ON THE QUENCHED CENTRAL LIMIT THEOREM FOR RANDOM DYNAMICAL SYSTEMS

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Abstract. We provide a necessary and sufficient condition under which the quenched central limit theorem without random centering holds for one-dimensional random systems that are uniformly expanding. This condition holds in particular when all the maps preserve a common measure. We also give a counter example which shows that this condition is not necessarily satisfied when the maps do not preserve a common measure.

1. Introduction

Precise understanding of the statistical properties and the limit theorems for sums of the form

$$S_n(x, \omega_1, \omega_2, \dots) = \sum_{k=0}^{n-1} \varphi(T_{\omega_k} \dots T_{\omega_1} x),$$

where the maps T_ω are randomly drawn transformations of a space X, has been the object of numerous works in the previous years, see [1, 3, 5, 9, 13] and references therein. Two different kinds of limit theorems can be distinguished: annealed results, which refer to the sums S_n seen as functions of both the variable x and the choice of the maps $\omega_1, \omega_2, \dots$, and quenched results, where S_n is considered as a function of the sole variable x, this for almost every sequence $\omega_1, \omega_2, \dots$ of maps. A similar analysis is also possible when the sequence of maps is deterministic, see for instance [6, 10], even though it often requires drastic conditions on the particular sequence, in contrast to the almost sure nature of quenched theorems.

In this note, we will review results from [3] on the annealed and quenched central limit theorem for random dynamical systems which present some uniform expansion. The setting, rather general and based on the quasi-compactness of a transfer operator, will be quickly specialized to one dimensional uniformly expanding systems. Building upon a strategy employed first by [1] and inspired by previous works on random walks in random environments, we will give a necessary and sufficient condition for the quenched central limit theorem without random centering to hold. More precisely, we will characterize when $\frac{S_n(x,\omega_1,\omega_2,\dots)}{\sqrt{n}}$ converges to a normal law for almost every sequence $\omega_1, \omega_2, \dots$, where φ is a BV observable with mean 0 with respect to the unique absolutely continuous stationary probability of the system. It should be stressed that, in contrast to [1] and [3], we do not assume that all the maps T_ω

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preserve a common measure. Nevertheless, we will show by a counter example that our necessary and sufficient condition can fail when the maps preserve different measures, which suggests to seek for a different formulation of the quenched central limit theorem, with a random centering. We will remark upon this in the last section of the paper.

2. Random dynamical systems: definition, assumptions and examples

2.1. **Basic definitions.**

We first give the definition of the class of random dynamical systems we will consider. Let $(\Omega, \mathbb{P})$ be a finite probability space, where $\mathbb{P} = \{p_\omega\}_{\omega \in \Omega}$ is a probability vector with $p_\omega > 0$ for all $\omega \in \Omega$. Let $T = \{T_\omega\}_{\omega \in \Omega}$ be a finite collection of measurable maps $T_\omega : X \to X$ on a Polish space X. We will refer to $(\Omega, \mathbb{P}, T)$ as a random dynamical system (RSD).

For a sequence $\underline{\omega} = (\omega_1, \omega_2, \dots) \in \Omega^{\mathbb{N}}$, we denote by $T^n_{\underline{\omega}}$ the composition $T_{\omega_n} \circ \dots \circ T_{\omega_1}$. The corresponding Markov chain (X_n) is defined by:

$$\begin{cases} X_0 \sim \mu \\ X_{n+1} = T_{\omega_{n+1}}(X_n), \end{cases}$$

where μ is probability measure on X and $(\omega_n)_n$ is a sequence of independent and identically distributed random variables with common law $\mathbb{P}$. The measure μ is stationary if X_n is distributed accordingly to μ for all n.

We can relate this stochastic process to a deterministic dynamical system as follows. We define a skew-product transformation

$$\begin{array}{rccc} S & : & \Omega^{\mathbb{N}} \times X & \to & \Omega^{\mathbb{N}} \times X \\ & & (\underline{\omega}, x) & \mapsto & (\sigma \underline{\omega}, T_{\omega_1} x) \end{array}$$

where $\sigma : \Omega^{\mathbb{N}} \to \mathbb{N}$ is the unilateral shift. We have $S^n(\underline{\omega}, x) = (\sigma^n \underline{\omega}, T^n_{\underline{\omega}} x)$ for any $n \geq 0$. A probability measure μ on X is stationary if and only if the measure $\mathbb{P}^{\otimes \mathbb{N}} \otimes \mu$ is invariant under S.

2.2. **A functional analytic framework.**

We assume the space X is endowed with a reference probability measure m such that each map T_ω is non singular with respect to m. We will be interested in the existence and properties of absolutely continuous stationary probability (acsp) with respect to m. In this case, each map T_ω admits a transfer operator $P_\omega : L^1(m) \to L^1(m)$ satisfying:

$$\int_X (P_\omega f) g \, dm = \int_X f (g \circ T_\omega) \, dm \text{ for all } f \in L^1(m) \text{ and } g \in L^\infty(m).$$

Definition 1. The annealed transfer operator $P : L^1(m) \to L^1(m)$ of the RDS $(\Omega, \mathbb{P}, T)$ is defined by

$$P = \sum_{\omega \in \Omega} p_\omega P_\omega.$$

An absolutely continuous probability measure μ is stationary if and only if its density $f = \frac{d\mu}{dm}$ verifies $Pf = f$.

We make the following hypothesis on P:

Assumptions. There exists a Banach space $(\mathcal{B}, \|\cdot\|)$ such that

(H1) $\mathcal{B}$ is compactly embedded in $L^1(m)$, with dense image;
(H2) Constant functions lie in $\mathcal{B}$;
(H3) $\mathcal{B}$ is a complex Banach lattice: for all $f \in \mathcal{B}$, $|f|$ and $\bar{f}$ belong to $\mathcal{B}$;
(H4) $\mathcal{B}$ is stable under P: $P(\mathcal{B}) \subset \mathcal{B}$ and P acts continuously on $(\mathcal{B}, \|\cdot\|)$;
(H5) Lasota-Yorke inequality: there exist $N \geq 1$, $\rho < 1$ and $K > 0$ such that

$$\|P^N f\| \leq \rho\|f\| + K\|f\|_{L^1_m}\,, \text{ for all } f \in \mathcal{B}.$$

It follows from Hennion's theorem [8] that P is a quasi-compact operator on $\mathcal{B}$ of spectral radius 1, with 1 as an isolated eigenvalue of finite mulitplicity. We also assume:

(H6) 1 is a simple eigenvalue of P on $\mathcal{B}$ and there is no other eigenvalue on the unit circle.

