(\mathbf{x}). \end{aligned} \tag{6.4}$$

So taking the mixed derivative of Eq. (6.4) with respect to x_1, \dots, x_d , we obtain

$$\frac{\partial^d V_\eta}{\partial x_1 \cdots \partial x_d} = -r^{-(d+1/\eta)} h_\eta(\mathbf{w})$$

where $r = \sum_{j=1}^d x_j$ and $w_i = x_i/r$ for $i = 1, \dots, d$.

Example 6.1. To illustrate the above, we provide a multivariate version of the η -asymmetric logistic model of Sec. 4. Consider the d -dimensional measure density

$$h_\eta(\mathbf{w}) = \frac{\prod_{i=1}^{d-1} (i\eta - \alpha)}{\eta^d \alpha^{d-1} N_\varrho} \left\{ \sum_{i=1}^d \left(\frac{w_i}{\varrho_i}\right)^{-1/\alpha} \right\}^{\alpha/\eta-d} \times \left(\prod_{i=1}^d w_i \right)^{-1/\alpha-1}$$

on $\mathbf{w} \in S_d$ where $\varrho_i > 0$, $\prod_{i=1}^d \varrho_i = 1$, $\eta, \alpha \in (0, 1]$, and $N_\varrho = \sum_{b \in B} (-1)^{|b|+1} (\sum_{i \in b} \varrho_i^{1/\alpha})^{\alpha/\eta}$ where B represents the set of all non empty subsets of $\{1, \dots, d\}$ and $|b|$ represents the number of elements in the set b . Then $h_\eta(\mathbf{w})$ as above satisfies the normalization condition (6.2). For example, when $d = 3$ we have that $h_\eta(w_1, w_2)$ is given by

$$\begin{aligned} h_\eta(w_1, w_2) &= \frac{(\eta - \alpha)(2\eta - \alpha)}{\eta^3 \alpha^2 N_\varrho} \left\{ (\varrho_1 \varrho_2 w_1)^{-1/\alpha} + \left(\frac{w_2}{\varrho_1}\right)^{-1/\alpha} + \left(\frac{1 - w_1 - w_2}{\varrho_2}\right)^{-1/\alpha} \right\}^{\alpha/\eta-3} \\ & \quad \times \{w_1 w_2 (1 - w_1 - w_2)\}^{-1/\alpha-1} \end{aligned}$$

where $\eta, \alpha \in (0, 1]$, $\varrho_1 > 0$, $\varrho_2 > 0$, and

$$\begin{aligned} N_\varrho &= (\varrho_1 \varrho_2)^{-1/\eta} + \varrho_1^{1/\eta} + \varrho_2^{1/\eta} - \{(\varrho_1 \varrho_2)^{-1/\alpha} + \varrho_1^{1/\alpha}\}^{\alpha/\eta} - \{(\varrho_1 \varrho_2)^{-1/\alpha} + \varrho_2^{1/\alpha}\}^{\alpha/\eta} \\ & \quad - \{\varrho_1^{1/\alpha} + \varrho_2^{1/\alpha}\}^{\alpha/\eta} + \{(\varrho_1 \varrho_2)^{-1/\alpha} + \varrho_1^{1/\alpha} + \varrho_2^{1/\alpha}\}^{\alpha/\eta}. \end{aligned}$$

Appendices

A Sufficient Conditions for an (η, H_η) -Pair Satisfying (2.8) to Define a Valid Limit Intensity for \mathcal{P}_n^* in (2.4)

These conditions require that for a large u there exists a small $\lambda = \Pr(X > us, Y > ut) \in (0, 1)$ for some fixed $s > 0$ and $t > 0$, such that

$$\begin{aligned} & \exp(-1/x) + \lambda \Lambda_\eta\{(x/u, \infty) \times (1, \infty)\} / \Lambda_\eta\{(s, \infty) \times (t, \infty)\} \quad \text{and} \\ & \exp(-1/x) + \lambda \Lambda_\eta\{(1, \infty) \times (x/u, \infty)\} / \Lambda_\eta\{(s, \infty) \times (t, \infty)\} \end{aligned}$$

