Invariants, Equivariants and Characters in Symmetric Bifurcation Theory

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In the analysis of stability in bifurcation problems it is often assumed that the (appropriate reduced) equations are in normal form. In the presence of symmetry, the truncated normal form is an equivariant polynomial map. Therefore, the determination of invariants and equivariants of the group of symmetries of the problem is an important step. In general, these are hard problems of invariant theory, and in most cases, they are tractable only through symbolic computer programs. Nevertheless, it is desirable to obtain some of the information about invariants and equivariants without actually computing them, for example, the number of linearly independent homogeneous invariants or equivariants of a certain degree. Generating functions for these dimensions are generally known as "Molien functions".

In this work we obtain formulas for the number of linearly independent homogeneous invariants or equivariants for Hopf bifurcation in terms of characters. We also show how to construct Molien functions for invariants and equivariants for Hopf bifurcation. Our results are then applied to the computation of the number of invariants and equivariants for Hopf bifurcation for several finite groups and the continuous group $\mathbf{O}(3)$.

1. Introduction

Symmetry appears naturally in several important physical models and in many cases the collection of all the symmetries of the problem forms a compact Lie group. Moreover, there is a fully symmetric solution that loses stability as a parameter is varied, and this loss of stability is due to the crossing of eigenvalues through the imaginary axis. When the eigenvalues are zero a steady-state bifurcation is expected to happen – that is, a bifurcation from the group-invariant equilibrium to equilibria with less symmetry. When the eigenvalues are imaginary, the bifurcation expected is a Hopf bifurcation to periodic solutions. A Lyapunov-Schmidt or center-manifold reduction reduces the bifurcation problem to equations on the sum of the generalised eigenspaces of these eigenvalues. Moreover, the generalised center subspace is invariant under the action of the symmetry group and the Lyapunov-Schmidt reduction or the center-manifold reduction can be performed in such a way that the reduced equations commute with the restricted action of the symmetry group. See for example Golubitsky and Schaeffer [14] and Carr [3]; see also the texts [5, 15, 16, 27] on symmetric bifurcation theory.

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We are led to consider the following situation. Let a compact Lie group G act linearly on $\mathbf{R}^n$ and let

$$\dot{x} \;=\; F(x,\lambda) \qquad \text{with} \qquad F(0,0) \;=\; 0 \tag{1.1}$$

be a G-equivariant bifurcation problem on $\mathbf{R}^n$. That is, $F : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is a family of smooth maps satisfying

$$F(g\,v,\lambda) \;=\; g\,F(v,\lambda)$$

for all $g \in G$ and the Jacobian matrix $J = \mathrm{d}F_{(0,0)}$ has only zero or purely imaginary eigenvalues.

The simplest case is when we have a one-parameter family in equation (1.1). In codimension-one steady-state bifurcation, the kernel of the Jacobian at the group-invariant solution is left invariant by the group of symmetries and generically the restriction of its action to the kernel is absolutely irreducible. See Golubitsky *et al.* [16, Proposition XIII 3.2]. For codimension-one Hopf bifurcation, the imaginary eigenspace of the Jacobian at the group-invariant solution is left invariant by the group of symmetries and generically the restriction of its action to the center subspace is simple – the sum of two absolutely irreducible representations, or irreducible but not absolutely irreducible. See [16, Proposition XVI 1.4]. In this case, there is a natural action of the circle group $\mathbf{S}^1$ on the center subspace that commutes with the action of G. See [16, Lemma XVI 3.2]. Then one is naturally led to consider the representation theory of $G \times \mathbf{S}^1$.

Because of the multiplicity of the eigenvalues, the linearised problem is highly degenerate – there is no preferred direction within the eigenspace. This degeneracy is partially resolved by the nonlinear terms, which are constrained by the symmetry; terms which respect the symmetry are said to be equivariant. In a particular problem, one can specify the action of the group on the center eigenspace and construct equivariant polynomials of a given degree. These equivariant polynomials are generally called “truncated normal forms” and are a fundamental tool in the study of the structure of the local bifurcations, e.g., existence, growth and stability of branches.

The notion of normal form of a vector field near a singularity is relatively old: it was developed for the purpose of simplifying Hamiltonian vector fields as occurring in celestial mechanics. The general definition of normal form for an arbitrary vector field near an equilibrium point was proposed by Takens [28] in 1974 and already used for the analysis of bifurcation phenomena. The intrinsic characterisation of a normal form of Elphick *et al.* [8] (see also [16, p. 284]) roughly says that a general polynomial vector field (of degree k) that commutes with the action of a certain group on the center eigenspace is a “generic vector field in normal form”.

In order to construct a generic vector field in normal form for a given group representation on a vector space V, one needs to know the invariant theory for that particular action. For a fixed (but arbitrary) k one has to construct a basis of the space of equivariant homogeneous polynomial mappings of degree i for every $i \leqslant k$ and then write a general equivariant polynomial mapping of degree k as a linear combination of those basis elements with real coefficients. An important requisite to achieve this is to be able to find the number of linearly independent invariant homogeneous polynomial functions of a certain fixed degree and the number of linearly independent equivariant homogeneous polynomial mappings of a certain fixed degree.

Formulas for the number of invariants and equivariants are useful, because in a specific problem they can be used to confirm that all possible invariants or equivariants have been found. This knowledge is particularly important in algorithmic invariant theory, where the normal form (up to an arbitrary but fixed degree) is constructed by symbolic computation. In those applications, the formulas for the numbers of invariants and equivariants are used to check completeness of the computed basis of invariants or equivariants up to a certain degree. For more details on algorithmic invariant theory with applications in equivariant bifurcation theory, see Gatermann [12, 13].

Formulas for the number of possible equivariant terms, using only the characters (traces) of the representations, are known for the case of a stationary bifurcation, see Sattinger [27]. In this paper we obtain new character formulas for the case of Hopf bifurcation, that will be helpful for those working in the area of equivariant bifurcation theory.

The use of character formulas has a number of advantages over working with the matrices of the representation. The characters of a representation are unique, but the matrices themselves are not. Secondly, the characters of the irreducible representations of many finite groups are tabulated, and are much easier to work with than the matrices. Finally, calculation with characters is now a standard feature in some computer algebra packages (e.g., GAP [11]). Thus, using our character formulas we are able to find the numbers of invariants and equivariants for Hopf bifurcation quickly and easily.

Another result of this paper is the adaptation of the usual formalism of Hilbert-Poincaré series and their underlying Molien formulas to work properly in complex coordinates since it is well known that one may simplify notation and computations by using the complex coordinates. See for example Menck [23].

Structure of the Paper

In Section 2 we review the two most important approaches to the calculation of the number of invariants and equivariants – character formulas and Hilbert-Poincaré series – and state our new results for the case of Hopf bifurcation. In Section 3 we introduce representation theory and characters and set up the notation for the rest of the paper. Our main new character formulas for the numbers of invariants and equivariants for Hopf bifurcation are proved in Section 4. In Section 5 we prove our generalisation of the Molien theorem to the case of Hopf bifurcation. Finally, in the Section 6, we present some application of our formulas, obtaining several (old and new) results for the numbers of invariants and equivariants for finite groups and for the symmetry group of the sphere.

2. Statement of the Main Results

In what follows we will review the two approaches to the calculation of the number of invariants and equivariants and state our new results. In the first part, we consider the character formulas and in the second part, the Hilbert-Poincaré series.

2.1. Character Formulas for Invariants and Equivariants

Let V be a finite-dimensional vector space over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. Let us denote by $\mathcal{P}_V^k$ the vector space of all homogeneous polynomials of degree k and by $\vec{\mathcal{P}}_V^k$ the vector space of all homogeneous polynomial maps of degree k on V. Define

$$\mathcal{P}_V = \bigoplus_{k=0}^{\infty} \mathcal{P}_V^k \qquad \text{and} \qquad \vec{\mathcal{P}}_V = \bigoplus_{k=0}^{\infty} \vec{\mathcal{P}}_V^k .$$

Under the point-wise product $\mathcal{P}_V$ becomes a graded commutative algebra over $\mathbf{K}$ and since the product of a polynomial mapping by a polynomial function is again a polynomial mapping it follows that $\vec{\mathcal{P}}_V$ is a module over the ring of polynomial functions $\mathcal{P}_V$.

Now suppose that a compact Lie group G acts linearly on V. The homomorphism $G \to \mathrm{GL}(V)$ that sends g to the linear transformation corresponding to the action of g on V is a (real or complex) representation of G on V. Here $\mathrm{GL}(V)$ is the group of invertible linear transformations $V \to V$. Throughout we denote by g the linear transformation corresponding to the action of $g \in G$ on V and by abuse of language we also call V a representation of G.

A polynomial function $f : V \to \mathbf{K}$ is G-*invariant* if

$$f(g\,v) = f(v)$$

for all $g \in G$ and $v \in V$. The set of all G-invariant homogeneous polynomial functions of degree k on V is denoted by $\mathcal{P}_V^k(G)$ and it is a subspace of $\mathcal{P}_V^k$, since it is the set of fixed-points of a linear action. The space $\mathcal{P}_V(G)$ of G-invariant polynomial functions is a sub-algebra of the algebra of all polynomial functions $\mathcal{P}_V$ on V and $\mathcal{P}_V^k(G) = \mathcal{P}_V(G) \cap \mathcal{P}_V^k$.

Similarly, a polynomial mapping $F : V \to V$ is G-*equivariant* if

$$g\,F(v) = F(g\,v)$$

for all $g \in G$ and $v \in V$. The set of all G-equivariant homogeneous polynomial mappings of degree k on V is denoted by $\vec{\mathcal{P}}_V^k(G)$ and it is a subspace of $\vec{\mathcal{P}}_V^k$, since it is also the set of fixed-points of a linear action. The space of G-equivariant polynomial mappings from V to V is a module over the ring $\mathcal{P}_V(G)$ and $\vec{\mathcal{P}}_V^k(G) = \vec{\mathcal{P}}_V(G) \cap \vec{\mathcal{P}}_V^k$. We note that for every $k \geqslant 0$ the vector spaces $\mathcal{P}_V^k(G)$ and $\vec{\mathcal{P}}_V^k(G)$ are finite-dimensional since $\mathcal{P}_V^k$ and $\vec{\mathcal{P}}_V^k$ are finite-dimensional vector spaces.