Proposition 2. *Under these assumptions, there exists a unique acsp μ. Its density $h = \frac{d\mu}{dm}$ belongs to $\mathcal{B}$. Furthermore, the skew-product system $(\Omega^{\mathbb{N}} \times X, S, \mathbb{P}^{\otimes\mathbb{N}} \otimes \mu)$ is exact, and hence ergodic and mixing.*

Existence and uniqueness is classic, see for instance the discussion in [3]. Exactness of the skew-product follows from arguments of Morita [11], see also Proposition 1.1.4 in [2].
The transfer operator P can be decomposed as

$$P = \Pi + Q,$$

where Π is the one-dimensional projector given by $\Pi(f) = \left(\int_X f\, dm\right) h$ and Q is a bounded operator on $\mathcal{B}$ with spectral radius strictly less than 1 and satisfying $\Pi Q = Q\Pi = 0$. We thus have $P^n = \Pi + Q^n$, where $\|Q^n\| \leq C\lambda^n$ for some $\lambda \in (0,1)$.

2.3. **Example: one-dimensional expanding maps.** Suppose $X = [0,1]$ and m is the Lebesgue measure. All maps T_ω are piecewise C^2 uniformly expanding maps on X, i.e. $\lambda(T_\omega) := \inf |T'_\omega| > 1$. Assume also that the system $(\Omega, \mathbb{P}, T)$ has the random covering property: for all non trivial interval $I \subset [0,1]$, there exist $n \geq 1$ and $\underline{\omega} \in \Omega^n$ such that $T^n_{\underline{\omega}}(I) = [0,1]$ up to finitely many points.
Then, as proved in [3, Example 2.1], the assumptions (H1)-(H6) are satisfied with the Banach space $\mathcal{B} = \mathrm{BV}$ of functions with bounded variation. Furthermore, the density of the unique acsp is bounded uniformly away from 0.

3. Annealed central limit theorem

In this section, we review results from [3] on annealed limit theorems. We assume that the RDS $(\Omega, \mathbb{P}, T)$ satisfied the assumptions (H1)-(H6) with a Banach space $\mathcal{B}$. Let $\mathcal{M}_{\mathcal{B}}$ denote the set of probability measures on X that are absolutely continuous with respect to m, with density in $\mathcal{B}$.
Let $\mathcal{B}_0 \subset L^1(m)$ be a Banach space with norm $\|\cdot\|_0$ for which there exists $C > 0$ such that

$$\text{(C)} \qquad \|fg\| \leq C\|f\|_0\|g\|, \ \forall f \in \mathcal{B}_0, \ \forall g \in \mathcal{B}.$$

If $\mathcal{B}$ is a Banach algebra, as it is the case for the space involved in example 2.3, we can choose $\mathcal{B}_0 = \mathcal{B}$.
We have the annealed central limit theorem for observables belonging to $\mathcal{B}_0$:

Theorem 3. [3, Proposition 3.2, Proposition 3.4, Theorem 3.5] *Let $\varphi : X \to \mathbb{R}$ be a bounded observable, with $\varphi \in \mathcal{B}_0$ and $\int_X \varphi \, d\mu = 0$. Define $S_n(\underline{\omega}, x) = \sum_{k=0}^{n-1} \varphi(T_{\underline{\omega}}^k x)$.*

1. *The limit $\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int_{\Omega^{\mathbb{N}}} \int_X S_n^2 \, d\mu \, d\mathbb{P}^{\otimes\mathbb{N}}$ exists.*
2. *The asymptotic variance σ^2 is 0 if and only if there exists $\psi \in L^2(\mu)$ such that $\varphi = \psi - \psi \circ T_\omega$, μ-a.e. for all $\omega \in \Omega$. In this case, we say that φ is a random coboundary.*
3. *For all $\nu \in \mathcal{M}_{\mathcal{B}}$, $\frac{S_n}{\sqrt{n}}$ converges in law to $\mathcal{N}(0, \sigma^2)$ under the probability $\mathbb{P}^{\otimes\mathbb{N}} \otimes \nu$.*

As a consequence of the proof of [3], we also have an estimation on the speed of convergence of the characteristic function of $\frac{S_n}{\sqrt{n}}$:

Lemma 4. [3, Lemma 3.10] *For all $\nu \in \mathcal{M}_{\mathcal{B}}$,*

$$\left| \mathbb{E}_{\mathbb{P}^{\otimes\mathbb{N}}\otimes\nu}\left(e^{i\frac{t}{\sqrt{n}}S_n}\right) - e^{-\frac{t^2\sigma^2}{2}} \right| = \mathcal{O}\left(\frac{1+|t|^3}{\sqrt{n}}\right)$$

for all $t \in \mathbb{R}$ and $n \geq 1$ such that $\frac{t}{\sqrt{n}}$ is small enough.