are both monotonic increasing in $x > us$ and $x > ut$, respectively. These are similar to the conditions in Theorem 1 of Ramos and Ledford (2009) where the $s = t = 1$ special case is considered. Under the above conditions, and providing (2.8) holds, then it is straightforward to show by a minor extension of the proof of Theorem 1 of Ramos and Ledford (2009) that there exists a bivariate random variable (X, Y) with unit Fréchet margins that satisfies Eq. (4.4). Simulating independent copies of (X, Y) , we construct the sequence of point processes \mathcal{P}_n^* as defined by Eq. (2.4). Kallenberg’s theorem (Ripley, 1987, Proposition 3.22) can then be used to show that \mathcal{P}_n^* converges weakly (i.e., in distribution) to a non homogeneous Poisson process, \mathcal{P}^* say. By construction \mathcal{P}^* has the required intensity on $(0, \infty) \times (0, \infty)$.

B The Simulation Scheme in Section 5.1

Changing variables in the conditional density function of (X, Y) for $x > s$ and $y > t$ in Eq. (5.1) according to $X^{-1} = \varrho Z^\eta \cos^{2\alpha} V$ and $\varrho Y^{-1} = Z^\eta \sin^{2\alpha} V$, the density of (Z, V) is given by

$$f_{ZV}(z, v) = \frac{\eta - \alpha}{\eta N_{st\varrho}} 2 \sin v \cos v$$

with domain

$$\begin{cases} 0 < z \leq (\varrho s)^{-1/\eta} \cos^{-2\alpha/\eta} v & \text{if } 0 < v \leq v^* \text{ and} \\ 0 < z \leq (t/\varrho)^{-1/\eta} \sin^{-2\alpha/\eta} v & \text{if } v^* < v < \pi/2, \end{cases}$$

where $v^* = \arctan \varrho^{1/\alpha}$. Noting that V has distribution function

$$F_V(v) = \begin{cases} N_{st\varrho}^{-1} (\varrho s)^{-1/\eta} (1 - \cos^{2(1-\alpha/\eta)} v) & \text{if } 0 < v \leq v^* \\ N_{st\varrho}^{-1} \left[(\varrho s)^{-1/\eta} - \left\{ (\varrho s)^{-1/\alpha} + \left(\frac{t}{\varrho}\right)^{-1/\alpha} \right\}^{\alpha/\eta} + \left(\frac{t}{\varrho}\right)^{-1/\eta} \sin^{2(1-\alpha/\eta)} v \right] & \text{if } v^* < v < \pi/2 \end{cases}$$

and that the conditional variable $Z | V = v$ is uniformly distributed with density function

$$f_{Z|V=v}(z) = \begin{cases} (\varrho s)^{1/\eta} \cos^{2\alpha/\eta} v & \text{if } 0 < v \leq v^* \\ \left(\frac{t}{\varrho}\right)^{1/\eta} \sin^{2\alpha/\eta} v & \text{if } v^* < v < \pi/2, \end{cases}$$

it follows that Z and V can be represented as in Eqs. (5.2) and (5.3).

C The Simulation Scheme in Section 5.3

Let f_η denote the density function associated with G_η in Eq. (4.3). Then

$$f_\eta(x, y) = N_\varrho^{-1} \eta^{-2} \exp \left[-N_\varrho^{-1} \left\{ (\varrho x)^{-1/\alpha} + \left(\frac{y}{\varrho} \right)^{-1/\alpha} \right\}^{\alpha/\eta} \right] (xy)^{-(1+1/\alpha)} \\ \times \left\{ (\varrho x)^{-1/\alpha} + \left(\frac{y}{\varrho} \right)^{-1/\alpha} \right\}^{\alpha/\eta-2} \left[N_\varrho^{-1} \left\{ (\varrho x)^{-1/\alpha} + \left(\frac{y}{\varrho} \right)^{-1/\alpha} \right\}^{\alpha/\eta} + \frac{\eta - \alpha}{\alpha} \right].$$

The transformation $X^{-1} = \varrho(N_\varrho Z)^\eta \cos^{2\alpha} V$ and $\varrho Y^{-1} = (N_\varrho Z)^\eta \sin^{2\alpha} V$ is such that the bivariate random variable (Z, V) has density

$$f_{ZV}(z, v) = \exp(-z) \left\{ \frac{\alpha}{\eta} z + \left(1 - \frac{\alpha}{\eta} \right) \right\} \sin 2v$$

for $z > 0$ and $0 < v < \pi/2$. From this representation the simulation approach of Sec. 5.3 follows immediately.

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