Rees matrix covers and the translational hull of a locally inverse semigroup

Francis J. Pastijn and Luís A. Oliveira*

Abstract

For any locally inverse semigroup S there exists a maximal dense ideal extension of S within the class **LI** of all locally inverse semigroups [13], [14]. Here we realize this maximal dense ideal extension in terms of a canonically constructed quotient of a regular Rees matrix semigroup over an inverse semigroup.

1 Preliminaries

A regular semigroup S is said to be locally inverse if eSe is an inverse semigroup for every idempotent e of S, or equivalently, if and only if Nambooripad's natural partial order $\leq$ on S turns S into a partially ordered semigroup. We refer to [6] as a general reference on semigroups and on locally inverse semigroups in particular.

The content of several papers referred to presently provides ample justification for the extensive study of the class **LI** of locally inverse semigroups in the past two decades. The question on how to construct ideal extensions within **LI** presents itself naturally and the very existence, for any given $S \in \mathbf{LI}$, of a maximal dense ideal extension of S with **LI** is helpful. We refer to [18] for the general background on ideal extensions of semigroups and all related concepts.

Given any $S \in \mathbf{LI}$ there exists a natural embedding $\pi_S : S \longrightarrow \Omega(S),\ s \longrightarrow \pi_s$ of S into the translational hull $\Omega(S)$ of S, and $S\pi_S$ is referred to as the inner part of $\Omega(S)$. It was discovered in [14] that there is a largest locally inverse subsemigroup $\Omega_{\mathbf{LI}}(S)$ of $\Omega(S)$ containing $S\pi_S$ as a subsemigroup. The order ideals of $(S, \leq)$ form a semigroup $O(S)$ for the multiplication of complexes and the mapping τ_S which associates with every $s \in S$ the principal order ideal $(s]$ of s in $(S, \leq)$ yields an embedding of S into $O(S)$ [13]. The idealizer of $S\tau_S$ in $O(S)$ contains a largest regular subsemigroup $T(S)$ which contains $S\tau_S$ [13]. It was found in [13], [14] that $\pi_S : S \longrightarrow \Omega_{\mathbf{LI}}(S)$ and $\tau_S : S \longrightarrow T(S)$ are maximal

*The second author was partially supported by the Centro de Matemática da Universidade do Porto (CMUP), financed by FCT through the programmes POCTI and POSI, with national and European Community structural funds. Pdf file available from http://cmup.fc.up.pt/cmup/.

dense ideal extensions of S within **LI**, and as such these ideal extensions are equivalent. We shall have numerous occasions to refer to [13] and [14].

The principal result of Section 4 brings yet another model isomorphic to $T(S)$ or $\Omega_{\mathbf{LI}}(S)$ in terms of a regular Rees matrix semigroup over an inverse semigroup. Rees matrix semigroups of this type were used in [3], [8], [9], [16] and the basic notions will be recalled in Section 3. There we show that S can be represented by such a Rees matrix semigroup if and only if the same is true for $T(S)$ (or for $\Omega_{\mathbf{LI}}(S)$).

If S is an inverse semigroup then $\Omega(S) = \Omega_{\mathbf{LI}}(S) \cong T(S)$ is again an inverse semigroup [18], [19], [22]. We shall assume that the reader is conversant with these two approaches for constructing maximal dense ideal extensions of inverse semigroups. In [22] $T(S)$ is seen as a subsemigroup of an inverse semigroup $C(S)$ which is itself contained in $O(S)$. In Section 2 we mimic this setting for locally inverse semigroups in general. We define the notion of a **thread** and let $C(S)$ be the set of all threads of the locally inverse semigroup S. Again then, $T(S) \subseteq C(S) \subseteq O(S)$, but unfortunately $C(S)$ need not constitute a subsemigroup of $O(S)$. We find transparent necessary and sufficient conditions for $C(S)$ to be a subsemigroup of $O(S)$ and find that these conditions are satisfied for some relevant subclasses of **LI**. More importantly, if $C(S)$ is a subsemigroup of $O(S)$ then $C(S) \in \mathbf{LI}$, and so $C(S)$ is structurally closer to S than the vastly larger $O(S)$. Another reason for considering $C(S)$ is for the sake of investigating completions as in [22]; we shall not make any explorations in that direction.

We want to mention some notions which are maybe less familiar. If S is a semigroup and $A \subseteq S$ then the **regular part** $Reg\, A$ of A consists of the $a \in A$ for which there exists $a' \in A$ such that a and a' are pairwise inverse elements of S. The set of idempotents of S is denoted by $E(S)$. We follow Nambooripad [10] in defining the quasi-orders ω^l and ω^r and the partial order ω on $E(S)$. If $S \in \mathbf{LI}$ and $e \in E(S)$ then we generally prefer to use $\leq$ instead of ω and $(e]$ instead of $\omega(e)$. If $e, f \in E(S)$ for some $S \in \mathbf{LI}$ then $e \wedge f$ is given by $\omega^r(e) \cap \omega^l(f) = \omega(e \wedge f)$ and $(E(S), \wedge)$ is said to be the **pseudosemilattice** of S. We refer to [10] and [13] for the required background on pseudosemilattices. We shall use the fact that for $e, f \in E(S)$, with $S \in \mathbf{LI}$, $e \wedge f = f \wedge e$ if and only if $ef = fe$ and then $e \wedge f = ef$.

It remains to state the following useful results. We leave the proof to the reader.

Theorem 1.1 *Let S be a regular semigroup. Then $S \in \mathbf{LI}$ if and only if every maximal subsemilattice of S is an order ideal of S.*

Corollary 1.2 *Let $S \in \mathbf{LI}$ and F a subsemilattice of S. Then $\{e \in E(S) \mid e \leq f$ for some $f \in F\}$ is a subsemilattice and an order ideal of S.*

2 Threads

Let S be a locally inverse semigroup and H, H' pairwise inverse elements in the semigroup $O(S)$ of order ideals of S. Then we shall say that H and H' are

pairwise inverse threads of S if HH' and $H'H$ are subsemilattices of S. If this is the case then of course HH' and $H'H$ are also order ideals of S. We call $H \in O(S)$ a **thread** if for some $H' \in O(S)$, H and H' are pairwise inverse threads. We denote the set of all threads of S by $C(S)$.

We collect some information concerning threads from [13].

Result 2.1 *Let S be a locally inverse semigroup.*

- (i) *If $H \in C(S)$ then no distinct elements of S are $\mathcal{L}$- or $\mathcal{R}$-related in S.*
- (ii) *For $F \in C(S)$, F is an idempotent of $O(S)$ if and only if it is a subsemilattice of S.*
- (iii) *Let H and H' be pairwise inverse threads and $E = HH'$. Then for every $a \in H$ there exists a unique inverse a' of a which belongs to H' and the mapping $H \longrightarrow H'$, $a \longrightarrow a'$ is an order isomorphism. For every $e \in E$ there exist unique $h \in H$ and $h' \in H'$ such that $h\mathcal{R}e\mathcal{L}h'$, and then h and h' are pairwise inverse elements of S with $e = hh'$.*

Proof: Let H and H' be pairwise inverse threads and $a \in H$. Then by Lemma 3.11 of [13] there exists an inverse a' of a which belongs to H'. Assume that $b \in H$ and $a\mathcal{L}b$ in S. Then $aa'\mathcal{L}ba'$ where both aa' and ba' belong to the semilattice HH', whence $aa' = ba'$. Thus $a = aa'a = ba'a = b$ since $a'a$ is an idempotent in L_b. We proved that a thread cannot contain distinct $\mathcal{L}$-related elements. The statement (i) follows by duality.

The remaining statements follow from Lemmas 3.2 and 3.11 of [13]. ■

Threads were considered for inverse semigroups by B. M. Schein in [22]. Threads are there called **permissible subsets**. If S is an inverse semigroup then $C(S)$ is an inverse subsemigroup of $O(S)$: for every $H \in C(S)$, $H^{-1} = \{a^{-1} \mid a \in H\}$ is the unique inverse of H in $C(S)$, and $E(C(S)) = C(E(S))$ is simply the $\cap$-semilattice of all ideals of the semilattice $E(S)$.

A thread of a completely simple semigroup is necessarily a singleton. More in general, if H is a thread of a normal band of groups S, then H intersects every $\mathcal{D}$-class of S in at most one element. If the normal band of groups S is a strong semilattice of completely simple semigroups $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ (in the sense of Section IV.1 of [21]), then H is a thread of S if and only if (i) $|H \cap S_\alpha| \leq 1$ for every $\alpha \in Y$, and (ii) if for $\alpha \in Y$, $a \in S_\alpha \cap H$, then $a\varphi_{\alpha,\beta} \in H$ for every $\beta \leq \alpha$ in Y. Threads were considered for normal bands of groups by M. Petrich in [20]. It was shown there (with a different notation) that $C(S)$ is a subsemigroup of $O(S)$ which is again a normal band of groups: if H is a thread of the normal band of groups S, then $H^{-1} = \{a^{-1} \mid a \in H\}$ is the inverse of H within the maximal subgroup of $C(S)$ which contains H.