The asymptotic variance σ^2 can be expressed in terms of any initial measure $\nu \in \mathcal{M}_{\mathcal{B}}$:

Lemma 5. *For all $\nu \in \mathcal{M}_{\mathcal{B}}$,*

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int_{\Omega^{\mathbb{N}}} \int_X S_n^2 \, d\nu \, d\mathbb{P}^{\otimes\mathbb{N}}$$

Proof. Let $\nu \in \mathcal{M}_{\mathcal{B}}$ with density f. Expanding the squares and using the duality property of the transfer operators, we obtain

$$\iint_{\Omega^{\mathbb{N}}\times X} S_n^2 \, d\nu \, d\mathbb{P}^{\otimes\mathbb{N}} - \iint_{\Omega^{\mathbb{N}}\times X} S_n^2 \, d\mu \, d\mathbb{P}^{\otimes\mathbb{N}} = \sum_{k=0}^{n-1} \int_X \varphi^2 P^k(f-h) \, dm + 2 \sum_{\substack{i,j=0 \\ i<j}}^{n-1} \int_X \varphi P^{j-i}(\varphi P^i(f-h)) \, dm.$$

Since $\int_X f \, dm = \int_X h \, dm = 1$, we have

$$\left| \int_X \varphi^2 P^k(f-h) \, dm \right| \leq \|\varphi\|_\infty^2 \|P^k(f-h)\|_{L^1_m} \leq C\|\varphi\|_\infty^2 \|Q^k(f-h)\| \leq C\|\varphi\|_\infty^2 \lambda^k,$$

On the other hand,

$$\int_X \varphi P^{j-i}(\varphi P^i(f-h)) \, dm = \int_X \varphi \left(\int_X \varphi P^i(f-h) \, dm \right) h \, dm + \int_X \varphi Q^{j-i}(\varphi P^i(f-h)) \, dm,$$

and since $\int_X \varphi h \, dm = \int_X \varphi \, d\mu = 0$ and $\|Q^{j-i}(\varphi P^i(f-h))\|_{L^1_m} \leq C\lambda^{j-i}\|\varphi\|_0 \|P^i(f-h)\| \leq C\|\varphi\|_0 \lambda^j$, we obtain that

$$\left| \int_X \varphi P^{j-i}(\varphi P^i(f-h)) \, dm \right| \leq C\|\varphi\|_0 \lambda^j,$$

whence the result after summation. □

We will also make use of an estimate for annealed large deviations:

Lemma 6. *For all $\nu \in \mathcal{M}_{\mathcal{B}}$, there exists $C > 0$ such that for all $\epsilon > 0$ small enough,*

$$\mathbb{P}^{\otimes \mathbb{N}} \otimes \nu(|S_n| > n\epsilon) \le Ce^{-C\epsilon^2 n}.$$

This lemma is a straightforward consequence of the large deviation principle [3, Theorem 3.6], see also [2, Lemma 1.2.17]. It can alternatively be deduced from the exponential decay of annealed correlations [3, Proposition 3.1] and the Gordin's martingale approach described in [4, Proposition 2.5] (see [3, Section 4] for the construction of the martingales).

4. Quenched central limit theorem

4.1. **A general approach.** We describe here an approach for the quenched central limit theorem originating from the fields of random walks in random environments, which was first employed in the context of random dynamical systems by Ayyer, Liverani and Stenlund [1].

Consider a RDS $(\Omega, \mathbb{P}, T)$ on a space X, with a stationary measure μ and an observable $\varphi : X \to \mathbb{R}$ with $\int_X \varphi \, d\mu = 0$. We introduce an auxiliary RDS $(\widetilde{\Omega}, \widetilde{\mathbb{P}}, \widetilde{T})$ on the space X^2 as follows. We set $\widetilde{\Omega} = \Omega$, $\widetilde{\mathbb{P}} = \mathbb{P}$, and for $\omega \in \Omega$, the map $\widetilde{T}_\omega(x, y) = (T_\omega x, T_\omega y)$. We define a new observable $\widetilde{\varphi} : X^2 \to \mathbb{R}$ by $\widetilde{\varphi}(x, y) = \varphi(x) - \varphi(y)$, and denote its associated Birkhoff sums by $\widetilde{S}_n$.

Theorem 7. [1] *Assume there exists $\sigma^2 > 0$ and a constant $C > 0$ such that for all $t \in \mathbb{R}$ and $n \ge 1$ with $\frac{t}{\sqrt{n}}$ small enough :*

(1) $\left|\mathbb{E}_{\mathbb{P}^{\otimes \mathbb{N}} \otimes \mu}(e^{i\frac{t}{\sqrt{n}} S_n}) - e^{-\frac{t^2\sigma^2}{2}}\right| \le C\frac{1+|t|^3}{\sqrt{n}}$,

(2) $\left|\mathbb{E}_{\mathbb{P}^{\otimes \mathbb{N}} \otimes (\mu \otimes \mu)}(e^{i\frac{t}{\sqrt{n}} \widetilde{S}_n}) - e^{-t^2\sigma^2}\right| \le C\frac{1+|t|^3}{\sqrt{n}}$.

Suppose also that for $n \ge 1$ and $\epsilon > 0$:

(3) $\mathbb{P}^{\otimes \mathbb{N}} \otimes \mu\left(\left|\frac{S_n}{n}\right| \ge \epsilon\right) \le Ce^{-C\epsilon^2 n}$.

Then, the quenched CLT without random centering holds: for $\mathbb{P}^{\otimes \mathbb{N}}$-a.e. $\underline{\omega} \in \Omega^{\mathbb{N}}$, $\frac{S_n(\underline{\omega}, \cdot)}{\sqrt{n}}$ converges in law to $\mathcal{N}(0, \sigma^2)$ under the probability measure μ.

We now present a procedure to check the assumptions of Theorem 7.

Step 1. One checks the RDS $(\Omega, \mathbb{P}, T)$ satisfies (H1)-(H6) with a Banach space $\mathcal{B}$. One then verifies that φ belongs to a Banach space $\mathcal{B}_0$ satisfying the condition (C). Thus the RDS $(\Omega, \mathbb{P}, T)$ admits a unique stationary probability μ absolutely continuous with respect to m. If φ is not a random coboundary, by Theorem 3, Lemma 4 and Lemma 6, hypothesis (1) and (3) of Theorem 7 hold true.

