On the real inversion formula for the bilateral Laplace transform

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Abstract

For the bilateral Laplace transform

$$F(x) = \int_{-\infty}^{\infty} e^{xt}\Phi(t)dt,\ x \in \mathbb{R},$$

which is well defined for any locally summable function $\Phi(t)$ such that $\Phi(t) = O(e^{-\cosh \alpha t}),\ t \to \infty,\ \alpha > 1$ we prove the following inversion formula

$$\Phi(x) = \lim_{n\to\infty} e^n \sqrt{\frac{n}{2\pi}} \sum_{k=0}^{\infty} \frac{(-n)^k e^{x(n+k+1)}}{k!} F(-n-k-1),\ x \in \mathbb{R}.$$

Here the convergence is with respect to L_p-norm, $\frac{3}{2} \le p < \infty$, or almost everywhere. Under additional condition on Φ to be represented by the Fourier transform of L_1-function the limit is uniform for all $x \in \mathbb{R}$. Special attention is given to the Hilbert case $p = 2$. As a consequence the related results are formulated for the Mellin transform of real variable.

Keywords: *Bilateral Laplace transform, Mellin transform, real inversion formula, convolution, Gamma-function, Stirling formula, Fourier transform, modified Bessel function, Hermite polynomials*

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1 Introduction and preliminary results

In this paper we mainly deal with the bilateral Laplace transform [3], [7] of the real variable

$$F(x) = \int_{-\infty}^{\infty} e^{xt}\Phi(t)dt, \quad x \in \mathbb{R}, \tag{1.1}$$

where a complex-valued locally summable function $\Phi(t)$ is defined on $\mathbb{R}$, such that integral (1.1) is convergent in a definite sense. We will establish integral and series expansions for a class of functions Φ, which will drive us to the so-called real inversion formula for transformation (1.1), i.e. we will find some inversion for the bilateral Laplace transform $F(x)$ when x is real.

As far as we aware there is a gap in real inversion theory for the case of the bilateral Laplace and Mellin transforms of real variables. The latter transformation (cf. [6]) can be easily obtained from (1.1) by the simple substitution $e^t = y$. Indeed, as a result we arrive at the following Mellin integral

$$F(x) = \int_{0}^{\infty} y^{x-1}\Phi(\log y)dy, \quad x \in \mathbb{R}. \tag{1.2}$$

If $\Phi(t)$ is an even (an odd) function on $\mathbb{R}$ then $F(x)$ is even (odd) and can be represented, correspondingly, by the following real transformations

$$F(x) = 2\int_{0}^{\infty} \cosh xt \ \Phi(t)dt, \tag{1.3}$$

$$F(x) = 2\int_{0}^{\infty} \sinh xt \ \Phi(t)dt. \tag{1.4}$$

Further, if we consider transformation (1.1) $F(z)$, where z belongs to the vertical strip $\sigma_1 < \mathrm{Re} z < \sigma_2$ in a complex plane we treat the bilateral Laplace transform under certain conditions as an analytic function in the interior of the strip. Moreover, for its inversion we may employ the theory of Fourier integrals to get an inversion formula with the integration over a vertical line in the complex plane. The boundedness properties of the bilateral Laplace transform of a complex variable can be deduced from those of the one-side Laplace transform, where the integration is realized over $\mathbb{R}_+$. For instance, when $0 < x < \infty$ then for the one-side Laplace transform

$$F(x) = \int_{0}^{\infty} e^{-xt}\Phi(t)dt$$

we have

$$\Phi(x) = \lim_{n\to\infty} \frac{(-1)^n}{n!}\left(\frac{n}{x}\right)^{n+1} F^{(n)}\left(\frac{n}{x}\right),$$

where $F^{(n)}$ means n-th derivative of F. This is the familiar Post-Widder real inversion formula for the one-side Laplace transform (see [3], [7]).

There are different approaches to obtain real inversion formulas for the one-side Laplace transform and its iterations, which structurally form a class of more general convolution transforms of the Fourier and Mellin type with hypergeometric functions as the kernel. We mention here the familiar Stieltjes transform, the Meijer transform, the Weierstrass transform, etc [3]. We note that, for example in [2], [3], [7] the method of the Mellin-Barnes integral representations [4] is applied, where the corresponding infinite product expansions for the ratio of Euler's Gamma-functions [1] are employed. Concerning the probabilistic approach to get real inversion formulas see [5].