Recall that if S is a locally inverse semigroup, then $\tau_S : S \longrightarrow O(S)$, $a \longrightarrow (a]$ is a faithful representation of S. If S is an inverse semigroup or a normal band of groups, then $C(S)$ is a locally inverse semigroup of the same type, canonically associated with the given S and moreover, $C(S)$ is an oversemigroup of $S\tau_S$. More in general we have the following.

Theorem 2.2 *Let S be a locally inverse semigroup such that $C(S)$ is a subsemigroup of $O(S)$. Then $C(S)$ is a locally inverse semigroup and $T(S)$ is the regular part of the idealizer of $S\tau_S$ in $C(S)$.*

Proof: If $C(S)$ is a subsemigroup of $O(S)$ then it is evidently a regular semigroup. For any idempotent E of $C(S)$, E is a subsemilattice and an order ideal of S, and by Lemma 3.10 of [13], the idempotents of the local submonoid $EC(S)E$ of $C(S)$ are precisely the ideals of E. Therefore the idempotents of $EC(S)E$ commute and $EC(S)E$ is an inverse semigroup, whence $C(S)$ is a locally inverse semigroup.

By Lemma 3.2 of [13], $T(S) \subseteq C(S)$. Therefore in particular $C(S)$ is an oversemigroup of $S\tau_S$ and $T(S)$ is the regular part of the idealizer of $S\tau_S$ in $C(S)$. ■

Unfortunately the subset $C(S)$ of $O(S)$ does not necessarily constitute a subsemigroup of $O(S)$ for any given locally inverse semigroup S. This is already not the case for completely 0-simple semigroups as we shall see later in this section. We set out to find necessary and sufficient conditions for $C(S)$ to be a subsemigroup of $O(S)$ and in Section 3 we shall see that these conditions apply to the class of straight locally inverse semigroups.

We shall first investigate the structural peculiarities for regular semigroups which are contained in $C(S)$.

Proposition 2.3 *Let S be a locally inverse semigroup and T a regular subsemigroup of $O(S)$ such that $T \subseteq C(S)$. Then*

- (i) *T is a locally inverse semigroup,*
- (ii) *$H \,\mathcal{R}\, K$ in T if and only if there exists a bijection $\varphi : H \longrightarrow K$ such that $h \,\mathcal{R}\, h\varphi$ for every $h \in H$,*
- (iii) *$H \leq K$ in T if and only if $H \subseteq K$.*

Proof: The proof of (i) follows the same argument as the first part of the proof of Theorem 2.2 and again uses Lemma 3.10 of [13]. In Proposition 3.12 of [13] we proved the above statements (ii) and (iii) for $T = T(S)$. The proof given there is valid for any regular subsemigroup T of $C(S)$ and uses Result 2.1. ■

Proposition 2.4 *Let S be a locally inverse semigroup and T a regular subsemigroup of $O(S)$ such that $T \subseteq C(S)$. The $\wedge$-operation in the pseudosemilattice of idempotents of T is given by: for $E, F \in E(T)$,*

$$E \wedge F = \{e \wedge f \mid e \in E,\ f \in F\}.$$

An idempotent h of S belongs to $E \wedge F$ if and only if there exist $e \in E$ and $f \in F$ such that $e \,\mathcal{R}\, h \,\mathcal{L}\, f$.

Proof: Let E and F be idempotents of T and consider the element $E \wedge F$ in the pseudosemilattice $(E(T), \wedge)$. By Result 2.1 E, F and $E \wedge F$ are each a subsemilattice and an order ideal of S. We let $h \in E \wedge F$, thus in particular, h is an idempotent of S, and $h = eh$ for some $e \in E$ since $E \wedge F = E(E \wedge F)$ in T. Then $h \mathcal{R} he \leq e$ in S, hence $he \in E$ since E is an order ideal. By duality we conclude that every $h \in E \wedge F$ is $\mathcal{R}$-related to some element of E and $\mathcal{L}$-related to some element of F.

Assume conversely that h is an idempotent and that $e \mathcal{R} h \mathcal{L} f$ for some $e \in E$ and $f \in F$. We set out to prove that $E(h] = (h]$. We have $E(h] \in O(S)$, $h = eh \in E(h]$ and thus $(h] \subseteq E(h]$. By Proposition 2.4 of [13], $E(h] = Eh$. For $g \in E$ we have $gh = g(h \wedge g)h = g(e \wedge g)h$, where in the semilattice E, $g(e \wedge g) = ge = eg \leq e$. Therefore $gh \leq h$ in T. We conclude that $E(h] = (h]$ indeed. Using duality we thus see that if $e \mathcal{R} h \mathcal{L} f$ for some $e \in E$ and $f \in F$, then $(h] \in \omega^r(E) \cap \omega^l(F) = \omega(E \wedge F)$, thus $(h] \leq E \wedge F$ in T. By Proposition 2.3 we then have that $(h] \subseteq E \wedge F$ and thus in particular that $h \in E \wedge F$.

We proved that

$$E \wedge F = \{h \in E(S) \mid e \mathcal{R} h \mathcal{L} f \text{ for some } e \in E,\ f \in F\}.$$

That

$$E \wedge F = \{e \wedge f \mid e \in E,\ f \in F\}$$

now follows from Lemma 3.3 of [13]. ∎

Theorem 2.5 *Let S be a locally inverse semigroup. The following are equivalent:*

(i) *$C(S)$ is a subsemigroup of $O(S)$,*

(ii) *for any idempotents $E, F \in C(S)$, $\{e \wedge f \mid e \in E,\ f \in F\}$ is an idempotent of $C(S)$,*

(iii) *S satisfies the conditions: for $e, f, g, h \in E(S)$,*

$$ef = fe,\ e \mathcal{R} g \mathcal{L} h \mathcal{R} f \quad \Longrightarrow \quad e = f,$$

$$ef = fe,\ e \mathcal{L} g \mathcal{R} h \mathcal{L} f \quad \Longrightarrow \quad e = f.$$

Proof: $(i) \Rightarrow (ii)$ is immediate from Proposition 2.4.

Assume that (ii) holds and let $e, f \in E(S)$ such that $ef = fe$. Then $E = (e] \cup (f]$ is a subsemilattice and an order ideal of S, that is, an idempotent of $C(S)$. Let $g, h \in E(S)$ such that $e \mathcal{R} g \mathcal{L} h \mathcal{R} f$ in S. Then also $(h]$ is an idempotent of $C(S)$ and so, according to our assumption, $\{p \wedge k \mid p \in E, k \leq h\}$ is an idempotent of $C(S)$, that is, a subsemilattice and order ideal of S. Since $g = e \wedge h$, $h = f \wedge h$ belong to this semilattice and $g\mathcal{L}h$, so $g = h$, whence $e\mathcal{R}f$, and since e and f commute, so $e = f$. Using duality we conclude that $(ii) \Rightarrow (iii)$.

Assume that (iii) holds. Let E and F be idempotents of $C(S)$ and $G = \{e \wedge f \mid e \in E,\ f \in F\}$. By Lemma 3.3 of [13], $G \in O(S)$ and G is a subpseudosemilattice of $(E(S), \wedge)$. Assume that $g, h \in G$ such that $g\mathcal{L}h$. Again by Lemma 3.3 of [13] there exist $e_1, e_2 \in E$ such that $e_1\mathcal{R}g$ and $e_2\mathcal{R}h$. Then $e_1e_2 = e_2e_1$ and $e_1 \,\mathcal{R}\, g \,\mathcal{L}\, h \,\mathcal{R}\, e_2$, hence $e_1 = e_2$ and thus also $g = h$. Using duality we have that G cannot contain distinct $\mathcal{L}$-related or $\mathcal{R}$-related idempotents of S. Therefore the subpseudosemilattice G of $(E(S), \wedge)$ is a subsemilattice of $(E(S), \wedge)$, and of S. Since G is also an order ideal of S, it follows that G is an idempotent of $C(S)$.

Let $H, K \in C(S)$. We need to show that $HK \in C(S)$. Therefore, let H and H' be pairwise inverse threads, K and K' be pairwise inverse threads, and put $E = KK'$, $F = H'H$ and $G = \{e \wedge f \mid e \in E,\ f \in F\}$. We shall show that HK and $K'GH'$ are pairwise inverse threads. From the foregoing we already know that G is an idempotent of $C(S)$. From Lemma 3.3 of [13] we know that $EG = G = GF$ and therefore

$$(HGK)(K'GH')(HGK) = HGK, \qquad (K'GH')(HGK)(K'GH') = K'GH',$$

whence HGK and $K'GH'$ are pairwise inverse elements of $O(S)$. Furthermore, $(HGK)(K'GH') = HGH'$ and $(K'GH')(HGK) = K'GK$ are idempotents of $O(S)$.