Given a linear action of a compact Lie group G on a finite-dimensional vector space V, the corresponding *character* is the function $\chi : G \to \mathbf{K}$ given by

$$\chi(g) = \mathrm{tr}(g) \qquad \text{for all} \quad g \in G .$$

The action of G on V induces a natural action on $\mathcal{P}_V^k$ and the corresponding character is denoted by $\chi_{(k)}$.

The following theorem is a well known result in invariant theory. See for example, Sattinger [27, Theorem 5.10].

Theorem 2.1 (Sattinger). *Let G be a compact Lie group acting linearly on a vector space V with corresponding character χ. Then*

$$\begin{aligned} \dim \mathcal{P}_V^k(G) &= \int_G \chi_{(k)}(g)\, \mathrm{d}\mu_G(g) \\ \dim \vec{\mathcal{P}}_V^k(G) &= \int_G \chi_{(k)}(g)\chi(g)\, \mathrm{d}\mu_G(g) \end{aligned} \tag{2.1}$$

where $\mathrm{d}\mu_G(g)$ is the normalised invariant measure of G.

In bifurcation theory, all representations are real with the complex representations arising due to extra structure such as the complex structure induced by a circle action in the Hopf case. Specifically, when considering symmetric Hopf bifurcation there is a natural action of the circle group $\mathbf{S}^1$ on V that commutes with the action of G, which in turn, induces a complex structure on V such that $G \times \mathbf{S}^1$ acts by unitary matrices.

Now there are two different ways to apply Theorem 2.1 in the case of Hopf bifurcation. In the first form, we consider the unitary action of $G \times \mathbf{S}^1$ on V (as a complex vector space) and we obtain the dimensions of the vector spaces $\mathcal{P}_V^k(G\times\mathbf{S}^1)$ and $\vec{\mathcal{P}}_V^k(G\times\mathbf{S}^1)$, of the complex valued invariant polynomial functions on a complex vector space V and V-valued equivariant polynomial mappings on a complex vector space V, respectively. In the second form, we consider V as a real vector space, which we denote by $V^{\mathbf{R}}$, carrying an orthogonal representation of $G \times \mathbf{S}^1$ and then we obtain the dimensions of the spaces $\mathcal{P}_{V^{\mathbf{R}}}^k(G \times \mathbf{S}^1)$ and $\vec{\mathcal{P}}_{V^{\mathbf{R}}}^k(G \times \mathbf{S}^1)$, of the real valued invariant polynomial functions on a real vector space $V^{\mathbf{R}}$ and $V^{\mathbf{R}}$-valued equivariant polynomial mappings on a real vector space $V^{\mathbf{R}}$, respectively.

However, the (truncated) normal forms for bifurcation problems are always real polynomial mappings and therefore, when computing the number of invariants and equivariants for normal form theory one should, in principle, disregard any extra structure and consider V as a real representation of G. Therefore, the second form of application of Theorem 2.1 is the one we are interested in.

The most common case of symmetric Hopf bifurcation is when $V^{\mathbf{R}} = U \oplus U$ with U a real representation of G [16, Chapter XVI]. In that case V (as a complex vector space) can be identified with the complexification $V = U \otimes_{\mathbf{R}} \mathbf{C}$ of U. The circle group $\mathbf{S}^1$ acts on $\mathbf{C}$ by

$$\theta \cdot z \;=\; e^{i\theta} z \qquad (\theta \in \mathbf{S}^1,\, z \in \mathbf{C}) \tag{2.2}$$

and $G \times \mathbf{S}^1$ acts on $V = U \otimes_{\mathbf{R}} \mathbf{C}$ by

$$(g,\theta)\cdot(u \otimes z) \;=\; (g\, u) \otimes (e^{i\theta} z) \qquad (g \in G,\, \theta \in \mathbf{S}^1,\, u \in U,\, z \in \mathbf{C})\,. \tag{2.3}$$

Due to this special structure it is possible to obtain character formulas for the number of invariants and equivariants that depend only on the representation of

G on U. Thus we do not need to worry about the form of the action of $G \times \mathbf{S}^1$ to compute the character.

In order to state our result we need to establish a couple of conventions. Let V be a complex vector space (not necessarily of the form $V = U \otimes_{\mathbf{R}} \mathbf{C}$) and denote by $V^{\mathbf{R}}$ the underlying real vector space of V. Denote by $\vec{\mathcal{P}}^k_{V^{\mathbf{R}},V}(G \times \mathbf{S}^1)$ the complex vector space of homogeneous polynomial mappings $h : V^{\mathbf{R}} \to V$ of degree k which are equivariant with respect to the actions of $G \times \mathbf{S}^1$ (by real matrices) on $V^{\mathbf{R}}$ and the action of $G \times \mathbf{S}^1$ (by complex matrices) on V, respectively. The main advantage of working with the space $\vec{\mathcal{P}}^k_{V^{\mathbf{R}},V}(G \times \mathbf{S}^1)$ rather than with $\vec{\mathcal{P}}^k_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)$, is that one can write a vector field in normal form as linear combination of homogeneous polynomial mappings with *complex* coefficients which is in complete agreement with the standard practice of writing normal forms for (non-symmetric) Hopf bifurcations. If we denote by $\dim_{\mathbf{C}} \vec{\mathcal{P}}^k_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)$ the complex dimension of the space $\vec{\mathcal{P}}^k_{V^{\mathbf{R}},V}(G \times \mathbf{S}^1)$ then

$$\dim_{\mathbf{R}} \vec{\mathcal{P}}^k_{V^{\mathbf{R}}}(G \times \mathbf{S}^1) \;=\; 2\,\dim_{\mathbf{C}} \vec{\mathcal{P}}^k_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)\,, \tag{2.4}$$

see Section 4 for details.

We state now our first main result.

Theorem 2.2. *Let G be a compact Lie group acting linearly on a real vector space U and denote by χ the corresponding character. Let $V = U \otimes_{\mathbf{R}} \mathbf{C}$ where $G \times \mathbf{S}^1$ acts on $V = U \otimes_{\mathbf{R}} \mathbf{C}$ by* (2.3)*. Then*

$$\begin{aligned} \dim_{\mathbf{R}} \mathcal{P}^{2k}_{V^{\mathbf{R}}}(G \times \mathbf{S}^1) \;&=\; \int_G \chi_{(k)}(g)^2 \,\mathrm{d}\mu_G(g)\,, \\ \dim_{\mathbf{R}} \mathcal{P}^{2k+1}_{V^{\mathbf{R}}}(G \times \mathbf{S}^1) \;&=\; 0 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \dim_{\mathbf{C}} \vec{\mathcal{P}}^{2k+1}_{V^{\mathbf{R}}}(G \times \mathbf{S}^1) \;&=\; \int_G \chi_{(k+1)}(g)\,\chi_{(k)}(g)\,\chi(g)\,\mathrm{d}\mu_G(g)\,, \\ \dim_{\mathbf{C}} \vec{\mathcal{P}}^{2k}_{V^{\mathbf{R}}}(G \times \mathbf{S}^1) \;&=\; 0\,, \end{aligned} \tag{2.6}$$

where $\mathrm{d}\mu_G(g)$ is the normalised invariant measure of G and $\chi_{(k)}$ is the corresponding character of the induced action of G on $\mathcal{P}^k_U$.

This result can be generalised to a group of the form $G \times \mathbf{T}^n$ where $\mathbf{T}^n$ is the n-dimensional torus. This is relevant for the study of *non-resonant* multiple Hopf mode-interaction where the n-torus action comes from n independent frequencies $w_1, \ldots, w_n$ associated to purely imaginary eigenvalues of the Jacobian matrix. For more details, see Remark 4.1 in Section 4.

In order to effectively apply Theorem 2.1 and Theorem 2.2, we need to calculate the character $\chi_{(k)}$ associated to the action of G on the space of homogeneous polynomials. We use the very simple and well known recursive formula

$$k\,\chi_{(k)}(g) \;=\; \sum_{i=0}^{k-1} \chi(g^{k-i})\chi_{(i)}(g)\,. \tag{2.7}$$

For completeness, we include the proof of this formula in Section 4 since we could not find it in the literature.

The character formulas for the dimensions of invariants and equivariants are very convenient when G is a finite group, since they can be explicitly evaluated using GAP [11]. Examples of the application of these formulas are given in Section 6.

2.2. Hilbert-Poincaré Series and Molien Formulas

The Hilbert-Poincaré series and their underlying Molien formulas are another method to compute the numbers of invariants and equivariants. In fact, they are widely used in algorithmic invariant theory [12, 13]. We start by reviewing the classical Molien theorem.

As before, let V be a finite-dimensional vector space over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. The *Hilbert-Poincaré series* Φ_G^V of the graded vector space $\mathcal{P}_V(G)$ is the generating function for the dimension of the vector space of invariants at each degree defined by

$$\Phi_G^V(t) \;=\; \sum_{k=0}^{\infty} \dim \mathcal{P}_V^k(G)\, t^k\,.$$

Similarly, the *Hilbert-Poincaré series* Ψ_G^V of the graded vector space $\vec{\mathcal{P}}_V(G)$ is defined by

$$\Psi_G^V(t) \;=\; \sum_{k=0}^{\infty} \dim \vec{\mathcal{P}}_V^k(G)\, t^k\,.$$

The following theorem goes back to Molien [24] for Φ_G^V when G is a finite group. Sattinger [27] extended to the case when G is a compact Lie group and introduced the formula for Ψ_G^V.