We start to give some sufficient conditions of the existence of the bilateral Laplace transform (1.1) and to obtain its estimate for all $x \in \mathbb{R}$.

Lemma 1. *Let $\Phi(t) \in L_{loc}(\mathbb{R})$ be such that $\Phi(t) = O(e^{-\cosh \alpha t}),\ t \to \infty,\ \alpha > 1$. Then the bilateral Laplace transform* (1.1) *exists, where the corresponding integral converges absolutely and uniformly on any compact set of* $\mathbb{R}$. *Moreover, the following estimate is true*

$$|F(x)| \le C \frac{2^{|x|/\alpha}}{\alpha} \Gamma\left(\frac{|x|}{\alpha}\right),\ x \in \mathbb{R}, \tag{1.5}$$

where $C > 0$ is a constant being independent of x and $\Gamma(z)$ is Euler's Gamma-function.

Proof. Indeed, by the straightforward estimation we derive

$$|F(x)| \le \int_{-\infty}^{\infty} e^{xt} |\Phi(t)|\, dt < C \int_{-\infty}^{\infty} e^{xt} e^{-\cosh \alpha t} dt$$

$$= \frac{2C}{\alpha} \int_0^{\infty} e^{-\cosh t} \cosh\left(\frac{xt}{\alpha}\right) dt = \frac{2C}{\alpha} K_{x/\alpha}(1), \tag{1.6}$$

where $K_\nu(z)$ is the modified Bessel function [1], which has the following integral representations

$$K_\nu(z) = \int_0^{\infty} e^{-z \cosh u} \cosh \nu u du, \tag{1.7}$$

$$K_\nu(z) = \frac{1}{2}\left(\frac{x}{2}\right)^{\nu} \int_0^{\infty} e^{-t-\frac{x^2}{4t}} t^{-\nu-1} dt. \tag{1.8}$$

Hence by using (1.8) we easily get the estimate $K_\nu(1) \le 2^{|\nu|-1}\Gamma\left(|\nu|\right), \nu \in \mathbf{R}$. Combining with (1.6) we get the desired estimate (1.5). □

We denote by $L_p(\mathbb{R}; \omega(t)dt)$ the Lebesgue spaces with respect to the measure $\omega(t)dt$ equipped with the norm

$$||f||_{L_p(\mathbb{R};\omega(t)dt)} = \left(\int_{-\infty}^{\infty} |f(t)|^p \omega(t) dt\right)^{1/p},\ 1 \le p < \infty. \tag{1.9}$$

Let us assume that $\Phi \in L_2(\mathbb{R}; e^{t^2}dt)$. This Hilbert space evidently contains all functions satisfying conditions of Lemma 1. In fact, we have

$$\int_{-\infty}^{\infty} |\Phi(t)|^2 e^{t^2} dt \le C \int_{-\infty}^{\infty} e^{t^2 - 2\cosh \alpha t} dt < \infty.$$

Lemma 2. *Let $\Phi \in L_2(\mathbb{R}; e^{t^2}dt)$. Then the bilateral Laplace transform* (1.1) *exists as a Lebesgue integral and defines an infinitely smooth function, i.e. $F \in C^\infty(\mathbb{R})$. Moreover, all derivatives $\frac{d^n}{dx^n}\left(e^{-x^2/2}F(x)\right)$, $n \in \mathbb{N}_0$ belong to $L_2(\mathbb{R}; dt)$ and satisfy the following inequality*

$$\int_{-\infty}^{\infty} \left| \frac{d^n}{dx^n}\left(e^{-x^2/2}F(x)\right)\right|^2 dx \le 2\pi n! \; ||\Phi||^2_{L_2(\mathbb{R}; e^{t^2}dt)}, \; n = 0, 1, 2, \ldots \quad . \tag{1.10}$$

Proof. We have

$$e^{-x^2/2}F(x) = \int_{-\infty}^{\infty} \Phi(t) e^{t^2/2} e^{-\frac{(x-t)^2}{2}} dt.$$