Step 2. One checks the auxiliary RDS $(\widetilde{\Omega}, \widetilde{\mathbb{P}}, \widetilde{T})$ satisfies (H1)-(H6) for a Banach space $\widetilde{\mathcal{B}}$ of functions on X^2, with reference measure $\widetilde{m} = m \otimes m$ and that $\widetilde{\varphi}$ belongs to a Banach space

$\widetilde{\mathcal{B}}_0$ satisfying *(*C) with $\widetilde{\mathcal{B}}$. The system $(\widetilde{\Omega}, \widetilde{\mathbb{P}}, \widetilde{T})$ admits then a unique acsp $\widetilde{\mu}$. It is easy to see that the marginals of $\widetilde{\mu}$ are absolutely continuous with respect to m and are stationary for $(\Omega, \mathbb{P}, T)$. Since μ is the unique acsp of $(\Omega, \mathbb{P}, T)$, we deduce the marginals of $\widetilde{\mu}$ are given by μ, and consequently, $\int_{X^2} \widetilde{\varphi} \, d\widetilde{\mu} = 0$. If the measure $\mu \otimes \mu$ belongs to $\mathcal{M}_{\widetilde{\mathcal{B}}}$, i.e. $h \otimes h \in \widetilde{\mathcal{B}}$, by Lemma 4, there exists $\widetilde{\sigma}^2 \geq 0$ such that

$$\left| \mathbb{E}_{\mathbb{P}^{\otimes \mathbb{N}} \otimes (\mu \otimes \mu)} (e^{i \frac{t}{\sqrt{n}} \widetilde{S}_n}) - e^{-\frac{t^2 \widetilde{\sigma}^2}{2}} \right| \leq C \frac{1 + |t|^3}{\sqrt{n}}$$

for all $t \in \mathbb{R}$ and $n \geq 1$ with $\frac{t}{\sqrt{n}}$ small enough.

Step 3. One checks that $\widetilde{\sigma}^2 = 2\sigma^2$. This implies assumption (2) of Theorem 7.

Steps 1 and 2 usually present no conceptual difficulties if the system at hand is uniformly expanding. The only subtlety resides in the fact that the auxiliary RDS acts on a space whose dimension is twice the dimension of the original system, which requires to work with a Banach space $\widetilde{\mathcal{B}}$ that might be slightly more complicated than $\mathcal{B}$. This is particularly true when $X = [0, 1]$, since $\mathcal{B}$ will be the space BV and $\widetilde{\mathcal{B}}$ the Quasi-Hölder space (or any other valid option in higher dimension, like the multidimensional version of BV). This implies to take into account the accumulation of discontinuities (the complexity), a phenomenon which only occurs in dimension 2 or higher. Nevertheless, the rather simple structure of the maps $\widetilde{T}_\omega$ (direct products) and the regularity partitions (made of rectangles) makes this study more accessible.
The only genuine difficulty then lies in step 3. Asymptotic variances can be compared with the help of Lemma 5. As surprising as the condition $\widetilde{\sigma}^2 = 2\sigma^2$ might seem, it is actually necessary for the quenched central limit theorem without random centering to hold:

Lemma 8. [3, Lemma 7.2] *Using the same notations introduced above, assume that there exists $\sigma^2 > 0$ and $\widetilde{\sigma}^2 > 0$ such that*

(1) $\frac{S_n}{\sqrt{n}}$ converges in law to $\mathcal{N}(0, \sigma^2)$ under the probability $\mathbb{P}^{\otimes \mathbb{N}} \otimes \mu$,
(2) $\frac{\widetilde{S}_n}{\sqrt{n}}$ converges in law to $\mathcal{N}(0, \widetilde{\sigma}^2)$ under the probability $\mathbb{P}^{\otimes \mathbb{N}} \otimes (\mu \otimes \mu)$,
(3) for a.e. $\underline{\omega}$, $\frac{S_n(\underline{\omega}, \cdot)}{\sqrt{n}}$ converges in law to $\mathcal{N}(0, \sigma^2)$ under the probability μ.

Then $\widetilde{\sigma}^2 = 2\sigma^2$.

4.2. **One-dimension systems: a necessary and sufficient condition.** In this section, we will make use of Theorem 7 to provide a concrete necessary and sufficient condition for the quenched central limit theorem without random centering to hold for one-dimensional expanding RDS. Even though the strategy could be equally applied to higher dimensional systems, we will focus on the one-dimensional case in order to maintain the technicality at a reasonable level.
As in example 2.3, let $(\Omega, \mathbb{P}, T)$ be a RDS on $X = [0, 1]$, where all maps T_ω are piecewise C^2 and uniformly expanding. Assume the RDS has the random covering property, and let μ be its unique acsp. Let $\varphi : X \to \mathbb{R}$ be an observable belonging to BV, with $\int_X \varphi \, d\mu = 0$, which

is not a random coboundary. By Theorem 3, φ satisfies the annealed CLT, with asymptotic variance $\sigma^2 > 0$.

Theorem 9. *The quenched CLT without random centering (i.e.* $\frac{S_n(\underline{\omega},\cdot)}{\sqrt{n}}$ *converges in law to* $\mathcal{N}(0,\sigma^2)$ *for a.e.* $\underline{\omega}$*) holds if and only if:*

$$\lim_{n\to\infty} \frac{1}{n}\int_{\Omega^{\mathbb{N}}} \left(\sum_{k=0}^{n-1}\int_X \varphi\circ T^k_{\underline{\omega}}\, d\mu\right)^2 d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}) = 0.$$

Remark 10.
(1) The condition of Theorem 9, even though looking difficult to check in practice, has the advantage over the condition $\widetilde{\sigma^2} = 2\sigma^2$ to involve only quantities associated to the RDS we are interested in, and not an auxiliary system.
(2) This condition is trivially satisfied if all the maps T_ω preserve the same measure μ. We thus generalize the result from [3].
(3) We will see in the next section that, for a large class of RDS, whenever the maps do not share a common invariant measure, we can always construct an observable for which this condition does not hold.

Proof. We follow the strategy described in section 4.1.

Step 1. By section 2.3, the transfer operator P of $(\Omega,\mathbb{P},T)$ satisfies (H1)-(H6) with $\mathcal{B} = \mathrm{BV}$. Since $\mathcal{B}$ is a Banach algebra and $\varphi\in\mathcal{B}_0 = \mathcal{B}$, this concludes the step 1.