Let $gb \in GK$ for some $g \in G$ and $b \in K$. By Result 2.1 there exists an inverse b' of b which belongs to K', and by Lemma 3.3 of [13] there exists $f \in F$ such that $f\mathcal{L}g$. We put $h = bb' \wedge g = bb' \wedge f \in G$. Since $g, h \in G$ with $h \in \omega^l(g)$, and since G is a semilattice, we have that $h \leq g$. Therefore $gb = ghb = hb \leq b$ and so $gb \in K$ since K is an order ideal. We proved that $GK \subseteq K$. Therefore also $K'GK \subseteq K'K$. Here $K'K$ is a subsemilattice of S since K and K' are pairwise inverse threads, whence $K'GK$ is also a subsemilattice of S, and thus $K'GK$ is an idempotent of $C(S)$. In a dual way we find that $HG \subseteq H$ and that HGH' is an idempotent of $C(S)$. From the above it follows that HGK and $K'GH'$ are pairwise inverse threads.

In order to show that $HK \in C(S)$ it suffices to show that $HK = HGK$. From $HG \subseteq H$ it follows that $HGK \subseteq HK$. Let $a \in H$, $b \in K$, and apply Result 2.1 to find an inverse a' of a in H' and an inverse b' of b in K'. Then $bb' \wedge a'a \in G$ and so $ab = a(bb' \wedge a'a)b \in HGK$. We conclude that $HK = HGK$, as required. We proved that $(iii) \Rightarrow (i)$. ■

We note that if S is an inverse semigroup or a normal band of groups then S obviously satisfies the condition of Theorem 2.5(iii).

Some special cases occur when requiring stronger conditions.

Theorem 2.6 *Let S be a locally inverse semigroup. Then*

- (i) $C(S) = T(S)$ *if and only if S satisfies the equivalent conditions of Theorem 2.5 and for every $e \in E(S)$, $(e]$ is a dually well-ordered chain,*
- (ii) $C(S) = T(S) = S\tau_S$ *if and only if every subsemilattice of S has a greatest element.*

Proof: Let $C(S) = T(S)$. Since $T(S)$ is a subsemigroup of $O(S)$, the equivalent conditions of Theorem 2.5 are satisfied. For every $e \in E(S)$, all the ideals of the semilattice $(e]$ should be in $T(S)$, and therefore by Corollary 2.16 of [22], $(e]$ is a dually well-ordered chain.

To prove the converse, assume that S is a locally inverse semigroup which satisfies the equivalent conditions of Theorem 2.5 and such that $(e]$ is dually well-ordered for all $e \in E(S)$. By Theorems 2.2 and 2.5 it suffices to show that every idempotent of $C(S)$ belongs to $T(S)$.

Let $e \in E(S)$ and F an idempotent of $C(S)$. Then in the pseudosemilattice of idempotents of the locally inverse semigroup $C(S)$ we have that $((e] \wedge F)(e] \leq (e]$, which by Proposition 2.3 entails that $((e] \wedge F)(e]$ is an ideal of $(e]$. Since $(e]$ is dually well-ordered it follows that $((e] \wedge F)(e] = (e']$ for some $e' \leq e$. Further, $(e] \wedge F \,\mathcal{R}\, (e']$ in $C(S)$ and so by Proposition 2.3 there exists $g \in (e] \wedge F$ such that $g\mathcal{R}e'$. Clearly then $(e] \wedge F \subseteq \omega^r(g)$, and since $(e] \wedge F$ is a subsemilattice and an order ideal of S we need to have $(e] \wedge F = (g]$. In a dual way we can show that $F \wedge (e] = (h]$ for some $h \in E(S)$. By Lemma 3.6 of [13] it follows that $F \in T(S)$. We proved (i).

Let S be a locally inverse semigroup such that $C(S) = S\tau_S$. Let L be a subsemilattice of S. Then by Corollary 1.2 $\overline{L} = \{f \in E(S) \mid f \leq g \text{ for some } g \in L\}$ is a subsemilattice and an order ideal of S, and thus $\overline{L}$ is an idempotent of $C(S)$. Since $C(S) = S\tau_S$, $\overline{L} = (e]$ for some $e \in \overline{L}$, whence e is the greatest element of L.

Assume that, conversely, every subsemilattice of S has a greatest element. Let $e, f, g, h \in E(S)$ such that $ef = fe$ and $e \,\mathcal{R}\, g \,\mathcal{L}\, h \,\mathcal{R}\, f$. Since $ef = fe$, we have that $\{e, f, ef\}$ is a semilattice, and thus it has a greatest element. Therefore, we may as well assume that $f \leq e$. Then $h \,\omega^r e$ since $h \,\mathcal{R}\, f \leq e$, so $f \,\mathcal{R}\, he \leq e$. Since also $e \,\mathcal{R}\, g \,\mathcal{L}\, h$, so $he \,\mathcal{L}\, e$. From $he \,\mathcal{L}\, e$ and $he \leq e$ it follows that $he = e$. Therefore $f\mathcal{R}e$ and $f \leq e$, thus $f = e$. In a dual way we can show that if $e, f, g, h \in E(S)$ such that $ef = fe$ and $e \,\mathcal{L}\, g \,\mathcal{R}\, h \,\mathcal{L}\, f$, then $e = f$. By Theorems 2.2 and 2.5 we conclude that $C(S)$ is a locally inverse subsemigroup of $O(S)$, and by (i), $C(S) = T(S)$. Hence $S\tau_S$ is an ideal of $C(S)$, and using Result 2.1(ii) we have that $E(C(S)) = E(S\tau_S)$. Therefore $C(S) = S\tau_S$. We proved (ii). ∎

Theorem 2.6 generalizes Corollary 2.16 and Proposition 1.34 of [22]. A non-trivial example of a locally inverse semigroup satisfying the condition of Theorem 2.6(ii) is the four-spiral semigroup investigated in [4]. The final example of [11] where the bicyclic semigroup is embedded into a bisimple idempotent generated locally inverse semigroup using a standard device also yields countably many examples of locally inverse semigroups satisfying the condition of Theorem 2.6(ii).

Corollary 2.7 *For a completely 0-simple semigroup S, the following are equivalent:*

(i) *$C(S)$ is a subsemigroup of $O(S)$,* (*ii*) $C(S) = T(S)$,

(*iii*) *S does not contain a copy of* $\mathcal{M}^0(3, G, 2; P)$ $(|G| = 1,\ P = \left(\begin{smallmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{smallmatrix}\right))$ *nor its dual, as a subsemigroup.*

Proof: The conditions (*i*) and (*iii*) are equivalent by Theorem 2.5, and (*ii*) obviously implies (*i*). If (*i*) holds, by Theorem 2.6(*i*), $C(S) = T(S)$ holds since for every idempotent e of the completely 0-simple semigroup S, $(e]$ is either a trivial or a two element chain. ■

Corollary 2.8 *For a completely 0-simple semigroup* S, $C(S) = T(S) = S\tau_S$ *if and only if* S *does not contain a copy of* $\mathcal{M}^0(2, G, 2; P)$ $(|G| = 1,\ P = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right))$ *as a subsemigroup.*

Proof: The proof follows immediately from Theorem 2.6(*ii*). ■

Example: Let $S = \mathcal{M}^0(4, G, 4; P)$ where $|G| = 1$ and

$$P = \begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1 \end{pmatrix} \quad .$$

Then S is a completely 0-simple semigroup where $S\tau_S \neq T(S) \neq C(S)$. S is idempotent-generated whereas $T(S)$ is not. S is locally a semilattice of groups whereas $T(S)$ is not.

3 Rees matrix semigroups over inverse semi groups

A **straight locally inverse semigroup** S is a locally inverse semigroup for which $E(S)$ is the disjoint union of the maximal subsemilattices of S. From [12], [15] we can derive further information about this important class of locally inverse semigroups. A locally inverse semigroup S is straight if and only if the maximal inverse subsemigroups of S are pairwise disjoint. Let L_2 and R_2 denote the two element left zero and right zero semigroup respectively. A regular semigroup S is locally inverse if and only if S does not contain a copy of L_2^1 nor of R_2^1, and if this is the case, then S is straight if and only if moreover, S does not contain a copy of L_2^0 and R_2^0.

Let S be a straight locally inverse semigroup. The relations $\rho_{\mathbf{LZ}}$ and $\rho_{\mathbf{RZ}}$ defined by, for $a, b \in S$,

$$a\rho_{\mathbf{LZ}}b \iff aS \cap bS \neq \emptyset, \tag{1}$$

$$a\rho_{\mathbf{RZ}}b \iff Sa \cap Sb \neq \emptyset, \tag{2}$$

are the least left zero semigroup and least right zero semigroup congruence on S, respectively. We shall use the notation

$$S/\rho_{\mathbf{LZ}} = I, \qquad S/\rho_{\mathbf{RZ}} = \Lambda, \tag{3}$$

and for $i \in I$, $\lambda \in \Lambda$

$$S_i = \{s \in S \mid s\rho_{\mathbf{LZ}} = i\}, \quad S_\lambda = \{s \in S \mid s\rho_{\mathbf{RZ}} = \lambda\}, \quad S_{i\lambda} = S_i \cap S_\lambda. \tag{4}$$

Then $\rho_{\mathbf{LZ}} \cap \rho_{\mathbf{RZ}}$ is the least rectangular band congruence, $S/\rho_{\mathbf{LZ}} \cap \rho_{\mathbf{RZ}} \cong I \times \Lambda$, and S is a rectangular band $I \times \Lambda$ of the semigroups $S_{i\lambda}$, $i \in I$, $\lambda \in \Lambda$. For $i \in I$, $\lambda \in \Lambda$, we put

$$E_i = E(S_i), \quad E_\lambda = E(S_\lambda), \quad E_{i\lambda} = E_i \cap E_\lambda. \tag{5}$$

Here the $E_{i\lambda}$ are the maximal subsemilattices of S and the $E_{i\lambda}SE_{i\lambda} = E_{i\lambda}S_{i\lambda}E_{i\lambda}$ are the maximal inverse subsemigroups of S. When dealing with a straight locally inverse semigroup S we shall always silently adopt the notation introduced above.