Hence, it is not difficult to verify that on any compact set of $\mathbb{R}$ we can differentiate through with respect to x in the latter integral. As a result we obtain

$$\frac{d^n}{dx^n}\left(e^{-x^2/2}F(x)\right) = \int_{-\infty}^{\infty} \Phi(t) e^{t^2/2} \frac{d^n}{dx^n}\left(e^{-\frac{(x-t)^2}{2}}\right) dt$$

$$= (-1)^n 2^{-n/2} \int_{-\infty}^{\infty} \Phi(t) e^{t^2/2} e^{-\frac{(x-t)^2}{2}} H_n\left(\frac{x-t}{\sqrt{2}}\right) dt,$$

where $H_n(y)$, $n \in \mathbb{N}_0$ is the system of Hermite polynomials [6]. Applying the Schwarz inequality, making elementary substitutions and taking into account the value of the normalized factor for the Hermite polynomials we derive the estimate

$$\left| \frac{d^n}{dx^n}\left(e^{-x^2/2}F(x)\right)\right|^2 \le 2^{-n} \int_{-\infty}^{\infty} |\Phi(t)|^2 e^{t^2} e^{-\frac{(x-t)^2}{2}} dt$$

$$\times \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} H_n^2\left(\frac{x-t}{\sqrt{2}}\right) dt = 2^{-n+\frac{1}{2}} \int_{-\infty}^{\infty} |\Phi(t)|^2 e^{t^2} e^{-\frac{(x-t)^2}{2}} dt$$

$$\times \int_{-\infty}^{\infty} e^{-y^2} H_n^2(y)\, dy = n!\sqrt{2\pi} \int_{-\infty}^{\infty} |\Phi(t)|^2 e^{t^2} e^{-\frac{(x-t)^2}{2}} dt. \tag{1.11}$$

Hence integrating through with respect to x in (1.11) we change the order of integration via Fubini's theorem and we get the inequality

$$\int_{-\infty}^{\infty} \left| \frac{d^n}{dx^n}\left(e^{-x^2/2}F(x)\right)\right|^2 dx \le n!\sqrt{2\pi} \int_{-\infty}^{\infty} |\Phi(t)|^2 e^{t^2} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} dx dt$$

$$= 2\pi n! \int_{-\infty}^{\infty} |\Phi(t)|^2 e^{t^2} dt,$$

which yields (1.10). □

2 Real inversion formulas

The main result of this section is the following

Theorem 1. *Let Φ satisfy conditions of Lemma 1. Then Φ admits the representation*

$$\Phi(x) = \lim_{n\to\infty} e^n \sqrt{\frac{n}{2\pi}} \sum_{k=0}^{\infty} \frac{(-n)^k e^{x(n+k+1)}}{k!} F(-n-k-1), \ x \in \mathbb{R}, \tag{2.1}$$

where $F(-n-k-1)$ are values of the bilateral Laplace transform (1.1) *in integers. The convergence in* (2.1) *is with respect to L_p-norm of the space $L_p(\mathbb{R}; dt)$, $\frac{3}{2} \le p < \infty$ and almost everywhere. If $\Phi \in L^*(\mathbb{R})$, i.e. can be represented by the Fourier transform of integrable function, then the convergence is uniform.*

Proof. We denote by

$$\Phi_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (ne^x)^{k+n+1}}{n!\ k!} F(-n-k-1). \tag{2.2}$$

Hence employing the asymptotic Stirling formula for factorials [1]

$$n! \sim \sqrt{2\pi n}\ n^n\ e^{-n}, \quad n \to \infty, \tag{2.3}$$

we observe that $\Phi_n(x)$ is equivalent to the expression under the limit sign in (2.1) when $n \to \infty$. Further we will appeal to (1.5) in order to estimate the series in (2.2) for all $x \in \mathbb{R}$, $n \in \mathbb{N}$. In fact, invoking more general asymptotic Stirling formula for Gamma-functions [1]

$$\Gamma(z) = O\left(z^{z-1/2} e^{-z}\right), \quad |z| \to \infty, \tag{2.4}$$

we obtain ($\alpha > 1$)