Theorem 2.3 (Molien). *Let G be a compact Lie group acting linearly on V. Then*

(i) the Hilbert-Poincaré series of $\mathcal{P}_V(G)$ is given by

$$\Phi_G^V(t) \;=\; \int_G \frac{1}{\det(1-gt)}\, \mathrm{d}\mu_G(g)\,, \tag{2.8}$$

(ii) the Hilbert-Poincaré series of $\vec{\mathcal{P}}_V(G)$ is given by

$$\Psi_G^V(t) \;=\; \int_G \frac{\chi(g^{-1})}{\det(1-gt)}\, \mathrm{d}\mu_G(g)\,, \tag{2.9}$$

where χ is the character afforded by the G-action on V and $\mathrm{d}\mu_G(g)$ is the normalised invariant measure of G. If the action of G on V is orthogonal then $g^{-1} = g^t$ and $\chi(g^{-1}) = \chi(g) = \mathrm{tr}(g)$.

We are interested in counting the number of real invariants and equivariants on a complex vector space V (not necessarily of the form $V = U \otimes_{\mathbf{R}} \mathbf{C}$) without changing to real coordinates. Let $\{v_1, \ldots, v_m\}$ be a basis of V (over $\mathbf{C}$). Then $\{v_1, \ldots, v_m; iv_1, \ldots, iv_m\}$ is a basis of $V^{\mathbf{R}}$ over $\mathbf{R}$. Denote by $\{x_1, \ldots, x_m; y_1, \ldots, y_m\}$ the coordinates of a vector $v \in V^{\mathbf{R}}$ relative to this basis and let $z_j = x_j + iy_j$ for $j = 1, \ldots, m$. Thus any polynomial f on $V^{\mathbf{R}}$ can be written either as a linear combination of monomials which are products of powers of the real coordinates x_j and y_j, or as a linear combination of monomials which are products of powers of the complex coordinates z_j and $\overline{z}_j$. Now one can modify the definition of the Hilbert-Poincaré series and obtain a new Molien theorem that counts invariants and equivariants in the coordinates $z, \overline{z}$. This was done by Forger [9] for the case of invariants.

Let $\mathcal{P}^{r,s}_{V^{\mathbf{R}}}(G)$ denote the vector space of G-invariant homogeneous polynomial functions on the variables $z, \overline{z}$ and bidegree (r,s). The *bigraded Hilbert-Poincaré series* of $\mathcal{P}_{V^{\mathbf{R}}}(G)$ is the generating function of two variables given by

$$\Phi^{V^{\mathbf{R}}}_G(z,\overline{z}) \;=\; \sum_{r,s=0}^{\infty} \dim_{\mathbf{R}} \mathcal{P}^{r,s}_{V^{\mathbf{R}}}(G)\, z^r \overline{z}^s\,.$$

Theorem 2.4 (Forger). *Let G be a compact Lie group acting linearly on a complex vector space V. Then the bigraded Hilbert-Poincaré series of $\mathcal{P}_{V^{\mathbf{R}}}(G)$ is given by*

$$\Phi^{V^{\mathbf{R}}}_G(z,\overline{z}) \;=\; \int_G \frac{1}{\det(1-gz)\det(1-\overline{gz})}\, \mathrm{d}\mu_G(g) \tag{2.10}$$

where $\mathrm{d}\mu_G(g)$ is the normalised invariant measure of G.

Let $\vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}},V}(G)$ be the vector space of V-valued G-equivariant homogeneous polynomial mappings on the variables $z, \overline{z}$ and bidegree (r,s). As before, we denote by $\dim_{\mathbf{C}} \vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}}}(G)$ the complex dimension of the space $\vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}},V}(G)$. The *bigraded Hilbert-Poincaré series* of $\vec{\mathcal{P}}_{V^{\mathbf{R}},V}(G)$ is the generating function of two variables given by

$$\Psi^{V^{\mathbf{R}}}_G(z,\overline{z}) \;=\; \sum_{r,s=0}^{\infty} \dim_{\mathbf{C}} \vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}}}(G) z^r \overline{z}^s\,.$$

Our second main result gives an integral formula for the bigraded Hilbert-Poincaré series of $\vec{\mathcal{P}}_{V^{\mathbf{R}},V}(G)$.

Theorem 2.5. *Let G be a compact Lie group acting linearly on a complex vector space V. Then the bigraded Hilbert-Poincaré series of $\vec{\mathcal{P}}_{V^{\mathbf{R}},V}(G)$ is given by*

$$\Psi^{V^{\mathbf{R}}}_G(z,\overline{z}) \;=\; \int_G \frac{\chi\big(g^{-1}\big)}{\det(1-gz)\det\big(1-\overline{gz}\big)}\, \mathrm{d}\mu_G(g) \tag{2.11}$$

where χ is the character afforded by the representation of G on V and $\mathrm{d}\mu_G(g)$ is the normalised invariant measure of G.

Finally, a combination of Theorems 2.4 and 2.5 to the special case where $V = U \otimes_{\mathbf{R}} \mathbf{C}$ with U a real representation of G gives our third main result, which has direct application to the computation of the number of invariants and equivariants for Hopf bifurcation.

Theorem 2.6. *Let G be a compact Lie group acting linearly on a real vector space U and let $V = U \otimes_{\mathbf{R}} \mathbf{C}$. Let $G \times \mathbf{S}^1$ act on V as in equation (2.3). Then:*

(i) the bigraded Hilbert-Poincaré series for $\mathcal{P}_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)$ is given by

$$\Phi^{V^{\mathbf{R}}}_{G\times\mathbf{S}^1}(z,\overline{z}) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \Phi^{V^{\mathbf{R}}}_{G}\left(e^{i\theta}z, e^{-i\theta}\overline{z}\right)\,\mathrm{d}\theta \tag{2.12}$$

where $\Phi^{V^{\mathbf{R}}}_{G}$ is the bigraded Hilbert-Poincaré series for $\mathcal{P}_{V^{\mathbf{R}}}(G)$.

(ii) the bigraded Hilbert-Poincaré series for $\vec{\mathcal{P}}_{V^{\mathbf{R}}}(G \times \mathbf{S}^1)$ is given by

$$\Psi^{V^{\mathbf{R}}}_{G\times\mathbf{S}^1}(z,\overline{z}) \;=\; \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}\Psi^{V^{\mathbf{R}}}_{G}\left(e^{i\theta}z, e^{-i\theta}\overline{z}\right)\,\mathrm{d}\theta \tag{2.13}$$

where $\Psi^{V^{\mathbf{R}}}_{G}$ is the bigraded Hilbert-Poincaré series for $\vec{\mathcal{P}}_{V^{\mathbf{R}}}(G)$.

We illustrate the above results with an example.

Example 2.7. Consider the symmetry group $\mathbf{D}_4$ of the square. This group is generated by the permutations $g = (1234)$ and $\kappa = (12)(34)$ and the conjugacy classes are $\{e\}$, $\{g^2\}$, $\{g, g^3\}$, $\{\kappa, g^2\kappa\}$ and $\{g\kappa, g^3\kappa\}$. A two-dimensional representation of $\mathbf{D}_4$ is obtained by considering the standard action of $\mathbf{D}_4$ as rotations and reflections in the plane: let $T(g)$ denote the matrix for the rotation through $2\pi/4$ and $T(\kappa)$ the matrix of the reflection in the y-axis. Thus

$$T(g) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T(\kappa) \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $\mathbf{D}_4$ acts on $U = \mathbf{R}^2$ irreducibly. The Hilbert-Poincaré series for $\mathcal{P}_U(\mathbf{D}_4)$ and $\vec{\mathcal{P}}_U(\mathbf{D}_4)$, are:

$$\begin{aligned}
\Phi^{U}_{\mathbf{D}_4}(t) &= \frac{1}{8}\left(\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2} + \frac{2}{1+t^2} + \frac{4}{1-t^2}\right), \\
\Psi^{U}_{\mathbf{D}_4}(t) &= \frac{1}{8}\left(\frac{2}{(1-t)^2} - \frac{2}{(1+t)^2}\right).
\end{aligned}$$

In order to apply Theorem 2.6 to the action of $\mathbf{D}_4 \times \mathbf{S}^1$ on $V = U \otimes_{\mathbf{R}} \mathbf{C}$, where $\theta \in \mathbf{S}^1$ acts on $\mathbf{C}$ by multiplication by $e^{i\theta}$, we need first to consider the action of $\mathbf{D}_4$ on $V = \mathbf{C}^2$ generated by the same matrices $T(g)$ and $T(\kappa)$. The bigraded Hilbert-Poincaré series for $\mathcal{P}_{V^{\mathbf{R}}}(\mathbf{D}_4)$ and $\vec{\mathcal{P}}_{V^{\mathbf{R}}}(\mathbf{D}_4)$ are:

$$\begin{aligned}
\Phi^{V^{\mathbf{R}}}_{\mathbf{D}_4}(z,\overline{z}) &= \frac{1}{8}\left(\frac{1}{(1-z)^2(1-\overline{z})^2} + \frac{1}{(1+z)^2(1+\overline{z})^2}\right. \\
&\qquad \left. + \frac{2}{(1+z^2)\left(1+\overline{z}^2\right)} + \frac{4}{(1-z^2)\left(1-\overline{z}^2\right)}\right), \\
\Psi^{V^{\mathbf{R}}}_{\mathbf{D}_4}(z,\overline{z}) &= \frac{1}{8}\left(\frac{2}{(1-z)^2(1-\overline{z})^2} - \frac{2}{(1+z)^2(1+\overline{z})^2}\right).
\end{aligned}$$