Step 2. We will show the annealed transfer operator $\widetilde{P}$ of the RDS $(\widetilde{\Omega},\widetilde{\mathbb{P}},\widetilde{T})$ satisfied (H1)-(H6) with the Banach space $V_1(X^2)$ of Quasi-Hölder functions. We first recall from Saussol [12] the definition of this space.
Let m_d denotes the Lebesgue measure on $\mathbb{R}^d$. Let $f:\mathbb{R}^d\to\mathbb{C}$ be a measurable function. For a Borel subset $A\subset\mathbb{R}^d$, define $\operatorname{osc}(f,A) = \operatorname*{ess\,sup}_{x,y\in A}|f(x)-f(y)|$. For any $\epsilon>0$, the map $x\mapsto \operatorname{osc}(f,B_\epsilon(x))$ is a positive lower semi-continuous function, so that the integral $\int_{\mathbb{R}^d}\operatorname{osc}(f,B_\epsilon(x))dx$ makes sense. For $f\in L^1(\mathbb{R}^d)$ and $0<\alpha\le 1$, define

$$|f|_\alpha = \sup_{0<\epsilon\le\epsilon_0}\frac{1}{\epsilon^\alpha}\int_{\mathbb{R}^d}\operatorname{osc}(f,B_\epsilon(x))dx.$$

For a regular compact subset $M\subset\mathbb{R}^d$, define

$$V_\alpha(M) = \{f\in L^1(\mathbb{R}^d)\,/\,\operatorname{supp}(f)\subset M,\, |f|_\alpha<\infty\},$$

endowed with the norm $\|f\|_\alpha = \|f\|_{L^1_m} + |f|_\alpha$, where m is the Lebesgue measure normalized so that $m(M)=1$. Note that while the norm depends on ϵ_0, the space V_α does not, and two choices of ϵ_0 give rise to two equivalent norms. This space is a Banach algebra.
Note that we will only work with $\alpha=1$. To emphasize the dependence on ϵ_0, we will also denote the semi-norm by $|\cdot|_{\epsilon_0}$.
Let $\widetilde{T}: M\to M$ be a piecewise C^2 map, with regularity partition $\{U_i\}_i$, satisfying the assumptions (PEi), $i=1,2,3,4$ of [12], with associated constants $\epsilon_0(\widetilde{T})>0$, $c(\widetilde{T})$ and

$s(\widetilde{T}) < 1$. It is easy to check that if T is a piecewise C^2 uniformly expanding map on $[0,1]$, then $\widetilde{T} = T \times T$ verifies these hypothesis with $M = [0,1]^2$ and $s(S) = \lambda(T)^{-1}$.
Define for $0 \le \delta \le \epsilon_0(\widetilde{T})$:

$$G_{\widetilde{T}}(\epsilon, \delta) = \sup_x \sum_i \frac{m_d(\widetilde{T}_i^{-1} B_\epsilon(\partial \widetilde{T} U_i) \cap B_{(1-s(\widetilde{T}))\delta}(x))}{m_d(B_{(1-s(\widetilde{T}))\delta}(x))},$$

where $\widetilde{T}_i$ is a smooth extension of $\widetilde{T}$ on a neighborhood of $\bar{U}_i$, and $B_\epsilon(A)$ denotes the ϵ-neighborhood in the euclidean metric of the set A. Set $\eta_{\widetilde{T}}(\delta) = s(\widetilde{T}) + 2\left(\sup_{0<\epsilon\le\delta} \frac{G_{\widetilde{T}}(\epsilon,\delta)}{\epsilon}\right)\delta$.
We have the following Lasota-Yorke inequality for the transfer operator $P_{\widetilde{T}}$ of $\widetilde{T}$ on $V_1(M)$:

Lemma 11. [12, Lemma 4.1] *For all $0 < \epsilon_0 \le \epsilon_0(\widetilde{T})$ and all $f \in V_1(M)$, one has*

$$|P_{\widetilde{T}} f|_{\epsilon_0} \le \left(1 + c(\widetilde{T}) s(\widetilde{T}) \epsilon_0\right) \eta_{\widetilde{T}}(\epsilon_0) |f|_{\epsilon_0} + A(\widetilde{T}, \epsilon_0) \|f\|_{L^1_{m_d}},$$

where $A(\widetilde{T}, \epsilon_0)$ depends only on $\widetilde{T}$ and ϵ_0.

We will estimate $G_{\widetilde{T}}$ when $\widetilde{T} = T \times T$, with T a piecewise C^2 uniformly expanding map on $[0,1]$. Remark that if $\{I_i\}_i$ is the regularity partition of the map T, the partition of $\widetilde{T}$ is given by $\{I_i \times I_j\}_{i,j}$.

Lemma 12. *Let $T : [0,1] \to [0,1]$ be a piecewise C^2 uniformly expanding map and $\widetilde{T} = T \times T : [0,1]^2 \to [0,1]$. There exists $\delta(T) > 0$ such that for all $0 < \epsilon \le \delta \le \delta(T)$, one has $G_{\widetilde{T}}(\epsilon, \delta) \le \frac{64 s(\widetilde{T})}{\pi(1-s(\widetilde{T}))} \frac{\epsilon}{\delta}$.*

Proof. Let denote by $B_\epsilon^\infty(A)$ the ϵ-neighborhood of the set A for the ℓ^∞-metric. We have $B_\epsilon(A) \subset B_\epsilon^\infty(A)$. For any i and j, one $\widetilde{T}(I_i \times I_j) = TI_i \times TI_j$, which is a rectangle whose sides are parallel to the coordinate axes. Hence, $B_\epsilon(\partial \widetilde{T}(I_i \times I_j)) \subset B_\epsilon^\infty(\partial \widetilde{T}(I_i \times I_j))$, which implies $\widetilde{T}_{i,j}^{-1} B_\epsilon(\partial \widetilde{T}(I_i \times I_j)) \subset B^\infty_{s(\widetilde{T})\epsilon}(\partial I_i \times I_j)$. This inclusion follows from the product-structure of $\widetilde{T}$. We then have $\widetilde{m}(\widetilde{T}_{i,j}^{-1} B_\epsilon(\partial \widetilde{T}(I_i \times I_j)) \cap B_{(1-s(\widetilde{T}))\delta}(x)) \le \widetilde{m}(B^\infty_{s(\widetilde{T})\epsilon}(\partial I_i \times I_j) \cap B_{(1-s(\widetilde{T}))\delta}(x)) \le 16 s(\widetilde{T}) \epsilon (1-s(\widetilde{T}))\delta$ [1]. For δ small enough, depending on T, any ball $B_{(1-s(\widetilde{T}))\delta}(x)$ will intersect at most 4 elements $\widetilde{T}_{i,j}^{-1} B_\epsilon(\partial \widetilde{T}(I_i \times I_j)$, whence the lemma. □