$$\left| \sum_{k=0}^{\infty} \frac{(-ne^x)^k}{k!} F(-n-k-1) \right| \le \sum_{k=0}^{\infty} \frac{(ne^x)^k}{k!} |F(-n-k-1)|$$

$$\le C 2^{(n+1)/\alpha - 1} \sum_{k=0}^{\infty} \frac{(2ne^x)^k}{k!} \Gamma\left(\frac{n+k+1}{\alpha}\right) < C_{n,\alpha} \sum_{k=1}^{\infty} \frac{(2ne^{x+1-1/\alpha})^k}{k^{k(1-1/\alpha)-n-1}} < \infty,$$

where $C_{n,\alpha} > 0$ is a constant. Therefore we substitute in (2.2) the value of coefficients $F(-n-k-1)$ via integral (1.1) and we change the order of integration and summation due to the absolute and uniform convergence. Calculating the inner series we arrive at the integral representation

$$\Phi_n(x) = \frac{(ne^x)^{n+1}}{n!} \sum_{k=0}^{\infty} \frac{(-ne^x)^k}{k!} \int_{-\infty}^{\infty} e^{-(n+k+1)t} \Phi(t) dt$$
$$= \frac{n^{n+1}}{n!} \int_{-\infty}^{\infty} e^{-ne^{x-t}} e^{(x-t)(n+1)} \Phi(t) dt. \tag{2.5}$$

Let us consider first the case $p = 2$. Since evidently $\Phi \in L_1(\mathbb{R}; dt) \cap L_2(\mathbb{R}; dt)$ and it possesses by the Fourier transform $\hat{\Phi}(\tau) = \frac{1}{\sqrt{2\pi}} F(i\tau) \in L_2(\mathbb{R}; d\tau)$ we treat (2.5) as a Fourier convolution with integrable function $e^{-ne^t} e^{t(n+1)} \in L_1(\mathbb{R}; dt)$ for each $n \in \mathbb{N}$. Hence taking into account the value of Euler's integral for Gamma-functions and applying the Plancherel identity for Fourier transforms [6, Theorem 65] we write

$$\Phi_n(x) = \frac{1}{n!\sqrt{2\pi}} \int_{-\infty}^{\infty} \Gamma(n+1+i\tau) \hat{\Phi}(\tau) e^{-(x+\log n)i\tau} d\tau. \tag{2.6}$$

Moreover, invoking the Parseval equality and elementary inequality $|\Gamma(n+1+i\tau)| \leq \Gamma(n+1) = n!$ we easily get

$$\int_{-\infty}^{\infty} |\Phi_n(x)|^2 dx = \int_{-\infty}^{\infty} \left| \frac{\Gamma(n+1+i\tau)}{n!} \right|^2 \left| \hat{\Phi}(\tau) \right|^2 d\tau$$
$$\leq \int_{-\infty}^{\infty} \left| \hat{\Phi}(\tau) \right|^2 d\tau = \int_{-\infty}^{\infty} |\Phi(x)|^2 dx.$$

In the same manner

$$\int_{-\infty}^{\infty} |\Phi_n(x) - \Phi(x)|^2 dx = \int_{-\infty}^{\infty} \left| \frac{\Gamma(n+1+i\tau)}{n^{i\tau} n!} - 1 \right|^2 \left| \hat{\Phi}(\tau) \right|^2 d\tau, \tag{2.7}$$

and the right-hand side of the latter integral tends to zero when $n \to \infty$ via the dominated convergence theorem and Stirling's formula (2.3). Consequently, we have proved (2.1) in the mean square convergence sense and we establish the real inversion formula for the bilateral Laplace transform for this case.