Then the bigraded Hilbert-Poincaré series for the number of invariants for $\mathbf{D}_4 \times \mathbf{S}^1$ is

$$\Phi^{V^{\mathbf{R}}}_{\mathbf{D}_4\times\mathbf{S}^1}(z,\overline{z}) = \frac{1}{2\pi}\int_0^{2\pi} \Phi^{V^{\mathbf{R}}}_{\mathbf{D}_4}\left(e^{i\theta}z, e^{-i\theta}\overline{z}\right)\, d\theta\,.$$

This integral can be evaluated by making the change of variable $u = e^{i\theta}$ and the applying Cauchy's Residue Theorem. The details are omitted here since this is a standard technique in complex variable theory. The result is

$$\begin{aligned}\Phi^{V^{\mathbf{R}}}_{\mathbf{D}_4\times\mathbf{S}^1}(z,\overline{z}) &= \frac{1+z^3\overline{z}^3}{(1-z\overline{z})(1-z^2\overline{z}^2)^2}\\ &= 1 + z\overline{z} + 3z^2\overline{z}^2 + 4z^3\overline{z}^3 + 7z^4\overline{z}^4 + \cdots\,.\end{aligned}$$

In the same way we can evaluate the bigraded Hilbert-Poincaré series for the number of equivariants for $\mathbf{D}_4 \times \mathbf{S}^1$

$$\Psi^{V^{\mathbf{R}}}_{\mathbf{D}_4\times\mathbf{S}^1}(z,\overline{z}) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}\,\Psi^{V^{\mathbf{R}}}_{\mathbf{D}_4}\left(e^{i\theta}z, e^{-i\theta}\overline{z}\right)\, d\theta$$

and the result is

$$\begin{aligned}\Psi^{V^{\mathbf{R}}}_{\mathbf{D}_4\times\mathbf{S}^1}(z,\overline{z}) &= \frac{z}{(1-z\overline{z})^3}\\ &= z + 3z^2\overline{z} + 6z^3\overline{z}^2 + 10z^4\overline{z}^3 + \cdots\,.\end{aligned}$$

Hence, the numbers of independent invariants of degree $(2,4,6,8)$ are $(1,3,4,7)$, and the numbers of independent equivariants of degree $(3,5,7)$ are $(3,6,10)$. In fact we can use the binomial theorem to show that the number of equivariants of degree $2k+1$ is $(k+1)(k+2)/2$. Also note that $\Phi^{V^{\mathbf{R}}}_{G\times\mathbf{S}^1}(z,\overline{z})$ is a function of the variable $z\overline{z}$ and $\Psi^{V^{\mathbf{R}}}_{G\times\mathbf{S}^1}(z,\overline{z})$ is z times a function of the variable $z\overline{z}$. This is simply a reflection of the general fact that a $G \times \mathbf{S}^1$-invariant polynomial has even (total) degree in the coordinates $z, \overline{z}$ and a $G \times \mathbf{S}^1$-equivariant polynomial has odd (total) degree in the coordinates $z, \overline{z}$ with z being of one degree higher than $\overline{z}$.

3. Background and Notation

In this section we review some important concepts concerning the representation theory of compact Lie groups. For details, see for example James *et al.* [21] for finite groups and Bröcker *et al.* [2] for compact Lie groups.

3.1. Representation Theory

Let G be a compact Lie group acting linearly on a finite-dimensional real or complex vector space V. Thus this action corresponds to a representation T of the group G on the vector space V through a linear homomorphism from G to the group $\mathrm{GL}(V)$ of invertible linear transformations on V. We denote by $g\,v$ the action of the linear transformation $T(g)$ of an element $g \in G$ on a vector $v \in V$.

A subspace W of V is *invariant* under G if $gW \subseteq W$ for all $g \in G$; in this case, we say that W is sub-representation and the action of G on V can be restricted to an action of G on W. The action is said to be *reducible* if V possesses a proper invariant subspace. Otherwise it is said to be *irreducible.* A representation T of G is *absolutely irreducible* if the only linear maps on V commuting with G are the scalar multiples of the identity. Two representations of a group G on the vector spaces V_1 and V_2 are called *equivalent* if there exists a non-singular linear transformation $S : V_1 \to V_2$ such that $S(g\, v_1) \;=\; g\, S(v_1)$ for all $g \in G$ and all $v_1 \in V_1$.

Let two representations of a group G be given on the vector spaces V_1 and V_2, respectively. Then there is natural representation of G on the direct sum $V_1 \oplus V_2$ given by $g\,(v_1 + v_2) = g\, v_1 + g\, v_2$ and a natural representation on the tensor product $V_1 \otimes V_2$ given by $g\,(v_1 \otimes v_2) = g\, v_1 \otimes g\, v_2$. By iteration of these constructions one obtains actions of G on the k-th direct sum $\bigoplus_k V_k$ and k-th tensor powers $V^{\otimes k}$ of a representation V of G. Also, by restriction, one obtains representations of G on the k-th symmetric tensor power $S^k V$ and k-th antisymmetric tensor power $A^k V$ of V, since these are always G-invariant subspaces of $V^{\otimes k}$. There is also a natural action on the dual space V^* of V given by $[g\,\psi](v) = \psi(g^{-1}\, v)$ for all $\psi \in V^*$.

Finally, we recall here an important isomorphism which will be used several times in the rest of the paper:

$$S^n(V \oplus W) \;\cong\; \bigoplus_{i=0}^{n} S^i V \otimes S^{n-i} W\,, \tag{3.1}$$

where $S^0(V)$ is the ground field. See for example Fulton and Harris [10, page 473] for a proof of (3.1).

3.2. The Haar Measure of a Compact Lie Group

Since G is a compact group there exists an *invariant measure* (unique up to a constant multiple) μ_G on G such that

$$\int_G f(hg)\,\mathrm{d}\mu(g) \;=\; \int_G f(gh)\,\mathrm{d}\mu(g) \;=\; \int_G f(g)\,\mathrm{d}\mu(g) \;=\; \int_G f(g^{-1})\,\mathrm{d}\mu(g)$$

for any continuous function f on G and for any $h \in G$. We assume that the measure is normalised so that $\int_G \mathrm{d}\mu(g) = 1$. In this case $\mathrm{d}\mu(g)$ is called the *Haar measure* of f and the integral with respect with this measure is called *normalised Haar integral.* See Bröcker and tom Dieck [2] for the proof and existence of the Haar measure on a compact Lie group. For finite groups the Haar integral reduces to the "averaging over the group" formula

$$\int_G f(g)\,\mathrm{d}\mu(g) \;=\; \frac{1}{|G|}\sum_{g\in G} f(g)\,.$$

Using the Haar measure it is possible to construct a G-invariant inner product $(\cdot,\cdot)$ on V, that is, $(g\,u, g\,v) \;=\; (u,v)$ for all $u, v \in V$ and $g \in G$. See for example [16, Proposition XII 1.3]. Moreover, we can choose an orthogonal basis with respect to such a G-invariant inner product where if M_g denotes the matrix representing g at this basis, then M_g is unitary for all $g \in G$, that is, $M_{g^{-1}} = M_g^* = \overline{M}_g^t$. Here, M_g^t denotes the transpose matrix of M_g. In particular, if V is real then M_g is orthogonal for all $g \in G$.

3.3. Real and Complex Representations

Usually in texts on representation theory, the complex representations are considered as most fundamental – mainly because of the algebraic completeness of the field of complex numbers – and then the real representations are defined as a special class of complex representations. On the other hand, in bifurcation theory, all the representations are real with the complex ones arising due to extra structure, as for example, in Hopf bifurcation where the circle action induces a complex structure on V. Hence we need to set up the notation and terminology in order to transfer results from complex representations and complex characters to real representations and real characters.

Let U be a real vector space. The *complexification* of U is the complex vector space $U^{\mathbf{C}}$ given by the following tensor product over $\mathbf{R}$

$$U^{\mathbf{C}} \;=\; U \otimes_{\mathbf{R}} \mathbf{C}\,.$$

The map $u \mapsto u \otimes 1$ allows us to identify U canonically with a subset of $U^{\mathbf{C}}$. If V is a complex vector space, we can restrict the definition of scalar multiplication to scalars in $\mathbf{R}$, thereby obtaining a vector space over $\mathbf{R}$. This vector space we denote by $V^{\mathbf{R}}$ and is called the *realification* of V. Note that the operations $(\cdot)^{\mathbf{C}}$ and $(\cdot)^{\mathbf{R}}$ are not inverse to each other: $(U^{\mathbf{C}})^{\mathbf{R}}$ has twice the real dimension of U, and $(V^{\mathbf{R}})^{\mathbf{C}}$ has twice the complex dimension of V. More precisely,

$$(U^{\mathbf{C}})^{\mathbf{R}} \;=\; U \oplus iU \;\cong\; U \oplus U \qquad \text{(as real vector spaces)} \tag{3.2}$$

where U means $U \otimes 1$ in $U \otimes_{\mathbf{R}} \mathbf{C}$ and i refers to the real linear mapping "multiplication-by-i".

For any complex vector space V let us denote by $\overline{V}$ the complex vector space which coincides with V as an additive group, but is endowed with the following multiplication by complex scalars: $c \cdot \bar{v} = \overline{cv}$ with $c \in \mathbf{C}$ and $v \in \overline{V}$. The vector space $\overline{V}$ is called the *complex conjugate* of V.