We are now in position to prove a Lasota-Yorke inequality for the transfer operator $\widetilde{P}$ of the RDS $(\widetilde{\Omega}, \widetilde{\mathbb{P}}, \widetilde{T})$ on the space $V_1(X^2)$. Remark first that $\widetilde{T}^n_{\underline{\omega}} = T^n_{\underline{\omega}} \times T^n_{\underline{\omega}}$. Let $\lambda > 1$ such that $\lambda(T_\omega) \ge \lambda$ for all $\omega \in \Omega$. Then $s(\widetilde{T}_\omega) = \lambda(T_\omega)^{-1} \le \lambda^{-1}$.

Lemma 13. *There exists $n \ge 1$, $\epsilon_0 > 0$, $\theta < 1$ and $K > 0$ such that for all $f \in V_1(X^2)$,*

$$|\widetilde{P}^n f|_{\epsilon_0} \le \theta |f|_{\epsilon_0} + K \|f\|_{L^1_{\widetilde{m}}}.$$

[1] For any disk D of radius R and any rectangle $\mathcal{R} = [a,b] \times [c,d]$ in the plane, one has $\widetilde{m}(D \cap B_r^\infty(\partial \mathcal{R}) \le 16rR$ for any $r > 0$.

Proof. Let $\epsilon_0^{(n)} > 0$ smaller than $\epsilon_0(\widetilde{T}^n_{\underline{\omega}})$, $\frac{1}{c(\widetilde{T}^n_{\underline{\omega}})}$ and $\delta(T^n_{\underline{\omega}})$ for any $\underline{\omega} \in \Omega^n$. Taking into account Lemma 12, if we sum the inequalities given by Lemma 11 for all maps $\widetilde{T}^n_{\underline{\omega}}$, we obtain:

$$|\widetilde{P}^n f|_{\epsilon_0^{(n)}} \leq (1+\lambda^{-n})\left(\lambda^{-n} + \frac{128\lambda^{-n}}{\pi(1-\lambda^{-n})}\right)|f|_{\epsilon_0^{(n)}} + A(\epsilon_0^{(n)})\|f\|_{L^1_{\widetilde{m}}},$$

where $A(\epsilon_0^{(n)})$ depends only on $\epsilon_0^{(n)}$. Since the term in front of $|f|_{\epsilon_0^{(n)}}$ goes to 0 as n goes to ∞, we can choose n large enough so that the lemma holds. □

This proves that $\widetilde{P}$ satisfies assumptions (H1)-(H5). In order to prove (H6), according to [3, Proposition 2.9], we need to prove that the RDS $(\widetilde{\Omega}, \widetilde{\mathbb{P}}, \widetilde{T})$ has the random covering property.

Lemma 14. *The RDS $(\widetilde{\Omega}, \widetilde{\mathbb{P}}, \widetilde{T})$ has the random covering property: for any non-trivial ball $B \subset [0,1]^2$, there exists $n \geq 1$ and $\underline{\omega} \in \Omega^n$ such that $\widetilde{T}^n_{\underline{\omega}}(B) = [0,1]^2$*

Proof. Let $B \subset [0,1]^2$ be a ball. There exist two non trivial subintervals I and J of $[0,1]$ such that $I \times J \subset B$. Since $(\Omega, \mathbb{P}, T)$ has the random covering property, there exist $n_1 \geq 1$ and $\underline{\omega}^{(1)} \in \Omega^{n_1}$ such that $T_{\underline{\omega}^{(1)}}(I) = [0,1]$. Let $K \subset T^{n_1}_{\underline{\omega}^{(1)}}(J)$ be an non trivial interval. By the random covering property, there exist $n_2 \geq 1$ and $\underline{\omega}^{(2)} \in \Omega^{n_2}$ such that $T^{n_2}_{\underline{\omega}^{(2)}}(K) = [0,1]$ and thus $T^{n_2}_{\underline{\omega}^{(2)}}(T^{n_1}_{\underline{\omega}^{(1)}}(J)) = [0,1]$. In particular, $T^{n_2}_{\underline{\omega}^{(2)}}$ is onto. Set $n = n_1 + n_2$ and $\underline{\omega} = \underline{\omega}^{(2)}\underline{\omega}^{(1)}$ (the concatenation of the two finite sequences). We have

$$\begin{aligned}
\widetilde{T}^n_{\underline{\omega}}(I \times J) &= T^n_{\underline{\omega}}(I) \times T^n_{\underline{\omega}}(J)\\
&= T^{n_2}_{\underline{\omega}^{(2)}}(T^{n_1}_{\underline{\omega}^{(1)}}(I)) \times T^{n_2}_{\underline{\omega}^{(2)}}(T^{n_1}_{\underline{\omega}^{(1)}}(J))\\
&= T^{n_2}_{\underline{\omega}^{(2)}}([0,1]) \times [0,1]\\
&= [0,1]^2,
\end{aligned}$$

since $T^{n_2}_{\underline{\omega}^{(2)}}$ is onto. □

This proves that $\widetilde{P}$ has a spectral gap on $V_1(X^2)$. Since $\varphi, h \in \mathrm{BV}$, it is easy to check that $\widetilde{\varphi} \in V_1(X^2) = \widetilde{\mathcal{B}}_0$ and $h \otimes h \in V_1(X^2)$, which achieves the step 2.