The following is a reformulation of Lemma 7 of [15]. Theorem 2.5 allows us to give a short independent proof.

Lemma 3.1 *Let S be a straight locally inverse semigroup. Then*

(i) $C(S)$ is a straight locally inverse semigroup,

(ii) for $H \in C(S)$ there exists a unique $(i, \lambda) \in I \times \Lambda$ such that $H \subseteq S_{i\lambda}$, and then

$$C(S) \longrightarrow I \times \Lambda, \quad H \longrightarrow (i, \lambda) \tag{6}$$

is a surjective homomorphism which induces the least rectangular band congruence on $C(S)$,

(iii) the $E(C(S)_{i\lambda}) = C(E_{i\lambda})$ are the maximal subsemilattices of $C(S)$.

Proof: Let $e, f, g, h \in E(S)$ such that $ef = fe$, $e\,\mathcal{R}\,g\,\mathcal{L}\,h\,\mathcal{R}\,f$. Since $ef = fe$, $e, f \in E_{i\lambda}$ for some $i \in I$, $\lambda \in \Lambda$, and since $e\,\mathcal{R}\,g\,\mathcal{L}\,h\,\mathcal{R}\,f$ we then have that $g, h \in E_{i\mu}$ for some $\mu \in \Lambda$. Since $g\mathcal{L}h$ in the semilattice $E_{i\mu}$ it follows that $g = h$. Then $e\mathcal{R}f$ in the semilattice $E_{i\lambda}$ and so $e = f$. Using duality and Theorem 2.5 we conclude that $C(S)$ is a locally inverse semigroup. That the $C(E_{i\lambda})$ are the maximal subsemilattices of $C(S)$ follows from Result 2.1(ii). Consequently, $C(S)$ is a straight locally inverse semigroup.

Let H and H' be pairwise inverse elements of $C(S)$. From (iii) it follows that there exist unique $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $HH' \subseteq E_{i\mu}$ and $H'H \subseteq E_{j\lambda}$, whence $H \in S_{i\lambda}$ for some unique $i \in I$ and $\lambda \in \Lambda$. One readily obtains that (6) is a homomorphism which induces a rectangular band congruence on $C(S)$. In each congruence class, the idempotents commute. Therefore, using Lallement's Lemma (Lemma 2.4.3 of [6]) one shows that this congruence relation is the least rectangular band congruence. We are now also allowed to write $E(C(S)_{i\lambda}) = C(E_{i\lambda})$ for all $i \in I$, $\lambda \in \Lambda$. ■

We shall need the following result (see Lemma 3.6 of [13]).

Result 3.2 *Let S be a locally inverse semigroup such that $C(S)$ is again a locally inverse semigroup. If $F \in E(C(S))$, then $F \in E(T(S))$ if and only if for all $e \in E(S)$ there exist $k, l \in E(S)$ such that $(e] \wedge F = (k]$ and $F \wedge (e] = (l]$.*

For an inverse semigroup S the idempotents of $C(S)$ are the order ideals of the semilattice $E(S)$, and for $F \in E(C(S)) = C(E(S))$ we have that $F \in E(T(S))$ if and only if for every $e \in E(S)$ there exists a $k \in E(S)$ such that $(e] \wedge F = (k]$. The elements of $E(T(S)) = T(E(S))$ are called the **retract ideals** of the semilattice $E(S)$ (see Lemma 2.8 of [22], or [16]). Here the $\wedge$-operation in $E(C(S))$ and in $E(T(S))$ is just the intersection.

Lemma 3.3 *Let S be a straight locally inverse semigroup. Then*

(i) $T(S)$ is a straight locally inverse semigroup,

(ii) for every $H \in T(S)$, there exists a unique $(i, \lambda) \in I \times \Lambda$ such that $H \subseteq S_{i\lambda}$, and then

$$T(S) \longrightarrow I \times \Lambda, \quad H \longrightarrow (i, \lambda) \tag{7}$$

is a surjective homomorphism which induces the least rectangular band congruence on $T(S)$,

(iii) the $E(T(S)_{i\lambda})$ are the maximal subsemilattices of $T(S)$ and $E(T(S)_{i\lambda}) \subseteq T(E_{i\lambda})$.

Proof: It is easy enough to show that a regular subsemigroup of a straight locally inverse semigroup is again a straight locally inverse semigroup. It thus follows from Theorem 2.2 and Lemma 3.1 that $T(S)$ is a straight locally inverse semigroup. For every $(i, \lambda) \in I \times \Lambda$, $C(S)_{i\lambda} \cap T(S)$ is nonempty since $S_{i\lambda}\tau_S \subseteq T(S)_{i\lambda}$ and we see that $T(S)$ is a rectangular band $I \times \Lambda$ of the $C(S)_{i\lambda} \cap T(S)$, $(i, \lambda) \in I \times \Lambda$. It is not difficult to see that (7) is the restriction to $T(S)$ of the homomorphism (6) and that this homomorphism induces the least rectangular band congruence on $T(S)$.

It remains to explain the inclusion in the statement (iii). From the above it follows that $T(S)_{i\lambda} = C(S)_{i\lambda} \cap T(S)$. Thus if $F \in E(T(S)_{i\lambda})$, then by Lemma 3.1, $F \in E(C(S)_{i\lambda}) = C(E_{i\lambda})$, that is, F is an order ideal of the semilattice $E_{i\lambda}$. From Result 3.2 it follows in particular that for every $e \in E_{i\lambda}$ there exists $f \in E_{i\lambda}$ such that $(e] \wedge F = (f]$, and thus again by Result 3.2, $F \in T(E_{i\lambda})$. ∎

When setting the statements of Lemmas 3.1 and 3.3 side by side one sees that the similarity in not complete: the inclusion mentioned in Lemma 3.3(iii) may well be strict as we shall soon discover. In other words, if S is a straight locally inverse semigroup, then not every retract ideal of a maximal subsemilattice of S need to be an idempotent of $T(S)$. In Theorem 3.6 we characterize the straight locally inverse semigroups S for which the inclusion of Lemma 3.3(iii) is always an equality.

In this section we shall be interested in a special class of straight locally inverse semigroups. Let I $[\Lambda]$ be a left [right] zero semigroup, V an inverse semigroup and $Q = (\omega_{\lambda i})$ a $\Lambda \times I$-matrix with entries $\omega_{\lambda i}$ which belong to the translational hull $\Omega(V)$ of V. On $I \times V \times \Lambda$ define a multiplication by:

$$(i, s, \lambda)(j, t, \mu) = (i, s\,\omega_{\lambda j}\,t, \mu).$$

We thus obtain a semigroup whose regular part we denote by $\mathcal{RM}(I, V, \Lambda; Q)$. Using this notation we collect the following information from Proposition 1 of [16].

Result 3.4 *$S = \mathcal{RM}(I, V, \Lambda; Q)$ is a straight locally inverse semigroup, and*

$$S = \{(i, \omega_{\mu i}^{-1}\,v\,\omega_{\lambda j}^{-1}, \lambda) \mid i, j \in I,\ \lambda, \mu \in \Lambda,\ v \in V\}.$$

The mapping

$$S \longrightarrow I \times \Lambda, \quad (i, s, \lambda) \longrightarrow (i, \lambda)$$

is a surjective homomorphism which induces the least rectangular band congruence on S. The maximal subsemilattices of S are given by

$$E_{i\lambda} = E(S_{i\lambda}) = \{(i, e\,\omega_{\lambda i}^{-1}, \lambda) \mid e \in E(V)\} = \{(i, \omega_{\lambda i}^{-1}\,e, \lambda) \mid e \in E(V)\}.$$

For a proper understanding of the above result recall that $\Omega(V)$ is an inverse semigroup, and for $\omega \in \Omega(V)$, ω^{-1} is the inverse of ω in $\Omega(V)$. Some authors (see e.g. [3], [8], [9]) take the entries of Q in V itself. This amounts to taking the $\omega_{\lambda i}$ in the inner part $V\pi_V$ of $\Omega(V)$.