Now suppose there is a group G acting linearly on the real vector space U. Then the action of $g \in G$ on U can be extended to an action on $V = U^{\mathbf{C}}$ by $g(u \otimes z) = gu \otimes z$ for all $z \in \mathbf{C}$ and $u \in U$. Moreover, if the circle group $\mathbf{S}^1$ acts on $\mathbf{C}$ by equation (2.2), this action commutes with the G-action on U and so we can extend the action of G on V to an action of $G \times \mathbf{S}^1$ on V. There is a natural representation of $G \times \mathbf{S}^1$ on $\overline{V}$ given by $(g,\theta)\,\bar{v} = \overline{(g,\theta)\,v}$. By choosing a $G \times \mathbf{S}^1$-invariant (hermitian) inner product on V it is easy to see that the representation of $G \times \mathbf{S}^1$ on $\overline{V}$ is equivalent to the representation of $G \times \mathbf{S}^1$ on V^*. We then may consider the action of $G \times \mathbf{S}^1$ on $V \oplus \overline{V}$ given by

$$\theta \cdot (v_1, \overline{v_2}) \;=\; (e^{i\theta} v_1, e^{-i\theta}\overline{v_2}) \qquad (\theta \in \mathbf{S}^1,\, v_1, v_2 \in V)\,. \tag{3.3}$$

Conversely, if G acts linearly on a complex vector space V then the realification $V^{\mathbf{R}}$ can be regarded as a real representation of G by "forgetting" the complex structure.

3.4. Invariant and Equivariant Polynomials

Recall that a function $f : V \to \mathbf{K}$ is called a *homogeneous polynomial function of degree* k on V if there exists a $\mathbf{K}$-valued symmetric k-multi-linear function

$$\hat{f} : \underbrace{V \times \cdots \times V}_{k \text{ times}} \longrightarrow \mathbf{K}$$

such that $f(v) = \hat{f}(v, \ldots, v)$ for all $v \in V$. Denote by $L_s^k(V)$ the space of all $\mathbf{K}$-valued symmetric multi-linear functions and $\mathcal{P}_V^k$ the vector space of all homogeneous polynomials of degree k on V. The mapping $\hat{f} \mapsto f$ is a natural isomorphism of $\mathbf{K}$-vector spaces $L_s^k(V) \cong \mathcal{P}_V^k$; the inverse mapping which associates to each polynomial a $\mathbf{K}$-valued symmetric k-multi-linear function is called *polarisation*. Similarly, a mapping $F : V \to V$ is called *homogeneous polynomial mapping of degree* k of V if there exists a V-valued symmetric k-multi-linear map

$$\hat{F} : \underbrace{V \times \cdots \times V}_{k \text{ times}} \longrightarrow V$$

such that $F(v) = \hat{F}(v, \ldots, v)$ for all $v \in V$. Denote by $L_s^k(V, V)$ the space of all $\mathbf{K}$-valued symmetric multi-linear functions and $\vec{\mathcal{P}}_V^k$ the vector space homogeneous polynomial mappings of degree k on V. The mapping $\hat{F} \mapsto F$ is a natural isomorphism of $\mathbf{K}$-vector spaces $L_s^k(V, V) \cong \vec{\mathcal{P}}_V^k$ which is compatible with the isomorphism $L_s^k(V) \cong \mathcal{P}_V^k$.

There are canonical isomorphisms (for a proof see Goodman and Wallach [17, page 621])

$$\mathrm{Hom}(S^k V) \;\cong\; L_s^k(V) \qquad \text{and} \qquad \mathrm{Hom}(S^k V, V) \;\cong\; L_s^k(V, V)\,. \tag{3.4}$$

If V^* is the dual space of V then there is the canonical isomorphism

$$V^* \otimes W \;\cong\; \mathrm{Hom}(V, W) \tag{3.5}$$

which maps $v^* \otimes w$ to the homomorphism $u \mapsto v^*(u)w$. Combining the isomorphisms (3.4) and (3.5) with the natural isomorphisms $L_s^k(V) \cong \mathcal{P}_V^k$ and $L_s^k(V, V) \cong \vec{\mathcal{P}}_V^k$ of $\mathbf{K}$-vector spaces we have the canonical identifications

$$\begin{aligned} &\text{(a) } \mathcal{P}_V^k \;\cong\; L_s^k(V) \;\cong\; \mathrm{Hom}(S^n V) \;\cong\; (S^n V)^* \\ &\text{(b) } \vec{\mathcal{P}}_V^k \;\cong\; L_s^k(V, V) \;\cong\; \mathrm{Hom}(S^n V, V) \;\cong\; (S^n V)^* \otimes V\,. \end{aligned} \tag{3.6}$$

Now suppose that a compact Lie group G acts linearly on V. This action induces natural actions of G on the spaces of k-multi-linear functions and mappings $L_s^k(V)$ and $L_s^k(V, V)$ respectively; on the spaces of k-homogeneous polynomial functions $\mathcal{P}_V^k$ and mappings $\vec{\mathcal{P}}_V^k$ respectively; and on the vector spaces $(S^n V)^*$ and $(S^n V)^* \otimes V$. Moreover, the identifications given by (3.6) are compatible with all these induced actions and therefore they provide equivalences of representations of G in all these spaces. In this way, one can identify all these representations and choose a particular form according to convenience.

3.5. Character Theory

For proofs of all statements in this subsection, consult [21] for finite groups and [2] for compact Lie groups.

Recall that two elements $g_1, g_2 \in G$ are *conjugate* if there is an element $h \in G$ such that $g_1 = hg_2h^{-1}$. Note that conjugacy is an equivalence relation on G and so partitions G into separate classes, called *conjugacy classes*. A function $f : G \to \mathbf{C}$ is called a *class function* if it is constant on the conjugacy classes. The *character* of a (real or complex) representation T of a group G is the trace

$$\chi_T(g) \;=\; \operatorname{tr} T(g) \qquad \text{for all} \quad g \in G\,.$$

Note that characters are constant on conjugacy classes. In fact, the characters of the complex irreducible representations form a basis for the vector space of complex class functions; therefore two complex representations are equivalent if and only if they have the same character. The character of a one-dimensional representation is said to be a *linear character*.

Since all representations of a compact Lie group are equivalent to unitary representations (by choosing an invariant inner product) we have $\operatorname{tr}(M_{g^{-1}}) = \operatorname{tr}\left(\overline{M}_g\right)$. Then an hermitian inner product can be defined on characters:

$$\langle \chi_1, \chi_2 \rangle \;=\; \int_G \chi_1(g)\overline{\chi_2(g)}\,\mathrm{d}\mu_G(g) \;=\; \int_G \chi_1(g)\chi_2(g^{-1})\,\mathrm{d}\mu_G(g)\,,$$

or for finite groups,

$$\langle \chi_1, \chi_2 \rangle \;=\; \frac{1}{|G|}\sum_{g\in G} \chi_1(g)\overline{\chi_2(g)} \;=\; \frac{1}{|G|}\sum_{g\in G} \chi_1(g)\chi_2(g^{-1})$$

for any two characters χ_1, χ_2. With respect to this hermitian inner product, the characters of irreducible inequivalent representations are orthonormal. In fact, a more subtle relation is true:

$$\langle \chi_1, \chi_2 \rangle \;=\; \langle \chi_2, \chi_1 \rangle\,. \tag{3.7}$$

Let V and W be two (real or complex) G-modules, with characters χ_V and χ_W, respectively. Then

$$\begin{array}{llll} \text{(a)} & \chi_{V\oplus W} \;=\; \chi_V + \chi_W\,, & \text{(b)} & \chi_{V\otimes W} \;=\; \chi_V \chi_W\,, \\[2ex] \text{(c)} & \chi_{V^*} \;=\; \overline{\chi}_V \;=\; \overline{\chi_V}\,, & \text{(d)} & \chi_V^2 \;=\; \chi_{S^2V} + \chi_{A^2V}\,, \\[2ex] \text{(e)} & \chi_{V^*}(g) \;=\; \chi_V(g^{-1})\,, & \text{(f)} & \chi_{S^2V}(g) \;=\; \tfrac{1}{2}\big(\chi_V(g)^2 + \chi_V(g^2)\big)\,. \end{array} \tag{3.8}$$

Usually the irreducible characters are defined for the complex representations and then the characters of the irreducible real representations are computed from the complex ones. In order to do this one should be able to decide when a complex representation has a real form, that is, it is a complexification of an absolutely irreducible real representation. A necessary condition is that the complex character

χ must be real valued. However this is not sufficient. The sufficient condition is supplied by the *Frobenius-Schur indicator*:

$$\iota_\chi \;=\; \int_G \chi(g^2)\,\mathrm{d}\mu(g)$$

for an irreducible complex character χ. According to the *Frobenius-Schur Theorem* we have:

$$\iota_\chi \;=\; \begin{cases} 0, & \text{if } \chi \text{ is not real valued,} \\ 1, & \text{if } \chi \text{ can be realised over } \mathbf{R}, \\ -1, & \text{if } \chi \text{ is real but cannot be realised over } \mathbf{R}. \end{cases}$$

3.6. Trace Formula

Recall that the *fixed-point subspace* of the action of G on V is defined by

$$\mathrm{Fix}(G, V) \;=\; \{v \in V : \; g\,v = v, \; \forall g \in G\}\,.$$

Since a polynomial function $f : V \to \mathbf{K}$ is invariant under G if and only if $g\,f = f$ where the action of G is defined by $(g\,f)(v) = f(g^{-1}\,v)$, it follows that $\mathcal{P}^k_V(G) = \mathrm{Fix}(G, \mathcal{P}^k_V)$. Applying the isomorphism (a) of (3.6) we have the canonical isomorphism

$$\mathcal{P}^k_V(G) \;\cong\; \mathrm{Fix}(G, (S^k V)^*)\,. \tag{3.9}$$

Since a polynomial mapping $F : V \to V$ is equivariant under G if and only if $g\,F = F$ where the action of G is defined by $(g\,F)(v) = g\,F(g^{-1}\,v)$, it follows that $\vec{\mathcal{P}}^k_V(G) = \mathrm{Fix}(G, \vec{\mathcal{P}}^k_V)$. Applying the isomorphism (b) of (3.6) we have the canonical isomorphism

$$\vec{\mathcal{P}}^k_V(G) \;\cong\; \mathrm{Fix}(G, (S^k V)^* \otimes V)\,. \tag{3.10}$$

The following theorem – whose proof can be found in [16, Theorem XIII 2.3] – together with the formulas (3.9) and (3.10), provides the fundamental link between character theory and invariant theory.