For general $p \in [3/2, \infty)$ we return to (2.5). Hence since $\Phi_n, \Phi \in L_p(\mathbb{R}; dx)$ we invoke the Schwarz inequality and take into account (2.7) to obtain

$$\int_{-\infty}^{\infty} |\Phi_n(x) - \Phi(x)|^p dx \leq \left(\int_{-\infty}^{\infty} |\Phi_n(x) - \Phi(x)|^{2(p-1)} dx \right)^{1/2}$$
$$\times \left(\int_{-\infty}^{\infty} |\Phi_n(x) - \Phi(x)|^2 dx \right)^{1/2}. \tag{2.8}$$

Meanwhile, employing (2.5), the Minkowski and generalized Minkowski inequalities with the value of the integral

$$\frac{n^{n+1}}{n!}\int_{-\infty}^{\infty} e^{-ne^t} e^{t(n+1)} dt = 1,$$

we find

$$\left(\int_{-\infty}^{\infty} |\Phi_n(x) - \Phi(x)|^{2(p-1)} dx\right)^{\frac{1}{2(p-1)}} \le ||\Phi||_{L_{2(p-1)}(\mathbb{R};dx)} + \left(\int_{-\infty}^{\infty} |\Phi_n(x)|^{2(p-1)} dx\right)^{\frac{1}{2(p-1)}}$$

$$\le 2||\Phi||_{L_{2(p-1)}(\mathbb{R};dx)} < \infty,\ p \ge \frac{3}{2}.$$

Consequently, combining with (2.7), (2.8) we get

$$\int_{-\infty}^{\infty} |\Phi_n(x) - \Phi(x)|^p dx \le \left[2||\Phi||_{L_{2(p-1)}(\mathbb{R};dx)}\right]^{p-1} \left(\int_{-\infty}^{\infty} |\Phi_n(x) - \Phi(x)|^2 dx\right)^{1/2} \to 0, n \to \infty,$$

and (2.1) is proved in the mean convergence sense for $p \ge 3/2$.

In order to establish the convergence almost everywhere we appeal to the Stirling formula (2.3) and we write

$$|\Phi_n(x) - \Phi(x)| \le \frac{n^{n+1}}{n!}\int_{-\infty}^{\infty} e^{-ne^{x-t}} e^{(x-t)(n+1)} |\Phi(t) - \Phi(x)| \, dt$$

$$\le M\sqrt{n}\int_{-\infty}^{\infty} e^{g_n(x-t)} |\Phi(t) - \Phi(x)| \, dt, \tag{2.9}$$

where $M > 0$ is an absolute constant and $g_n(y) = n(1-e^y)+(n+1)y,\ g_n(0) = 0$. It is easy to prove by ordinary calculus that $-\infty < g_n(y) < g_n(y_0)\ y \in \mathbb{R}$, where $y_0 = \log\left(1 + \frac{1}{n}\right)$ is the point where it attains its maximum value $g_n(y_0) = (n+1)\log\left(1 + \frac{1}{n}\right) - 1 < \frac{1}{n}, n \in \mathbb{N}$. Moreover, $g_n(y)$ is steadily increasing for $-\infty < y \le y_0$ and steadily decreasing for $y_0 < y < \infty$. Thus splitting up the latter integral in (2.9) we have

$$|\Phi_n(x) - \Phi(x)| \le M\left[\sqrt{n}\int_{|t-x|<1/n} + \sqrt{n}\int_{|t-x|>1/n}^{|t-x|<\delta} + \sqrt{n}\int_{|t-x|>\delta}\right] e^{g_n(x-t)} |\Phi(t) - \Phi(x)| \, dt$$

$$= M\left[I_1 + I_2 + I_3\right] \tag{2.10}$$

for any finite $\delta > 2/n,\ n \in \mathbb{N}$. For the first integral we find that

$$I_1 \le \sqrt{n}\int_{|t-x|<1/n} e^{1/n} |\Phi(t) - \Phi(x)| \, dt \le \frac{e}{\sqrt{n}}\, n\int_{|t-x|<1/n} |\Phi(t) - \Phi(x)| \, dt$$

$$= o\left(\frac{1}{\sqrt{n}}\right), \ n \to \infty$$

almost for all $x \in \mathbb{R}$ since $\Phi \in L_1(\mathbb{R})$. Further, we show that when $|y| > \delta$ then $g_n(y) < 0$. Indeed, for $y < -\delta$ it follows from the discussion above. When $y > \delta$ we write

$$g_n(y) = y + n(1 + y - e^y) = y - n\sum_{k=2}^{\infty} \frac{y^k}{k!} < y(1 - \frac{ny}{2}) < 0.$$