The semigroup $\mathcal{RM}(I, V, \Lambda; Q)$ of Result 3.4 is called a **regular Rees matrix semigroup over the inverse semigroup** V. Not every straight locally inverse semigroup has such a representation: those that have are characterized in Theorem 7 of [16], and we give further characterizations in Theorem 3.6. By Theorem 10 of [16], a locally inverse semigroup S can be represented as a $\mathcal{RM}(I, V, \Lambda; Q)$ where Q has entries in V if and only if S is a locally inverse semigroup whose maximal subsemilattices each have a greatest element. Such is the case for $C(S)$ whenever S is a straight locally inverse semigroup, because here each maximal subsemilattice of $C(E_{i\lambda})$ has a greatest element $E_{i\lambda}$.

Lemma 3.5 *Let $S = \mathcal{RM}(I, V, \Lambda; Q)$ be a regular Rees matrix semigroup over the inverse semigroup V. For $(i, \lambda) \in I \times \Lambda$, let $\gamma \in \Omega(V)$ such that $\gamma \leq \omega_{\lambda i}$ in $\Omega(V)$. Then*

$$\{(i, e\gamma^{-1}, \lambda) \mid e \in E(V)\}$$

and

$$\{(i, \gamma^{-1}e, \lambda) \mid e \in E(V)\}$$

are retract ideals of $E_{i\lambda}$, and conversely, every retract ideal of $E_{i\lambda}$ can be written in either such way.

Proof: The mapping

$$E_{i\lambda} \longrightarrow E(V),\ (i, \omega_{\lambda i}^{-1} e, \lambda) \longrightarrow \omega_{\lambda i}\omega_{\lambda i}^{-1} e, \qquad e \in E(V)$$

is an isomorphism of $E_{i\lambda}$ onto the subsemilattice $\omega_{\lambda i}\omega_{\lambda i}^{-1}E(V)$ of $E(V)$. Since $\omega_{\lambda i}\omega_{\lambda i}^{-1}E(V)$ is itself a retract ideal of $E(V)$, the retract ideals of $\omega_{\lambda i}\omega_{\lambda i}^{-1}E(V)$ are precisely the retract ideals of $E(V)$ which are contained in $\omega_{\lambda i}\omega_{\lambda i}^{-1}E(V)$, that is, the retract ideals which are of the form $\varepsilon E(V)$, where $\varepsilon \leq \omega_{\lambda i}\omega_{\lambda i}^{-1}$ in $\Omega(V)$. Therefore F is a retract ideal of $E_{i\lambda}$ if and only if

$$F = \{(i, \omega_{\lambda i}^{-1}\varepsilon e, \lambda) \mid e \in E(V)\}$$

for some $\varepsilon \leq \omega_{\lambda i}\omega_{\lambda i}^{-1}$. Since $\varepsilon \leq \omega_{\lambda i}\omega_{\lambda i}^{-1}$ if and only if $\varepsilon = \gamma\gamma^{-1}$ for some $\gamma \leq \omega_{\lambda i}$ in $\Omega(V)$, it follows that F is a retract ideal of $E_{i\lambda}$ if and only if

$$F = \{(i, \gamma^{-1}e, \lambda) \mid e \in E(V)\}$$

for some $\gamma \leq \omega_{\lambda i}$ in $\Omega(V)$.

The other statement can be proved in a dual way. ■

Theorem 3.6 *For a straight locally inverse semigroup S the following are equivalent:*

- (i) *S can be represented as a regular Rees matrix semigroup $\mathcal{RM}(I, V, \Lambda; Q)$ over an inverse semigroup V where the entries of Q are in $\Omega(V)$,*
- (ii) *the maximal subsemilattices of S are idempotents of $T(S)$,*
- (iii) *the idempotents of $T(S)$ are the retract ideals of the maximal subsemilattices of S,*
- (iv) *for every $(i, \lambda) \in I \times \Lambda$, the largest regular [inverse] subsemigroup of $T(S)_{i\lambda}$ is given by $T(E_{i\lambda}SE_{i\lambda})$.*

Proof: The equivalence of the statements (i) and (ii) follows from Result 3.2 and Theorem 7 of [16]. (iii) implies (ii) trivially.

Assume that (i) holds. We may as well assume that $S = \mathcal{RM}(I, V, \Lambda; Q)$ is as in (i), with $Q = (\omega_{\lambda i})$. Let F be any retract ideal of a maximal subsemilattice of S and $(j, a, \mu) \in S$. By Lemma 3.5,

$$F = \{(i, \gamma^{-1}e, \lambda) \mid e \in E(V)\}$$

for some $(i, \lambda) \in I \times \Lambda$ and for some $\gamma \leq \omega_{\lambda i}$. Then $(i, \gamma^{-1}\omega_{\lambda j}aa^{-1}\omega_{\lambda j}^{-1}, \lambda) \in F$ and

$$(i, \gamma^{-1}\omega_{\lambda j}aa^{-1}\omega_{\lambda j}^{-1}, \lambda)(j, a, \mu) = (i, \gamma^{-1}\omega_{\lambda j}a, \mu)$$

and so $(i, \gamma^{-1}\omega_{\lambda j}a, \mu)\tau_S \subseteq F(j, a, \mu)$. Let $(i, \gamma^{-1}e, \lambda) \in F$ for some $e \in E(V)$. Then

$$\begin{aligned}(i, \gamma^{-1}e, \lambda)(j, a, \mu) &= (i, \gamma^{-1}e\omega_{\lambda j}a, \mu)\\ &= (i, \gamma^{-1}e\omega_{\lambda j}\omega_{\lambda j}^{-1}\omega_{\lambda j}aa^{-1}a, \mu)\\ &= (i, \gamma^{-1}e\omega_{\lambda j}aa^{-1}\omega_{\lambda j}^{-1}\omega_{\lambda j}a, \mu)\\ &= (i, \gamma^{-1}e\omega_{\lambda j}aa^{-1}\omega_{\lambda j}^{-1}, \lambda)(j, a, \mu)\\ &\leq (i, \gamma^{-1}\omega_{\lambda j}aa^{-1}\omega_{\lambda j}^{-1}, \lambda)(j, a, \mu)\\ &= (i, \gamma^{-1}\omega_{\lambda j}a, \mu).\end{aligned}$$

Therefore $F(j, a, \mu) = (i, \gamma\omega_{\lambda j}a, \mu)\tau_S \in S\tau_S$, whence $F((j, a, \mu)\tau_S) \in S\tau_S$ by Proposition 2.4 of [13]. From this and its dual it follows that F is in the idealizer of $S\tau_S$ in $O(S)$. By Result 2.1(ii) F is an idempotent of $O(S)$. Therefore F is an idempotent of $T(S)$. In view of Lemma 3.3(iii) we may now conclude that (iii) holds true. We proved that the statements (i), (ii) and (iii) are equivalent.

Assume that (iii) is satisfied. Then for $(i, \lambda) \in I \times \Lambda$, $E(T(S)_{i\lambda}) = T(E_{i\lambda})$ and so by the remarks preceding Lemma 3.1, $T(E_{i\lambda})T(S)T(E_{i\lambda})$ is the largest regular [inverse] subsemigroup of $T(S)_{i\lambda}$. If

$$H \in T(E_{i\lambda})\,T(S)T(E_{i\lambda}),$$

then $H \in O(E_{i\lambda}SE_{i\lambda})$ and since H is in the idealizer of $S\tau_S$ in $O(S)$, so H is also in the idealizer of $(E_{i\lambda}SE_{i\lambda})\tau_S$ in $O(E_{i\lambda}SE_{i\lambda})$. The same holds true for the inverse H^{-1} of H in the inverse semigroup $T(E_{i\lambda})T(S)T(E_{i\lambda})$. Therefore $H \in T(E_{i\lambda}SE_{i\lambda})$. If conversely $H \in T(E_{i\lambda}SE_{i\lambda})$, we may take the inverse H^{-1} of H in the inverse semigroup $T(E_{i\lambda}SE_{i\lambda})$ and obtain pairwise inverse threads H and H^{-1} of S. The idempotents HH^{-1} and $H^{-1}H$ are retract ideals of $E_{i\lambda}$ and so HH^{-1} and $H^{-1}H$ are idempotents of $T(S)$ by (iii). Therefore $H, H^{-1} \in T(S)$ by Proposition 2.10(i) of [14]. Obviously then,

$$H = (HH^{-1})H(H^{-1}H) \in T(E_{i\lambda})T(S)T(E_{i\lambda}).$$

Thus $T(E_{i\lambda}SE_{i\lambda}) = T(E_{i\lambda})T(S)T(E_{i\lambda})$. We proved that ($iii$) implies ($iv$).

Assume that (iv) holds. Then for $(i, \lambda) \in I \times \Lambda$, $E_{i\lambda} \in T(E_{i\lambda}SE_{i\lambda}) \subseteq T(S)$, and we conclude that ($ii$) holds. ∎

Example: Let L be the four element semilattice $\{a, b, c, d\}$ with a and b incomparable and d the identity element. Let $R = \{1, 2\}$ be a two-element right zero semigroup and S be the right normal band $(L \times R) \setminus \{(d, 1)\}$. Then S is a straight locally inverse semigroup which does not satisfy the equivalent conditions of Theorem 3.6.