Theorem 3.1 (Trace Formula). *Let a compact group G act linearly on a vector space V. Then*

$$\dim \mathrm{Fix}(G, V) \;=\; \int_G \chi_V(g)\,\mathrm{d}\mu_G(g) \;=\; \langle \chi_V\,, \mathbf{1}\rangle \tag{3.11}$$

where χ_V is the character of the representation of G on V, $\mathbf{1}$ is the trivial character of G and $\mathrm{d}\mu_G(g)$ is the normalised invariant measure of G.

4. Character Formulas for Invariants and Equivariants

In this section we prove the character formulas for the dimensions of the vector spaces of polynomial functions of degree k that are invariant or equivariant with respect to the action of $G \times \mathbf{S}^1$. The symbol $\int_G$ is used to denote the normalised Haar integral.

4.1. Proof of Theorem 2.2

Let $\{v_1,\dots,v_m\}$ be a basis of V (over $\mathbf{C}$). Then $\{v_1,\dots,v_m;iv_1,\dots,iv_m\}$ is a basis of $V^{\mathbf{R}}$ over $\mathbf{R}$. Denote by $\{x_1,\dots,x_m;y_1,\dots,y_m\}$ the coordinates of a vector $v\in V^{\mathbf{R}}$ relative to this basis and let $z_j=x_j+iy_j$ for $j=1,\dots,m$. Thus any polynomial f on $V^{\mathbf{R}}$ can be written either as a linear combination of monomials which are products of powers of the real coordinates x_j and y_j, or as a linear combination of monomials which are products of powers of the complex coordinates z_j and $\overline{z}_j$. Let us write $z=(z_1,\dots,z_m)$ and $\overline{z}=(\overline{z}_1,\dots,\overline{z}_m)$. Then using multi-indices, any polynomial function $f:V^{\mathbf{R}}\to\mathbf{R}$ can be written as

$$f(z,\overline{z}) \;=\; \sum_{\alpha,\beta} a_{\alpha\beta} z^{\alpha}\overline{z}^{\beta}$$

where $\alpha,\beta\in(\mathbf{Z}_0^+)^m$, $z^{\alpha}=z_1^{\alpha_1}z_2^{\alpha_2}\dots z_m^{\alpha_m}$ and the coefficients $a_{\alpha\beta}$ may be complex.

Now the action of $\mathbf{S}^1$ on $V\oplus\overline{V}$ is given by equation (3.3) and $V^{\mathbf{R}}$ is the subspace of $V\oplus\overline{V}$ such that $v_1=v_2$ and it is invariant under the action of $G\times\mathbf{S}^1$. Then f is $\mathbf{S}^1$-invariant if for each α,β such that $a_{\alpha\beta}\neq 0$ we have $|\alpha|=|\beta|$ (where $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_m$). Therefore f has bidegree (r,r) in $z,\overline{z}$.

Similarly, the mapping $F:V^{\mathbf{R}}\to V^{\mathbf{R}}$ has components

$$F_j(z,\overline{z}) \;=\; \sum_{\alpha,\beta} b_{\alpha\beta} z^{\alpha}\overline{z}^{\beta}$$

where the coefficients $b_{\alpha\beta}$ may be complex. In this case, the $\mathbf{S}^1$-equivariance is equivalent to having $|\alpha|=|\beta|+1$ if $b_{\alpha\beta}\neq 0$. This is [16, Lemma XVI 9.3]. Therefore F has components of bidegree $(r+1,r)$ in $z,\overline{z}$.

From these observations we have that

$$\dim\mathcal{P}^{2k+1}_{V^{\mathbf{R}}}(G\times\mathbf{S}^1) \;=\; 0 \qquad\text{and}\qquad \dim\vec{\mathcal{P}}^{2k}{}_{V^{\mathbf{R}}}(G\times\mathbf{S}^1) \;=\; 0\,,$$

for all $k\geqslant 0$. Therefore we just need to compute $\dim\mathcal{P}^{2k}_{V^{\mathbf{R}}}(G\times\mathbf{S}^1)$ and $\dim\vec{\mathcal{P}}^{2k+1}_{V^{\mathbf{R}}}(G\times\mathbf{S}^1)$ for $k\in\mathbf{Z}_0^+$.

Let us consider first the formula for the invariants. By equation (3.9) the space $\mathcal{P}^{2k}_{V^{\mathbf{R}}}(G\times\mathbf{S}^1)$ can be identified with the real vector space

$$\mathrm{Fix}\left(G\times\mathbf{S}^1,(S^{2k}(V^{\mathbf{R}}))^*\right).$$

Since $V\oplus\overline{V}=U^{\mathbf{C}}\oplus\overline{U^{\mathbf{C}}}$ and $V^{\mathbf{R}}=U\oplus iU$ we have $V\oplus\overline{V} \;=\; (U\oplus iU)^{\mathbf{C}}$ as representations of $G\times\mathbf{S}^1$. Therefore,

$$\begin{aligned}\dim_{\mathbf{R}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^k\left(V^{\mathbf{R}}\right)\right)^*\right) \;&=\; \dim_{\mathbf{R}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^k(U\oplus iU)\right)^*\right)\\ &=\; \dim_{\mathbf{C}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^k\left(V\oplus\overline{V}\right)\right)^*\right).\end{aligned}$$

Recall that a polynomial on V is invariant under $G\times\mathbf{S}^1$ if and only if it is invariant under G and $\mathbf{S}^1$. From the formula

$$S^{2k}(V\oplus\overline{V}) \;\cong\; \bigoplus_{a=0}^{2k} S^aV\otimes S^{2k-a}\overline{V}$$

and because of the $\mathbf{S}^1$-action, an invariant polynomial function f has bidegree (r,r) in $z, \overline{z}$. This is equivalent to

$$\operatorname{Fix}\left(\mathbf{S}^1, \left(S^{2k}\left(V \oplus \overline{V}\right)\right)^*\right) \;\cong\; \left(S^k V \otimes S^k \overline{V}\right)^* .$$

Therefore

$$\operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k}\left(V \oplus \overline{V}\right)\right)^*\right) \;\cong\; \operatorname{Fix}\left(G, \left(S^k V \otimes S^k \overline{V}\right)^*\right) .$$

Using the Trace Formula (3.11) we have

$$\dim_{\mathbf{C}} \operatorname{Fix}\left(G, \left(S^k V \otimes S^k \overline{V}\right)^*\right) \;=\; \int_G \overline{\chi}_{(k)}(g)\chi_{(k)}(g) .$$

Now, χ is also the character of the representation of G on U and hence it is real valued. Thus

$$\dim_{\mathbf{R}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^k V^{\mathbf{R}}\right)^*\right) \;=\; \int_G \chi_{(k)}(g)^2 .$$

For the equivariants, by equation (3.10) we have

$$\vec{\mathcal{P}}^{2k+1}_{V^{\mathbf{R}}}(G \times \mathbf{S}^1) \;\cong\; \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V^{\mathbf{R}}\right)\right)^* \otimes_{\mathbf{R}} V^{\mathbf{R}}\right) .$$

Now we make three observations that follow from the trace formula. The first one is

$$\begin{aligned}&\dim_{\mathbf{R}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V^{\mathbf{R}}\right)\right)^* \otimes_{\mathbf{R}} V^{\mathbf{R}}\right) \\ &\qquad = \dim_{\mathbf{C}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V \oplus \overline{V}\right)\right)^* \otimes_{\mathbf{C}} \left(V \oplus \overline{V}\right)\right) .\end{aligned}$$

The second observation is

$$\begin{aligned}&\dim_{\mathbf{C}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V \oplus \overline{V}\right)\right)^* \otimes_{\mathbf{C}} \left(V \oplus \overline{V}\right)\right) \\ &\qquad = 2\dim_{\mathbf{C}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V \oplus \overline{V}\right)\right)^* \otimes_{\mathbf{C}} V\right)\end{aligned}$$

and finally

$$\begin{aligned}&\dim_{\mathbf{C}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V^{\mathbf{R}}\right)\right)^* \otimes_{\mathbf{R}} V\right) \\ &\qquad = \dim_{\mathbf{C}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V \oplus \overline{V}\right)\right)^* \otimes_{\mathbf{C}} V\right) .\end{aligned}$$

In particular, these observations justify the relation (2.4). Thus

$$\begin{aligned}&\dim_{\mathbf{R}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V^{\mathbf{R}}\right)\right)^* \otimes_{\mathbf{R}} V^{\mathbf{R}}\right) \\ &\qquad = 2\dim_{\mathbf{C}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V \oplus \overline{V}\right)\right)^* \otimes_{\mathbf{C}} V\right) .\end{aligned} \tag{4.1}$$

We compute now

$$\dim_{\mathbf{C}} \operatorname{Fix}\left(G \times \mathbf{S}^1, \left(S^{2k+1}\left(V \oplus \overline{V}\right)\right)^* \otimes_{\mathbf{C}} V\right) .$$

Proceeding as before, from the formula

$$S^{2k+1}\left(V\oplus\overline{V}\right) \;\cong\; \bigoplus_{a=0}^{2k+1} S^aV\otimes_{\mathbf{C}} S^{2k+1-a}\overline{V}$$

and because of the $\mathbf{S}^1$-action, an equivariant polynomial mapping F has bidegree $(r+1,r)$ in $z,\overline{z}$. This is equivalent to

$$\mathrm{Fix}\left(\mathbf{S}^1,\left(S^{2k+1}\left(V\oplus\overline{V}\right)\right)^*\otimes_{\mathbf{C}} V\right) \;\cong\; (S^{k+1}V^*\otimes_{\mathbf{C}} S^k\overline{V}^*)\otimes_{\mathbf{C}} V\,.$$