Hence for $0 < \delta \leq 1, \ n = 1, 2, \ldots,$ we obtain

$$n \ e^{g_n(y)} < \frac{n}{1 - g_n(y)} = \frac{1}{e^y - 1 - y + (1-y)/n} < \frac{1}{e^{-\delta} - 1 + \delta}, \quad y < -\delta,$$

$$n \ e^{g_n(y)} < \frac{1}{e^{\delta} - 1 - \delta}, \quad y > \delta,$$

and when $\delta > 1$ it gives correspondingly

$$n \ e^{g_n(y)} < e^{\delta}, \quad y < -\delta,$$

$$n \ e^{g_n(y)} < \frac{1}{e^{\delta} - 2\delta} \quad y > \delta,$$

for all $n \in \mathbb{N}$. Therefore for the third integral we derive

$$I_3 \leq \frac{1}{\sqrt{n}} \int_{|t-x|>\delta} n \ e^{g_n(x-t)} |\Phi(t)| dt + \sqrt{n} |\Phi(x)| \int_{|t-x|>\delta} e^{g_n(x-t)} dt.$$

Hence invoking the above estimates of the kernel $e^{g_n(x-t)}$ we get that the first integral of the latter inequality is less or equal of $C(x,\delta)/\sqrt{n}$, where $C(x,\delta) > 0$ is a constant depending on x and δ. Thus it tends to zero when $n \to \infty$. The second integral can be treated as follows. In fact, since $g_n'(y) = 1 + n - ne^y \neq 0$ for $|y| > \delta$ with integration by parts we deduce

$$\sqrt{n}|\Phi(x)| \int_{|t-x|>\delta} e^{g_n(x-t)} dt = \frac{|\Phi(x)|}{\sqrt{n}} \int_{|y|>\delta} e^{g_n(y)} \frac{g_n'(y)dy}{1 + 1/n - e^y}$$

$$= \frac{|\Phi(x)|}{\sqrt{n}} \left. \frac{e^{g_n(y)}}{1 + 1/n - e^y} \right|_{|y|>\delta} + \frac{|\Phi(x)|}{\sqrt{n}} \int_{|y|>\delta} e^{g_n(y)+y} \frac{dy}{(1 + 1/n - e^y)^2}$$

$$\leq |\Phi(x)| \left[O\left(\frac{1}{\sqrt{n}}\right) + \frac{e}{\sqrt{n}} \int_{|y|>\delta} \frac{e^y dy}{(1 + 1/n - e^y)^2} \right] = |\Phi(x)| \left[O\left(\frac{1}{\sqrt{n}}\right) \right.$$

$$\left. - \frac{e}{\sqrt{n}} \left. \frac{1}{1 + 1/n - e^y} \right|_{|y|>\delta} \right] = |\Phi(x)| \ O\left(\frac{1}{\sqrt{n}}\right), \ n \to \infty.$$

The second integral remains

$$I_2 \le \frac{e}{\sqrt{n}}\, n \int_{|y|>1/n}^{|y|<\delta} |\Phi(x+y)-\Phi(x)|\, dy = \frac{1}{\sqrt{n}} o\left(1\right),\ \delta \to 0.$$

Now we first find a positive δ such that I_2 is sufficiently small and then we let $n \to \infty$. Returning to (2.10) we conclude the convergence almost everywhere in (2.1).

However, if the Fourier transform $\hat{\Phi}(\tau) \in L_1(\mathbb{R}; d\tau)$ then we find

$$|\Phi_n(x) - \Phi(x)| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| \frac{\Gamma(n+1+i\tau)}{n^{i\tau} n!} - 1 \right| \left| \hat{\Phi}(\tau) \right| d\tau \le \sqrt{\frac{2}{\pi}} ||\hat{\Phi}||_{L_1(\mathbb{R}; d\tau)}.$$