If S satisfies the equivalent conditions of Theorem 3.6 then each maximal subsemilattice $T(E_{i\lambda})$ of the straight locally inverse semigroup $T(S)$ contains a greatest element $E_{i\lambda}$. Therefore by Theorem 10 of [16], $T(S)$ can be represented as a regular Rees matrix semigroup over an inverse semigroup where the entries of the sandwich matrix belong to this inverse semigroup. We set out to find such a representation.

Lemma 3.7 *Let $S = \mathcal{RM}(I, V, \Lambda; Q)$ be as in Result 3.4. For $i, j \in I$ and $\lambda, \mu \in \Lambda$, let $H \in C(V)$ be such that $H \subseteq \omega_{\lambda j}^{-1}\omega_{\lambda j}V\omega_{\mu i}\omega_{\mu i}^{-1}$. Then the sets*

$$K = \{(j, v, \mu) \mid v \in H\} \tag{8}$$

and

$$K' = \{(i, \omega_{\mu i}^{-1}v^{-1}\omega_{\lambda j}^{-1}, \lambda) \mid v \in H\} \tag{9}$$

are pairwise inverse threads of S. Conversely, any pairwise inverse threads of S can be so obtained.

Proof: Let H, K and K' be as stated above. Let $a = (j, v, \mu) \in K$ for some $v \in H$ and consider $a' = (i, \omega_{\mu i}^{-1}v^{-1}\omega_{\lambda j}^{-1}, \lambda) \in K'$. One readily verifies that a and a' are pairwise inverse elements of S. Consequently $K \subseteq KK'K$ and $K' \subseteq K'KK'$.

If $b = (j, w, \mu) \in K$ for some $w \in H$, then $a'b = (i, \omega_{\mu i}^{-1}v^{-1}w, \mu) \in E_{i\mu}$, where $v^{-1}w \in E(V)$ since $H \in C(V)$. It follows that $K'K \subseteq E_{i\mu}$ and consequently also $K' \subseteq E_{i\mu}K'$. If $g = (i, \omega_{\mu i}^{-1}e, \mu) \in E_{i\mu}$ for some $e \in E(V)$, then $ga' = (i, \omega_{\mu i}^{-1}ev^{-1}\omega_{\lambda j}^{-1}, \lambda)$ since $\omega_{\mu i}\omega_{\mu i}^{-1}v^{-1} = v^{-1}$. Here $ga' \in K'$ because $ev^{-1} \in H^{-1}$ since H^{-1} is an order ideal. We proved that $K' = E_{i\mu}K'$. If $c' \leq a'$ in S, then $c' = ha'$ for some $h \leq a'a \in E_{i\mu}$, whence $c' \in E_{i\mu}K' = K'$. We proved that K' is an order ideal of S. From $K' \subseteq K'KK' \subseteq E_{i\mu}K' = K'$ we also have that $K' = K'KK'$. By duality we find that $KK' \subseteq E_{j\lambda}$.

We have that $K \subseteq KK'K \subseteq KE_{i\mu}$. For $a = (j, v, \mu) \in K$ and $g = (i, \omega_{\mu i}^{-1}e, \mu) \in E_{i\mu}$ we have $ag = (j, ve, \mu) \in K$ since $H \in C(V)$. If follows that $K = KE_{i\mu}$ and therefore also that $K = KK'K$. It is now easy to prove that K is an order ideal of S. Consequently KK' and $K'K$ are order ideals of the maximal subsemilattices $E_{j\lambda}$ and $E_{i\mu}$ of S, respectively. We proved that K and K' are pairwise inverse threads.

To prove the converse, assume that K and K' are pairwise inverse threads of S. By Lemma 3.1(ii), $K \subseteq S_{j\mu}$ and $K' \in S_{i\lambda}$ for some $i, j \in I$ and some $\lambda, \mu \in \Lambda$, and then $KK' \subseteq E_{j\lambda}$ and $K'K \subseteq E_{i\mu}$. By Result 2.1(iii) there exists for every $a \in K$ a unique inverse a' of a in K'. One readily verifies that such pairwise inverse elements of S must be of the form $a = (j, v, \mu)$ and $a' = (i, \omega_{\mu i}^{-1}v^{-1}\omega_{\lambda j}^{-1}, \lambda)$ for some $v \in \omega_{\lambda j}^{-1}\omega_{\lambda j}V\omega_{\mu i}\omega_{\mu i}^{-1}$. We let H consist of the elements $v \in \omega_{\lambda j}^{-1}\omega_{\lambda j}V\omega_{\mu i}\omega_{\mu i}^{-1}$ such that $(j, v, \mu) \in K$. In view of Result 2.1(iii) we then have

$$K = \{(j, v, \mu) \mid v \in H\}$$

and

$$K' = \{(i, \omega_{\mu i}^{-1}v^{-1}\omega_{\lambda j}^{-1}, \lambda) \mid v \in H\},$$

where $H \subseteq \omega_{\lambda j}^{-1}\omega_{\lambda j}V\omega_{\mu i}\omega_{\mu i}^{-1}$. It remains to prove that $H \subseteq C(V)$.

Let $v \in H$ and $e \le vv^{-1}$ in V. Then $ev \in \omega_{\lambda j}^{-1}\omega_{\lambda j}V\omega_{\mu i}\omega_{\mu i}^{-1}$. Since

$$(j, e\omega_{\lambda j}^{-1}, \lambda) \le (j, vv^{-1}\omega_{\lambda j}^{-1}, \lambda) = (j, v, \mu)(i, \omega_{\mu i}^{-1}v^{-1}\omega_{\lambda j}^{-1}, \lambda) \in KK'$$

it follows that $(j, e\omega_{\lambda j}^{-1}, \lambda) \in KK'$ and therefore that

$$(j, ev, \mu) = (j, e\omega_{\lambda j}^{-1}, \lambda)(j, v, \mu) \in KK'K = K.$$

Thus $ev \in H$, and so H is an order ideal of V.

Let $v, w \in H$. Then $(j, v, \mu) \in K$ and $(i, \omega_{\mu i}^{-1}w^{-1}\omega_{\lambda j}^{-1}, \lambda) \in K'$, and so

$$(j, vw^{-1}\omega_{\lambda j}^{-1}, \lambda) = (j, v, \mu)(i, \omega_{\mu i}^{-1}w^{-1}\omega_{\lambda j}^{-1}, \lambda)$$

belongs to $KK' \subseteq E_{j\lambda}$. Therefore $vw^{-1}\omega_{\lambda j}^{-1} = e\omega_{\lambda j}^{-1}$ for some $e \in E(V)$. Consequently $vw^{-1} = vw^{-1}\omega_{\lambda j}^{-1}\omega_{\lambda j} = e\omega_{\lambda j}^{-1}\omega_{\lambda j} \in E(V)$. From this and its dual follows that HH^{-1}, $H^{-1}H \subseteq E(V)$. Therefore $H \in C(V)$. ■

Lemma 3.8 *Let $S = \mathcal{RM}(I, V, \Lambda; Q)$ be as in Result 3.4. For $i, j \in I$ and $\lambda, \mu \in \Lambda$, let $H \in T(V)$ such that $H \subseteq \omega_{\lambda j}^{-1}\omega_{\lambda j}V\omega_{\mu i}\omega_{\mu i}^{-1}$. Then the sets K and K' which are given by (8) and (9) are pairwise inverse elements of $T(S)$. Conversely, any pairwise inverse elements of $T(S)$ can be so obtained.*

Proof: By Proposition 2.10.(i) of [14] and Lemma 3.7, K and K' are pairwise inverse elements of $T(S)$ if and only if K and K' are given by (8) and (9) and moreover, KK' and $K'K$ are idempotents of $T(S)$. In view of Theorem 3.6 and duality it thus suffices to prove that if H, K and K' are as in the statement of Lemma 3.7, then $K'K$ is a retract ideal of $E_{i\mu}$ if and only if $H^{-1}H$ is a retract ideal of $E(V)$.

One verifies that

$$K'K = \{(i, \omega_{\mu i}^{-1}e, \mu) \mid e \in H^{-1}H\} \subseteq E_{i\mu}$$

and that

$$E_{i\mu} \longrightarrow \omega_{\mu i}\omega_{\mu i}^{-1}E(V),\ (i, \omega_{\mu i}^{-1}f, \mu) \longrightarrow \omega_{\mu i}\omega_{\mu i}^{-1}f, \qquad f \in E(V),$$

is an isomorphism which maps $K'K$ onto $H^{-1}H$. Since $\omega_{\mu i}\omega_{\mu i}^{-1}E(V)$ is itself a retract ideal of $E(V)$ it follows that $H^{-1}H$ is a retract ideal of $E(V)$ if and only if $H^{-1}H$ is a retract ideal of $\omega_{\mu i}\omega_{\mu i}^{-1}E(V)$, that is, if and only if $K'K$ is a retract ideal of $E_{i\mu}$. ■

For a regular Rees matrix semigroup $S = \mathcal{RM}(I, V, \Lambda; Q)$ with $Q = (\omega_{\lambda i})$ where the $\omega_{\lambda i}$ are in the translational hull $\Omega(V)$ of the inverse semigroup V we put

$$H_{\lambda i} = E(V)\omega_{\lambda i} = \omega_{\lambda i}E(V), \quad P = (H_{\lambda i}). \tag{10}$$

Then for all $(i, \lambda) \in I \times \Lambda$, $H_{\lambda i} \in T(V) \subseteq C(V)$ and $\mathcal{RM}(I, T(V), \Lambda; P)$ and $\mathcal{RM}(I, C(V), \Lambda; P)$ are regular Rees matrix semigroups over the inverse semigroups $T(V)$ and $C(V)$, respectively.