Hence by the fact that $\overline{V}\;\cong\;V^*$ (as representations of G) we have

$$\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^{2k+1}\left(V\oplus\overline{V}\right)\right)^*\otimes_{\mathbf{C}} V\right) \;\cong\; \mathrm{Fix}\left(G,S^{k+1}\overline{V}\otimes_{\mathbf{C}} S^kV\otimes_{\mathbf{C}} V\right)\,.$$

Again, by the Trace Formula we obtain

$$\dim_{\mathbf{C}}\mathrm{Fix}(G,S^{k+1}\overline{V}\otimes_{\mathbf{C}} S^kV\otimes_{\mathbf{C}} V) \;=\; \int_G \overline{\chi}_{(k+1)}(g)\chi_{(k)}(g)\chi(g)\,.$$

Since χ is also the character of the representation of G on U, it is real valued and we have

$$\dim_{\mathbf{C}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^{2k+1}\left(V\oplus\overline{V}\right)\right)^*\otimes_{\mathbf{C}} V\right) \;=\; \int_G \chi_{(k+1)}(g)\,\chi_{(k)}(g)\,\chi(g)\,.$$

By equation (4.1) we have

$$\dim_{\mathbf{R}}\mathrm{Fix}\left(G\times\mathbf{S}^1,\left(S^{2k+1}(V^{\mathbf{R}})^*\otimes_{\mathbf{R}} V^{\mathbf{R}}\right)\right) \;=\; 2\int_G \chi_{(k+1)}(g)\,\chi_{(k)}(g)\,\chi(g)$$

and by the relation (2.4) we have

$$\dim_{\mathbf{C}}\vec{\mathcal{P}}^{2k+1}_{V^{\mathbf{R}}}(G\times\mathbf{S}^1) \;=\; \int_G \chi_{(k+1)}(g)\,\chi_{(k)}(g)\,\chi(g)\,.$$

□

Remark 4.1. Theorem 2.2 can be generalised to a group of the form $G\times\mathbf{T}^n$ where $\mathbf{T}^n$ is the n-dimensional torus. In this case, G acts on a direct sum $U=U_1\oplus\cdots\oplus U_n$ of real vector spaces and $G\times\mathbf{T}^n$ acts on $V_1\oplus\cdots\oplus V_n$ with $V_i=U_i\otimes_{\mathbf{R}}\mathbf{C}$ $(i=1,\ldots,n)$ by

$$(g,\theta_1,\ldots,\theta_n)(u_1\otimes z_1,\ldots,u_n\otimes z_n) \;=\; \left((gu_1)\otimes(e^{i\theta_1}z_1),\ldots,(gu_n)\otimes(e^{i\theta_n}z_n)\right)\,.$$

A general formula for arbitrary n and k is very complicated, but it is relatively easy to write down formulas for small values of n by inductively applying the formula to the decomposition of the symmetric tensor power (3.1). For example, if $n=2$ we have $U=U_1\oplus U_2$ and $V=V_1\oplus V_2$. Then

$$S^{2k}(V_1\oplus\overline{V}_1\oplus V_2\oplus\overline{V}_2) \;\cong\; \bigoplus_{i=0}^{2k}\left(S^i(V_1\oplus\overline{V}_1)\otimes S^{2k-i}(V_2\oplus\overline{V}_2)\right)\,.$$

From the $\mathbf{T}^2$-action we have that an invariant polynomial $p(v_1, \bar{v_1}, v_2, \bar{v}_2)$ on $V^{\mathbf{R}}$ has total degree $2k$ and bidegree (i,i) in $v_1, \bar{v}_1$ and bidegree (j,j) in $v_2, \bar{v}_2$ where $i+j=k$. That is,

$$\mathrm{Fix}(\mathbf{T}^2, S^{2k}(V_1 \oplus \overline{V}_1 \oplus V_2 \oplus \overline{V}_2)) \;\cong\; \bigoplus_{i=0}^{k} \big(S^i(V_1) \otimes S^i(\overline{V}_1)\big) \otimes \big(S^{k-i}(V_2) \otimes S^{k-i}(\overline{V}_2)\big)\,.$$

Hence, if χ_i denotes the character of G on U_i $(i=1,2)$ then the number of $G \times \mathbf{T}^2$-invariant polynomials of degree $2k$ is

$$\dim_{\mathbf{R}} \mathcal{P}^{2k}_{V^{\mathbf{R}}}(G \times \mathbf{T}^2) \;=\; \sum_{i=0}^{k} \int_G \chi_{1,(i)}(g)^2\, \chi_{2,(k-i)}(g)^2\,.$$

4.2. Proof of the Recursive Formula

Formula (2.7) is known. Since we did not find its proof in the literature, we include it here for completeness. Let G act unitarily on a finite-dimensional complex vector space, say $W \equiv \mathbf{C}^n$. Denote by T the representation, χ the corresponding character and $\chi_{(k)}$ the character of the induced action of G on the k-th symmetric power $S^k W$. Fix $g \in G$. We have that $T(g)$ is diagonalised. Suppose that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $T(g)$. It follows then that

$$\chi_{(k)}(g) \;=\; \sum \lambda_1^{m_1} \dots \lambda_n^{m_n}$$

where the sum is over all non-negative integers m_j satisfying $m_1 + \cdots + m_n = k$. We introduce the generating function

$$f(t) \;=\; \frac{1}{(1-\lambda_1 t)\cdots(1-\lambda_n t)}\,.$$

Observe that f is well defined for t in a sufficiently small neighbourhood of $t=0$. Moreover, all the k-th derivatives at $t=0$ exist, and

$$\chi_{(k)}(g) \;=\; \frac{f^k(0)}{k!}\,.$$

By induction, it can be shown that

$$k\frac{f^k(0)}{k!} \;=\; \sum_{j=0}^{k-1} (\lambda_1^{k-j} + \cdots + \lambda_n^{k-j}) \frac{f^j(0)}{j!}$$

where $f^0(0) = 1$ and observe that

$$\chi(g^j) \;=\; \lambda_1^j + \cdots + \lambda_n^j\,.$$

Therefore

$$k\chi_{(k)}(g) \;=\; \sum_{j=0}^{k-1} \chi(g^{k-j}) \frac{f^j(0)}{j!} \;=\; \sum_{j=0}^{k-1} \chi(g^{k-j})\chi_{(j)}(g)\,.$$

$\square$

Remark 4.2. There are other formulas for the character $\chi_{(k)}$:

(i) an explicit but rather unwieldy formula (Sattinger [27, p. 110])

$$\chi_{(k)}(g) \;=\; \sum \frac{\chi^{i_1}(g)\chi^{i_2}(g^2)\cdots\chi^{i_k}(g^k)}{1^{i_1}i_1!\,2^{i_2}i_2!\cdots k^{i_k}i_k!} \tag{4.2}$$

where the sum is over all non-negative integers i_j satisfying $\sum_{j=1}^{k} j i_j = k$.

(ii) an alternative explicit formula that makes clear the link between $\chi_{(k)}$ and the symmetric group $\mathbf{S}_k$ (GAP Reference Manual [11, p. 770])

$$\chi_{(k)}(g) \;=\; \frac{1}{k!}\sum_{\sigma\in\mathbf{S}_k}\prod_{n=1}^{k}(\chi(g^n))^{c_n(\sigma)} \tag{4.3}$$

where $c_n(\sigma)$ is the number of cycles of length n in σ.

5. Hilbert-Poincaré Series and Molien Formulas

In this section we prove the integral formulas for the bigraded Hilbert-Poincaré Series of $\vec{\mathcal{P}}_{V^{\mathbf{R}}}(G)$ and the special case when the representation is of real type. We use the symbol $\int_G$ to denote the normalised Haar integral.

5.1. Proof of Theorem 2.5

Recall that taking the $z, \overline{z}$ coordinates, we obtain that any polynomial f on $V^{\mathbf{R}}$ can be written as a linear combination of monomials which are products of powers of the complex coordinates z_j and $\overline{z}_j$, that is,

$$f(z,\overline{z}) \;=\; \sum_{\alpha,\beta} a_{\alpha\beta} z^{\alpha}\overline{z}^{\beta}$$

where $\alpha, \beta \in (\mathbf{Z}_0^+)^m$, $z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_m^{\alpha_m}$ and the coefficients $a_{\alpha\beta}$ may be required to be complex. On the other hand, f can be decomposed as

$$f(z,\overline{z}) \;=\; \sum_{k\geq 0} f_k(z,\overline{z})$$

where

$$f_k(z,\overline{z}) \;=\; \textstyle\sum_{r+s=k} f_{r,s}(z,\overline{z})$$

$$f_{r,s}(z,\overline{z}) \;=\; \textstyle\sum_{|\alpha|=r,\,|\beta|=s} a_{\alpha\beta}\, z^{\alpha}\overline{z}^{\beta}\,.$$

Here, the polynomial function $f_{r,s}(z,\overline{z})$ is homogeneous of degree k (if $r+s=k$) and of *bidegree* (r,s).

Now we define a G-equivariant homogeneous polynomial mapping of bidegree (r,s) as a G-equivariant homogeneous polynomial mapping whose components are homogeneous polynomial functions of bidegree (r,s). Let $\vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}}}(G)$ be the vector space of G-equivariant homogeneous polynomial mappings on $V^{\mathbf{R}}$ of bidegree (r,s). Then we have a decomposition into a direct sum of G-invariant subspaces

$$\vec{\mathcal{P}}^{k}_{V^{\mathbf{R}}}(G) \;=\; \bigoplus_{k=r+s} \vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}}}(G)\,.$$

As before we denote by $\dim_{\mathbf{C}} \vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}}}$ the complex dimension of the space $\vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}},V}$ of V-valued mappings on $V^{\mathbf{R}}$ with polynomial components homogeneous of bidegree (r,s). We have

$$\dim_{\mathbf{C}}(S^rV\otimes S^s\overline{V})^*\otimes V \;=\; \dim_{\mathbf{C}} \vec{\mathcal{P}}^{r,s}_{V^{\mathbf{R}}}\,.$$

The action of the group G on V induces an action of G on $(S^rV\otimes S^s\overline{V})^*\otimes V$ and therefore the G-equivariant mappings from $V^{\mathbf{R}}$ to V with polynomial components homogeneous of bidegree (r,s) are in one-to-one correspondence with the G-invariant elements of $(S^rV\otimes S^s\overline{V})^*\otimes V$ under this induced action. By the Trace Formula we have

$$\dim_{\mathbf{C}}\operatorname{Fix}\big(G,(S^rV\otimes S^s\overline{V})^*\otimes V\big) \;=\; \int_G \overline{\chi}_{(r,s)}(g)\chi(g) \;=\; \int_G \chi_{(r,s)}(g)\overline{\chi}(g)$$

where $\chi_{(r,s)}$ is the character of the induced action of G on $S^rV\otimes S^s\overline{V}$.