Step 3. By Lemma 8, the quenched CLT without random centering is true if and only if $\widetilde{\sigma}^2 = 2\sigma^2$. The next lemma shows this condition is equivalent to the condition given in Theorem 9, which concludes the proof. □

Lemma 15. *We have $\widetilde{\sigma}^2 = 2\sigma^2$ if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\int_{\Omega^{\mathbb{N}}} \left(\sum_{k=0}^{n-1}\int_X \varphi \circ T^k_{\underline{\omega}}\, d\mu\right)^2 d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}) = 0.$$

Proof. Since the density $h \otimes h$ of the measure $\mu \otimes \mu$ belongs to $\widetilde{\mathcal{B}} = V_1(X^2)$, one has by Lemma 5:

$$\widetilde{\sigma}^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathbb{P}^{\otimes\mathbb{N}}\otimes\mu\otimes\mu}(\widetilde{S}_n^2).$$

We have

$$\mathbb{E}_{\mathbb{P}^{\otimes\mathbb{N}}\otimes\mu\otimes\mu}(\widetilde{S}_n^2) = \sum_{k,l=0}^{n-1}\int_{\Omega^{\mathbb{N}}}\int_X\int_X \widetilde{\varphi}(\widetilde{T}_{\underline{\omega}}^k(x,y))\widetilde{\varphi}(\widetilde{T}_{\underline{\omega}}^l(x,y))d\mu(x)d\mu(y)d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega})$$
$$= \sum_{k,l=0}^{n-1}\int_{\Omega^{\mathbb{N}}}\int_X\int_X \left(\varphi(T_{\underline{\omega}}^k x)-\varphi(T_{\underline{\omega}}^k y)\right)\left(\varphi(T_{\underline{\omega}}^l x)-\varphi(T_{\underline{\omega}}^l y)\right)d\mu(x)d\mu(y)d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}).$$

But

$$\int_{\Omega^{\mathbb{N}}}\int_X\int_X \left(\varphi(T_{\underline{\omega}}^k x)-\varphi(T_{\underline{\omega}}^k y)\right)\left(\varphi(T_{\underline{\omega}}^l x)-\varphi(T_{\underline{\omega}}^l y)\right)d\mu(x)d\mu(y)d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega})$$
$$= \int_{\Omega^{\mathbb{N}}}\int_X \varphi(T_{\underline{\omega}}^k x)\varphi(T_{\underline{\omega}}^l x)d\mu(x)d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}) - \int_{\Omega^{\mathbb{N}}}\left(\int_X \varphi(T_{\underline{\omega}}^k x)d\mu(x)\right)\left(\int_X \varphi(T_{\underline{\omega}}^l y)d\mu(y)\right)d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega})$$
$$- \int_{\Omega^{\mathbb{N}}}\left(\int_X \varphi(T_{\underline{\omega}}^k y)d\mu(x)\right)\left(\int_X \varphi(T_{\underline{\omega}}^l x)d\mu(y)\right)d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}) + \int_{\Omega^{\mathbb{N}}}\int_X \varphi(T_{\underline{\omega}}^k y)\varphi(T_{\underline{\omega}}^l y)d\mu(y)d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}).$$

We then obtain

$$\mathbb{E}_{\mathbb{P}^{\otimes\mathbb{N}}\otimes\mu\otimes\mu}(\widetilde{S}_n^2) = 2\int_{\Omega^{\mathbb{N}}}\int_X\left(\sum_{k=0}^{n-1}\varphi(T_{\underline{\omega}}^k x)\right)^2 d\mu(x)d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}) - 2\int_{\Omega^{\mathbb{N}}}\left(\sum_{k=0}^{n-1}\int_X\varphi(T_{\underline{\omega}}^k x)d\mu(x)\right)^2 d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega})$$
$$= 2\mathbb{E}_{\mathbb{P}^{\otimes\mathbb{N}}\otimes\mu}(S_n^2) - 2\int_{\Omega^{\mathbb{N}}}\left(\sum_{k=0}^{n-1}\int_X \varphi\circ T_{\underline{\omega}}^k d\mu\right)^2 d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}),$$

which concludes the proof, since $\sigma^2 = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\mathbb{P}^{\otimes\mathbb{N}}\otimes\mu}(S_n^2)$ by Lemma 5. □

4.3. **A counter-example.** We present here a construction that suggests the quenched CLT without random centering holds for all observables in BV only if all the maps share a common invariant measure. We consider the case where $\Omega = \{0,1\}$, $p_0 = p_1 = \frac{1}{2}$, and the two maps T_0 and T_1 are piecewise C^2, uniformly expanding, both having the covering property. They hence admit both a unique absolutely continuous invariant probability, denoted respectively by μ_0 and μ_1. We assume that $\mu_0 \neq \mu_1$. We also suppose there exist $C > 0$ and $\lambda < 1$ such that for all $\underline{\omega} \in \Omega^{\mathbb{N}}$, all $n \geq 1$ and all $f \in \mathrm{BV}$ with $\int_X f\,dm = 0$,

$$\|P_{\omega_n}\dots P_{\omega_1} f\|_{\mathrm{BV}} \leq C\lambda^n\|f\|_{\mathrm{BV}}.$$

This assumption is for instance verified if the two maps are sufficiently close in a convenient topology, or if they are β-transformations with β very close, see [6] for more details.
The RDS admits a unique acsp, denoted by μ. Let ψ be a C^∞ observable for which $\int_X \psi\,d\mu_0 \neq \int_X \psi\,d\mu_1$ and set $\varphi = \psi - \int_X \psi\,dm$. Then φ does not satisfy the quenched CLT without random centering.
By Theorem 9, it is equivalent to show that

$$\int_{\Omega^{\mathbb{N}}}\left(\frac{\sum_{k=1}^{n-1}\int_X \varphi\circ T_{\omega_k}\circ\dots T_{\omega_1}\,d\mu}{\sqrt{n}}\right)^2 d\mathbb{P}^{\otimes\mathbb{N}}(\underline{\omega}) \tag{1}$$

does not go to 0.
Changing the time direction, i.e. replacing $(\omega_1, \dots, \omega_n)$ by $(\omega_n, \dots, \omega_1)$, and applying the transfer operator, 1 can be rewritten as