We arrive at the main theorem of this section.

Theorem 3.9 *Let $S = \mathcal{RM}(I, V, \Lambda; Q)$ be a regular Rees matrix semigroup over the inverse semigroup V with entries $\omega_{\lambda i}$ of Q in $\Omega(V)$. Then with the notation (10), the mappings*

$$\varphi : C(S) \longrightarrow \mathcal{RM}(I, C(V), \Lambda; P),\ \{j\} \times H \times \{\mu\} \longrightarrow (j, H, \mu)$$

and

$$\psi : T(S) \longrightarrow \mathcal{RM}(I, T(V), \Lambda; P),\ \{j\} \times H \times \{\mu\} \longrightarrow (j, H, \mu)$$

are isomorphisms.

Proof: By Lemma 3.7, $\{j\} \times H \times \{\mu\} \in C(S)$ if and only if $H \in C(V)$ with $H = \omega_{\lambda j}^{-1}\omega_{\lambda j}H\omega_{\mu i}\omega_{\mu i}^{-1}$ for some $i \in I$ and $\lambda \in \Lambda$. By Result 3.4, $(i, H, \lambda) \in \mathcal{RM}(I, C(V), \Lambda; P)$ if and only if $H = H_{\lambda j}^{-1}H_{\lambda j}HH_{\mu i}H_{\mu i}^{-1}$ for some $i \in I$ and $\lambda \in \Lambda$. Since

$$\omega_{\lambda j}^{-1}\omega_{\lambda j}H\omega_{\mu i}\omega_{\mu i}^{-1} = H_{\lambda j}^{-1}H_{\lambda j}HH_{\mu i}H_{\mu i}^{-1}$$

it follows that the mapping φ is a bijection.

Let $K_1, K_2 \in C(S)$ with $K_1 = \{i\} \times H_1 \times \{\lambda\}$ and $K_2 = \{j\} \times H_2 \times \{\mu\}$ for some $H_1, H_2 \in C(V)$. Then

$$K_1K_2 = \{(i, v_1\omega_{\lambda j}v_2, \mu) \mid v_1 \in H_1,\ v_2 \in H_2\}$$

$$(K_1\varphi)(K_2\varphi) = (i, H_1, \lambda)(j, H_2, \mu) = (i, H_1H_{\lambda j}H_2, \mu)$$

where

$$H_1H_{\lambda j}H_2 = H_1\omega_{\lambda j}H_2 = \{v_1\omega_{\lambda j}v_2 \mid v_1 \in H_1,\ v_2 \in H_2\}.$$

Therefore φ is an isomorphism.

Using Lemma 3.8 one shows in a similar way that ψ is an isomorphism. ■

Corollary 3.10 *Let $S = \mathcal{RM}(I, V, \Lambda; Q)$ be a regular Rees matrix semigroup over the inverse semigroup V with entries with entries of Q in $\Omega(V)$. Then with the notation (10),*

$$S \longrightarrow \mathcal{RM}(I, T(V), \Lambda; P),\ (i, v, \lambda) \longrightarrow (i, (v], \lambda)$$

and

$$S \longrightarrow \mathcal{RM}(I, \Omega(V), \Lambda; P),\ (i, v, \lambda) \longrightarrow (i, \pi_v, \lambda)$$

are each a maximal dense ideal extension of S within the class of all locally inverse semigroups.

A straight locally inverse semigroup S is called a **perfect rectangular band of inverse semigroups** if, with the notation (1)-(5), each $S_{i\lambda}$ is an inverse semigroup and, $S_{i\lambda}S_{j\mu} = S_{i\mu}$ for all $i, j \in I$ and $\lambda, \mu \in \Lambda$. From [16] it follows that this will be the case if and only if S has a representation of the form $\mathcal{RM}(I, V, \Lambda; Q)$ where the entries of Q are in the group of units of $\Omega(V)$. By Theorem 3.9 it then follows that $T(S)$ is again a prefect rectangular band of inverse semigroups. It is easy to show that an ideal of a perfect rectangular band of inverse semigroups is again a perfect rectangular band of inverse semigroups. We thus have

Corollary 3.11 *A locally inverse semigroup S is a perfect rectangular band of inverse semigroups if and only if the same is true for $T(S)$.*

4 A Rees matrix cover

In this section we return to general locally inverse semigroups. For every locally inverse semigroup S we realize $T(S) \cong \Omega_{\mathbf{LI}}(S)$ as a homomorphic image of a canonically constructed regular Rees matrix semigroup over an inverse semigroup.

Let S be a locally inverse semigroup. We let $U(S)$ be the set consisting of the triples of the form (E, s, F) where $E, F \in E(T(S))$ and $s \in S$ is such that $e\mathcal{R}s\mathcal{L}f$ for some $e \in E$ and $f \in F$. On $U(S)$ we define a multiplication by: for $(E_1, s_1, F_1), (E_2, s_2, F_2) \in U(S)$, put $(E_1, s_1, F_1)(E_2, s_2, F_2) = (E_1, s_1s_2, F_2)$. We verify that this multiplication is well-defined. If $e_1\mathcal{R}s_1$ for some $e_1 \in E_1$, then since $R_{s_1s_2} \leq R_{s_1} = R_{e_1}$ there exists a unique $g_1 \in R_{s_1s_2}$ such that $g_1 \leq e_1$ and then $g_1 \in E_1$ since E_1 is an order ideal of S. Therefore $g_1\mathcal{R}s_1s_2$ for some $g_1 \in E_1$. In a dual way one finds that $s_1s_2\mathcal{L}g_2$ for some $g_2 \in F_2$.

Let $(E, s, F) \in U(S)$ such that $e\mathcal{R}s\mathcal{L}f$ for some $e \in E$ and $f \in F$. There exists an inverse s' of s in S such that $e = ss'$ and $f = s's$ and one verifies that $((f], s', (e])$ is an inverse of (E, s, F) in $U(S)$. Thus $U(S)$ is a regular semigroup which is also a subdirect product of the rectangular band $E(T(S))\times E(T(S))$ and the semigroup S. Therefore $U(S)$ is a locally inverse semigroup and incidentally, the existence variety of locally inverse semigroups (in the sense of [1], [2], [5], [7]) generated by $U(S)$ coincides with the existence variety generated by S as long as the latter contains all rectangular bands.

With the notation of Proposition 2.4 we have

Lemma 4.1 *For $E, F \in E(T(S))$, $\{E\} \times (E \wedge F) \times \{F\}$ is a maximal subsemilattice of $U(S)$, and conversely, every maximal subsemilattice of $U(S)$ is of this form.*

Proof: Let L be a maximal subsemilattice of $U(S)$. Since the projection

$$U(S) \longrightarrow E(T(S)) \times E(T(S))$$

induces a rectangular band congruence, there exist $E, F \in E(T(S))$ such that L consists of idempotents of the form (E, g, F). Therefore by Proposition 2.4,

$L \subseteq \{E\} \times (E \wedge F) \times \{F\}$. Again by Proposition 2.4, $E \wedge F \in E(T(S))$ and therefore $E \wedge F$ is a subsemilattice of S. It follows that $\{E\} \times (E \wedge F) \times \{F\}$ is a maximal subsemilattice of $U(S)$. ■

Corollary 4.2 *$U(S)$ is a straight locally inverse semigroup.*

Lemma 4.3 *The locally inverse semigroup $U(S)$ can be represented as a regular Rees matrix semigroup $\mathcal{RM}(I, V, \Lambda; Q)$ over an inverse semigroup V where the entries of Q are in $\Omega(V)$ and $I = \Lambda = E(T(S))$.*

Proof: In view of Theorem 3.6 it suffices to verify whether the maximal subsemilattices of $U(S)$ belong to $T(U(S))$. Let $E, F, H, K \in E(T(S))$ and $g \in E(S)$ such that $(H, g, K) \in U(S)$. By Proposition 2.4, $E \wedge F \in E(T(S))$ and so by Lemma 3.6 of [13] there exists $k \in E(S)$ such that

$$\{h \wedge g \mid h \in E \wedge F\} = (k].$$

From this we infer that

$$\{(E, h, F) \wedge (H, g, K) \mid h \in E \wedge F\} = ((E, k, K)]$$

in $E(U(S))$. Using Lemma 3.6 of [13] and duality we conclude that the maximal subsemilattice $\{E\} \times (E \wedge F) \times \{F\}$ of $U(S)$ belongs to $T(U(S))$ as required. ■