This yields immediately the uniform convergence in (2.1). Finally we substitute $\Phi_n(x)$ given by integral (2.5) into (1.1) and we change the order of integration by virtue of Fubini's theorem. Taking $n > -x-1$ and making elementary substitutions we arrive at the equality

$$F_n(x) = \int_{-\infty}^{\infty} e^{xt} \Phi_n(t) dt = \frac{n^{n+1}}{n!} \int_{-\infty}^{\infty} \Phi(y) \int_{-\infty}^{\infty} e^{-ne^{t-y}} e^{(t-y)(n+1)+xt} dt dy$$

$$= \frac{\Gamma(n+1+x)}{n!\ n^x} F(x).$$

Consequently, via Stirling's formula we have the limit equality $\lim_{n\to\infty} F_n(x) = F(x)$, which holds for all $x \in \mathbb{R}$. □

As a corollary we formulate the corresponding result for the Mellin transform of real variable (see (1.2))

$$G(x) = \int_0^{\infty} h(t) t^{x-1} dt. \tag{2.11}$$

Theorem 2. *Let h be locally integrable on $\mathbb{R}_+$ and satisfy the condition $h(t) = O(e^{-\frac{1}{2}(t^{\alpha}+t^{-\alpha})})$, $\log t \to \infty$, $\alpha > 1$. Then the Mellin transform (2.11) exists, where the corresponding integral converges absolutely and uniformly on any compact set of $\mathbb{R}$. Besides $h(x)$ admits the representation*

$$h(x) = \lim_{n\to\infty} e^n \sqrt{\frac{n}{2\pi}} \sum_{k=0}^{\infty} \frac{(-n)^k x^{n+k+1}}{k!} G(-n-k-1),\ x \in \mathbb{R}_+.$$

Here the series converges absolutely and the limit is with respect to L_p-norm in $L_p(\mathbb{R}_+; dt)$, $\frac{3}{2} < p < \infty$ and almost everywhere. Finally, if $h(e^x) \in L^*(\mathbb{R})$, then the limit is uniform.

Finally we prove

Theorem 3. *Let Φ, Ψ satisfy conditions of Lemma* 1 *with $\alpha > 2$ and F, G be their Laplace transforms* (1.1). *Then the following equality takes place*

$$\int_{-\infty}^{\infty} \Phi(x)\Psi(x)dx = \lim_{n\to\infty} e^n \sqrt{\frac{n}{2\pi}} \sum_{k=0}^{\infty} \frac{(-n)^k}{k!} F(-n-k-1)G(n+k+1), \tag{2.12}$$

where the integral and series converge absolutely. In particular, if $\Phi(x)$ is even (odd) on $\mathbb{R}$ then for transforms (1.3), (1.4) *it holds*

$$\int_{0}^{\infty} |\Phi(x)|^2 dx = \lim_{n\to\infty} e^n \sqrt{\frac{n}{8\pi}} \sum_{k=0}^{\infty} \frac{(-n)^k}{k!} |F(n+k+1)|^2. \tag{2.13}$$

Proof. Appealing to equalities (2.1), (2.2) and inverting the order of integration and summation we derive

$$\int_{-\infty}^{\infty} \Phi_n(x)\Psi(x)dx = e^n \sqrt{\frac{n}{2\pi}} \sum_{k=0}^{\infty} \frac{(-n)^k}{k!} F(-n-k-1)G(n+k+1). \tag{2.14}$$

This is indeed possible since by virtue of Lemma 1 for each $n \in \mathbb{N}$ the series

$$\sum_{k=0}^{\infty} \frac{n^k}{k!} |F(-n-k-1)G(n+k+1)| = O\left(\sum_{k=1}^{\infty} \frac{(4en)^k}{k^{k(1-2/\alpha)-n-1}} \right) < \infty, \ \alpha > 2.$$

Hence taking into account the estimate

$$\left| \int_{-\infty}^{\infty} [\Phi_n(x) - \Phi(x)]\, \Psi(x)dx \right| \leq ||\Psi||_{L_2(\mathbb{R};dx)} ||\Phi_n - \Phi||_{L_2(\mathbb{R};dx)} \to 0, \ n \to \infty,$$

we pass to the limit through the equality (2.14) and we prove (2.12). Equality (2.13) follows immediately letting $\Psi = \overline{\Phi}$ and using the evenness properties of the original and appropriate transforms (1.3), (1.4). $\square$

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