$$\int_{\Omega^{\mathbb{N}}} \left(\frac{\sum_{k=1}^{n-1} \int_X \varphi P_{\omega_k} \dots P_{\omega_n} h \, dm}{\sqrt{n}} \right)^2 d\mathbb{P}^{\otimes \mathbb{N}}(\underline{\omega}), \tag{2}$$

where h is the density of μ.
For any sequence $\underline{\omega}$, by assumption we have

$$\begin{aligned} \|P_{\omega_1} \dots P_{\omega_n} h - P_{\omega_1} \dots P_{\omega_n} P_{\omega_{n+1}} h\|_{\mathrm{BV}} = \|P_{\omega_1} \dots P_{\omega_n}(h - P_{\omega_{n+1}} h)\|_{\mathrm{BV}} &\le C\lambda^n \|h - P_{\omega_{n+1}} h\|_{\mathrm{BV}} \\ &\le C\lambda^n. \end{aligned}$$

Consequently, $P_{\omega_1} \dots P_{\omega_n}$ converges exponentially fast to a function $h_{\underline{\omega}} \in \mathrm{BV}$: there exists $C > 0$ and $\lambda < 1$ such that for any $n \ge 1$ and any $\underline{\omega}$,

$$\|P_{\omega_1} \dots P_{\omega_n} h - h_{\underline{\omega}}\|_{\mathrm{BV}} \le C\lambda^n.$$

Equation 2 can then be rewritten as

$$\int_{\Omega^{\mathbb{N}}} \left(\frac{\sum_{k=1}^{n-1} \left[\int_X \varphi h_{\sigma^{k-1}\underline{\omega}} \, dm + \mathcal{O}(\lambda^{n-k}) \right]}{\sqrt{n}} \right)^2 d\mathbb{P}^{\otimes \mathbb{N}}(\underline{\omega}).$$

Set $G(\underline{\omega}) = \int_X \varphi \, h_{\underline{\omega}} \, dm$. This function is Lipschitz on $\Omega^{\mathbb{N}}$ for the symbolic metric $d_\lambda(\underline{\omega}, \underline{\omega}') = \lambda^{s(\underline{\omega}, \underline{\omega}')}$ with

$$s(\underline{\omega}, \underline{\omega}') = \inf\{n \ge 1 \, : \, \omega_n \neq \omega'_n\}.$$

Indeed, if $\omega_1 = \omega'_1, \dots, \omega_n = \omega'_n$, then

$$\begin{aligned} |G(\underline{\omega}) - G(\underline{\omega}')| &\le \|\varphi\|_\infty \|h_{\underline{\omega}} - h_{\underline{\omega}'}\|_{\mathrm{BV}} \\ &\le \|h_{\underline{\omega}} - P_{\omega_1} \dots P_{\omega_n} h\|_{\mathrm{BV}} + \|(P_{\omega_1} \dots P_{\omega_n} - P_{\omega'_1} \dots P_{\omega'_n}) h\|_{\mathrm{BV}} + \|P_{\omega'_1} \dots P_{\omega'_n} h - h_{\underline{\omega}'}\|_{\mathrm{BV}} \\ &\le C\lambda^n. \end{aligned}$$

We can then rewrite 2 as

$$\int_{\Omega^{\mathbb{N}}} \left(\frac{\mathcal{O}(1) + \sum_{k=0}^{n-1} G(\sigma^k \underline{\omega})}{\sqrt{n}} \right)^2 d\mathbb{P}^{\otimes \mathbb{N}}(\underline{\omega}).$$

If G was not of mean 0, then Birkhoff's ergodic theorem for σ would imply that the previous equation blows up, and we would prove that the equation 1 does not go to 0. Howerer, G is of mean 0 with respect to $\mathbb{P}^{\otimes \mathbb{N}}$, and one has to work a little bit more. Since G is Lipschitz, it satisfies a central limit theorem for the deterministic system $(\Omega^{\mathbb{N}}, \sigma)$. If the corresponding asymptotic variance is non zero, we obtain the desired conclusion. Otherwise, G is L^2 coboundary, and even a Hölder coboundary. In particular, G must vanish on the fixed point of σ and thus $G(0, 0, \dots) = G(1, 1, \dots) = 0$. Since $h(0, 0, \dots) = \lim_n P_0^n h$ is the density of the invariant measure μ_0 for T_0, we have $G(0,0,\ldots) = \int_X \varphi\, d\mu_0$. Similarly, $G(1,1,\ldots) = \int_X \varphi\, d\mu_1$. We thus have

$$\int \varphi\, d\mu_0 = \int \varphi\, d\mu_1 (= 0),$$

which contradicts our choice of the function φ.

4.4. **Concluding remarks.** The previous counter example strongly suggests that, in the generic situation where the stationary measure μ is not preserved by all the maps T_ω, the quenched central limit theorem without random centering does not hold. One might then be tempted to conjecture that a random centering is necessary, i.e. that for $\mathbb{P}^{\otimes \mathbb{N}}$-a.e. $\underline{\omega}$,

$$\frac{S_n(\underline{\omega}, \cdot) - \mu(S_n(\underline{\omega}, \cdot))}{\sqrt{n}} \Longrightarrow_\mu \mathcal{N}(0, \sigma^2).$$

A clue in this direction is given by Theorem 9, since it affirms that the quenched CLT without random centering is valid if and only if $\frac{\mu(S_n(\underline{\omega}, \cdot))}{\sqrt{n}}$ goes to 0 in $L^2(\mathbb{P}^{\otimes \mathbb{N}})$, i.e. if and only if the difference between the two formulations is negligible in L^2.

In a recent paper by Dobbs and Stenlund [7], the importance of the centering has been highlighted in the context of quasistatic dynamical systems, and in particular the fact that the regularity of the initial distribution plays a role to determine whether a centering is admissible or not, see section 3.2 and the discussion after Theorem 3.6. It would be interesting to investigate if this phenomenon occurs also in the different but related topic of random dynamical systems.

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