Lemma 4.4 *Let K and K' be pairwise inverse elements of $T(S)$ and $E, F \in E(T(S))$ such that $KK' \subseteq E$ and $K'K \subseteq F$. Then $\{E\} \times K \times \{F\} \in T(U(S))$ and conversely, every element of $T(U(S))$ can be so obtained.*

Proof: Let K, K', E and F be as stated. Then $K, K' \in O(S)$, from which we derive that $\{E\} \times K \times \{F\}$, $\{F\} \times K' \times \{E\}$ are pairwise inverse elements of $O(U(S))$. Since

$$(\{E\} \times K \times \{F\})(\{F\} \times K' \times \{E\}) = \{E\} \times KK' \times \{E\}$$

and

$$(\{F\} \times K' \times \{E\})(\{E\} \times K \times \{F\}) = \{F\} \times K'K \times \{F\}$$

are subsemilattices of $U(S)$ it follows that these two elements of $O(U(S))$ are pairwise inverse threads of $U(S)$. By Lemma 3.6 of [13], KK' is a retract ideal of E and so $\{E\} \times KK' \times \{E\}$ is a retract ideal of the maximal subsemilattice $\{E\} \times E \times \{E\}$ of $U(S)$. From Theorem 3.6 and Lemma 4.3 it follows that $\{E\} \times KK' \times \{E\}$ is an idempotent of $T(U(S))$. By symmetry, $\{F\} \times K'K \times \{F\}$ is an idempotent of $T(U(S))$. From Theorem 2.2 and Lemma 3.1 it follows that $\{E\} \times K \times \{F\}$ and $\{F\} \times K' \times \{E\}$ are pairwise inverse elements of $T(U(S))$.

By Lemma 3.3(ii) and Corollary 4.2 every element of $T(U(S))$ is of the form $\{E\} \times K \times \{F\}$ for some appropriate subset K of $\{k \in S \mid (E, k, F) \in U(S)\}$. Since the projection α of $U(S)$ onto S is a surjective homomorphism it follows from Corollary 4.4 of [13] that $K = (\{E\} \times K \times \{F\})\alpha \in T(S)$. ■

Corollary 4.5 *Let α be the projection of $U(S)$ onto S. Then $\tau_{U(S)}^{-1}\alpha\tau_S$ can be extended in a unique way to a homomorphism $\overline{\alpha} : T(U(S)) \longrightarrow T(S)$. Moreover, $\overline{\alpha}$ is surjective.*

Proof: From Corollary 4.4 of [13] and Lemma 4.4 we have that

$$\overline{\alpha} : T(U(S)) \longrightarrow T(S),\ \{E\} \times K \times \{F\} \longrightarrow K$$

is the unique homomorphism which extends $\tau_{U(S)}^{-1}\alpha\tau_S$. It follows from Lemma 4.4 that $\overline{\alpha}$ is surjective. ∎

We are ready for the main theorem. In view of the fact that $\pi_S : S \longrightarrow \Omega_{\mathbf{LI}}(S)$ and $\tau_S : S \longrightarrow T(S)$ are equivalent ideal extensions and the alternatives offered in the statement of Corollary 3.10, there are several variant statements possible. We make one choice.

In the statement of the following theorem α will be the projection of $U(S)$ onto S, $\varphi : U(S) \longrightarrow \mathcal{RM}(I, V, \Lambda; Q)$ a representation of $U(S)$ as in Lemma 4.3 and $\pi : \mathcal{RM}(I, V, \Lambda; Q) \longrightarrow \mathcal{RM}(I, \Omega(V), \Lambda; Q)$ the ideal extension as in Corollary 3.10.

Theorem 4.6 *Let S be a locally inverse semigroup. Then there exists a unique homomorphism $\psi : \mathcal{RM}(I, \Omega(V), \Lambda; Q) \longrightarrow \Omega_{\mathbf{LI}}(S)$ which makes the diagram*

$$\begin{array}{ccc} \mathcal{RM}(I, \Omega(V), \Lambda; Q) & \xrightarrow{\psi} & \Omega_{\mathbf{LI}}(S) \\ {\scriptstyle\pi}\Big\uparrow & & \Big\uparrow{\scriptstyle\pi_S} \\ \mathcal{RM}(I, V, \Lambda; Q) & \xrightarrow[\varphi^{-1}\alpha]{} & S \end{array}$$

commutative. Moreover, ψ is surjective.

Proof: The proof follows immediately from Corollary 4.5 since on the one hand $\varphi\pi : U(S) \longrightarrow \mathcal{RM}(I, \Omega(V), \Lambda; Q)$ and $\tau_{U(S)} : U(S) \longrightarrow T(U(S))$ are equivalent dense ideal extensions, and on the other hand $\tau_S : S \longrightarrow T(S)$ and $\pi_S : S \longrightarrow \Omega_{\mathbf{LI}}(S)$ are equivalent dense ideal extensions. ∎

The surjective homomorphism $\varphi^{-1}\alpha$ induces an isomorphism on every maximal inverse subsemigroup of $\mathcal{RM}(I, V, \Lambda; Q)$ and is therefore called a **local isomorphism**. A similar statement can be made for ψ and we leave the proof of this nontrivial fact to the reader. The Rees matrix semigroups mentioned in Theorem 4.6 are called **Rees matrix covers** for S and $\Omega_{\mathbf{LI}}(S)$. Rees matrix covers for locally inverse semigroups have been constructed before (see e.g. [9], [16]).

References

[1] K. Auinger, On the lattice of existence varieties of locally inverse semigroups, *Canad. Math. Bull.* **37** (1994), 13–20.

[2] K. Auinger, On existence varieties of regular semigroups, in: Gomes, Gracinda M. S. (ed.) et al., *Semigroups, Algorithms, Automata and Languages*. Proceedings of workshops held at the International Centre of Mathematics, CIM, Coimbra, Portugal, May, June and July 2001. Singapore, World Scientific (2002); 65–81.

[3] K. Byleen, Regular four-spiral semigroups, idempotent-generated semigroups and the Rees construction, *Semigroup Forum* **22** (1981), 97–100.

[4] K. Byleen, J. Meakin and F. Pastijn, The fundamental four-spiral semigroup, *J. Algebra* **54** (1978), 6–26.

[5] T. E. Hall, Identities for existence varieties of regular semigroups, *Bull. Austral. Math. Soc.* **40** (1989), 59–77.

[6] J. M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, London, 1995.

[7] P. R. Jones, An introduction to existence varieties of regular semigroups, *Southeast Asian Bull. Math.* **19** (1995), 107–118.

[8] D. B. McAlister, Regular Rees matrix semigroups and regular Dubreil-Jacotin semigroups, *J. Austral. Math. Soc. Ser. A* **31** (1981), 325–336.

[9] D. B. McAlister, Rees matrix covers for locally inverse semigroups, *Trans. Amer. Math. Soc.* **277** (1983), 727–737.

[10] K. S. S. Nambooripad, Pseudo-semilattices and biordered sets I, *Simon Stevin* **55** (1981), 103–110; Pseudo-semilattices and biordered sets II, *Simon Stevin* **56** (1982), 143–159.

[11] F. J. Pastijn, Embedding semigroups in semibands, *Semigroups Forum* **14** (1977), 247–263.

[12] F. J. Pastijn, The structure of pseudo-inverse semigroups, *Trans. Amer. Math. Soc.* **273** (1982), 631–655.

[13] F. J. Pastijn and L. Oliveira, Maximal dense ideal extensions of locally inverse semigroups, preprint.

[14] F. J. Pastijn and L. Oliveira, Ideal extensions of locally inverse semigroups, preprint.

[15] F. J. Pastijn and M. Petrich, straight locally inverse semigroups, *Proc. London Math. Soc.* (3) **49** (1984), 307–328.

[16] F. J. Pastijn and M. Petrich, Rees matrix semigroups over inverse semigroups, *Proc. Roy. Soc. Edinburgh Sect. A* **102** (1986), 61–90.

[17] M. Petrich, On ideals of semilattices, *Czechoslovak Math. J.* **22** (1972), 361–367.

[18] M. Petrich, *Introduction to Semigroups*, Merrill, Columbus, 1973.

[19] M. Petrich, *Inverse Semigroups*, Wiley, New York, 1984.

[20] M. Petrich, The translational hull of a normal cryptogroup, *Math. Slovaca* **44** (1994), 245–262.

[21] M. Petrich and N. R. Reilly, *Completely Regular Semigroups*, Wiley, New York, 1999.

[22] B. M. Schein, Completions, translational hulls and ideal extensions of inverse semigroups, *Czechoslovak Math. J.* **23** (98) (1973), 575–610.

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201-1881, USA
E-mail address: francisp@mscs.mu.edu

Departamento de Matemática Pura, Faculdade de Ciências da Universidade do Porto, R. Campo Alegre 687, 4169-007 Porto, Portugal
E-mail address: loliveir@fc.up.pt