The rest of the proof consists in calculating the character $\chi_{(r,s)}$ and we follow [9]. Fix $g\in G$ and as before let g denote the linear transformation corresponding to the action of $g\in G$ on V. Since g is an unitary matrix it can be diagonalised. Suppose that V has complex dimension m, and let $w_1,\ldots,w_m$ be a basis of V consisting of eigenvectors of g, with eigenvalues $\lambda_1,\ldots,\lambda_m$. The monomials $z^\alpha\overline{z}^\beta$ where $|\alpha|=r$ and $|\beta|=s$ form a basis of the space of homogeneous polynomials on $V\oplus\overline{V}$ of bidegree (r,s). Moreover, they correspond to the eigenvectors associated with the eigenvalues $\lambda^\alpha\overline{\lambda}^\beta$ of the induced action of G. Here we use multi-index notation for λ and $\overline{\lambda}$. Recall that $\chi_{(r,s)}$ is the character of the representation of G on the space of polynomials on $V\oplus\overline{V}$ of bidegree (r,s). We obtain

$$\chi_{(r,s)}(g) \;=\; \sum_{|\alpha|=r,\,|\beta|=s} \lambda^\alpha\overline{\lambda}^\beta$$

In what follows, we shall use $z\lambda$ to denote $(z\lambda_1,\ldots,z\lambda_m)$ and $\overline{z\,\lambda}$ to denote $\big(\overline{z\,\lambda_1},\ldots,\overline{z\,\lambda_m}\big)$. Multiplying by $z^r\overline{z}^s$ and summing over r and s, we obtain the formal power series

$$\begin{aligned}
\sum_{r,s=0}^{\infty}\chi_{r,s}(g)z^r\overline{z}^s \;&=\; \sum_{r,s=0}^{\infty}\sum_{|\alpha|=r,\,|\beta|=s}(z\lambda)^\alpha\left(\overline{z\,\lambda}\right)^\beta\\
&=\; \prod_{j=1}^{m}\frac{1}{(1-z\lambda_j)}\prod_{j=1}^{m}\frac{1}{\left(1-\overline{z\,\lambda_j}\right)}\\
&=\; \frac{1}{\det(1-zg)}\frac{1}{\det\left(1-\overline{z\,g}\right)}\,.
\end{aligned}$$

Finally, multiplying by $\overline{\chi}$ and using the Trace Formula we obtain the result. $\square$

5.2. Proof of Theorem 2.6

Given $(g,\theta) \in G \times \mathbf{S}^1$ and recalling (2.3), we have

$$\det(1-(g,\theta)z) \;=\; \det\left(1-g\left(e^{i\theta}z\right)\right)$$

and

$$\chi\big((g,\theta)^{-1}\big) \;=\; e^{-i\theta}\chi\left(g^{-1}\right) \;.$$

Applying Theorems 2.4, 2.5 and using the fact that the normalised Haar measure on the circle group $\mathbf{S}^1$ is $\frac{1}{2\pi}\mathrm{d}\theta$, we obtain the formulas of Theorem 2.6. □

6. Applications

In this section we present several applications of our formulas to finite groups and continuous groups. In the case of finite groups we illustrate the use of the computer algebra package GAP [11] and obtain some general results about invariants and equivariants of $S_N \times \mathbf{S}^1$. In the continuous group case we show that it is possible to explicitly evaluate the integrals appearing in the character formulas and in the Hilbert-Poincaré series for the groups $\mathbf{SO}(3)$ and $\mathbf{O}(3)$. The symbol $\int_G$ is used to denote the normalised Haar integral.

Let us state some useful facts that will be used to simplify calculations in the rest of the paper.

Proposition 6.1. *Let U be a real representation of G with corresponding character χ and $V = U \otimes_{\mathbf{R}} \mathbf{C}$ the representation of $G \times \mathbf{S}^1$ given by* (2.3)*. Then for each nontrivial real linear character λ of G the dimensions of the spaces of invariants and equivariants of the representation corresponding to $\lambda\chi$ are equal to the dimensions of the spaces of invariants and equivariants of the representation corresponding to χ.*

Proof. Let λ be a nontrivial linear character of G and let $\phi = \lambda\chi$. Let $k \geq 1$. Since λ is a linear character, we have

$$\phi_{(k)}(g) \;=\; \lambda(g)^k \chi_{(k)}(g).$$

Thus

$$\begin{aligned} \phi^2_{(k)} \;&=\; \lambda^{2k}\chi^2_{(k)} \;=\; \chi^2_{(k)}, \\ \phi_{(k+1)}\,\phi_{(k)}\,\phi \;&=\; \lambda^{2k+2}\chi_{(k+1)}\,\chi_{(k)}\,\chi \;=\; \chi_{(k+1)}\,\chi_{(k)}\,\chi \end{aligned}$$

since $\lambda(g) = \pm 1$ and so λ^{2k} is the trivial character. Hence we have shown that

$$\begin{aligned} \int_G \chi_{(k)}(g)^2 \;&=\; \int_G \phi_{(k)}(g)^2 \;, \\ \int_G \chi_{(k+1)}(g)\,\chi_{(k)}(g)\,\chi(g) \;&=\; \int_G \phi_{(k+1)}(g)\,\phi_{(k)}(g)\,\phi(g) \,. \end{aligned}$$

Remark 6.2. A number of well known results regarding the number of invariants and equivariants in the absolutely (and nontrivial) irreducible case follow from

Class	1	(12)	(123)	(12)(34)	(1234)
|Class|	1	6	8	3	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

Table 1. *Character table of S_4. The rows are indexed by the irreducible characters of S_4 and the columns are indexed by the conjugacy class representatives.*

Class	1	(12)	(123)	(12)(34)	(1234)
|Class|	1	6	8	3	6
$\chi(g)$	3	1	0	-1	-1
$\chi(g^2)$	3	3	0	3	-1
$\chi(g^3)$	3	1	3	-1	-1
$\chi(g^4)$	3	3	0	3	3
$\chi_{(2)}(g)$	6	2	0	2	0
$\chi_{(3)}(g)$	10	2	1	-2	0
$\chi_{(4)}(g)$	15	3	0	3	1

Table 2. *Irreducible character $\chi(g)$ and derived characters $\chi(g^k)$ and $\chi_{(k)}(g)$ for the group S_4, for the natural character labelled χ_4 in Table 1.*

Theorem 2.1 and the properties of characters (3.8). There can be no invariant of degree 1, and there is only one equivariant of first degree, which is simply the identity mapping. There is a unique independent quadratic invariant, which in the orthogonal case is $\sum_{k=1}^{m} x_k^2$. This last result follows from the fact that $\int_G \chi_{(2)}(g) = 1$, which in turn follows from the Frobenius-Schur theorem.

6.1. Finite Groups

6.1.1. *Computer Calculations*

We first go through an example of the calculation of the dimensions of the spaces of invariants and equivariants using characters for a particular group action, and then summarise the results obtained computationally for several other groups.

Consider the permutation group S_4, which has five conjugacy classes containing elements of the same cycle type. The character table for S_4 is given in Table 1. The group acts on $\mathbf{R}^3$ via the 'natural' irreducible representation in which the character $\chi(g)$ for each class is obtained by subtracting one from the number of elements fixed by each permutation; this character is denoted by χ_4 in Table 1. From this character we can find the characters $\chi(g^k)$ and $\chi_{(k)}(g)$. These are listed in Table 2 for $k \leq 4$. From the information in that table it is then possible to calculate the dimensions of the spaces of invariants and equivariants for stationary and Hopf bifurcation, by finding the appropriate sums using Theorems 2.1 and 2.2 respectively.

Since χ is irreducible of real type there is only one quadratic invariant (we refer to 'number of invariants' as an abbreviation for the dimension of the space of invariants). From the first formula of (2.1), the number of cubic invariants is

$$I(3) \;=\; \frac{1}{24}\sum_g \chi_{(3)}(g) \;=\; \frac{1}{24}(10-6+12+8) \;=\; 1$$

and the number of quartic invariants is

$$I(4) \;=\; \frac{1}{24}\sum_g \chi_{(4)}(g) \;=\; \frac{1}{24}(15+6+9+18) \;=\; 2\;.$$

The number of quadratic equivariants is, using the second formula in Theorem 2.1,

$$E(2) \;=\; \frac{1}{24}\sum_g \chi_{(2)}(g)\chi(g) \;=\; \frac{1}{24}(18-6+12) \;=\; 1$$

and the numbers of cubic and quartic equivariants are

$$E(3) \;=\; \frac{1}{24}\sum_g \chi_{(3)}(g)\chi(g) \;=\; 2, \qquad E(4) \;=\; \frac{1}{24}\sum_g \chi_{(4)}(g)\chi(g) \;=\; 2\;.$$

For the case of Hopf bifurcation, there is one quadratic invariant and the numbers of invariants of degree 4, 6, 8 are found from (2.5) to be

$$I_H(4) \;=\; \frac{1}{24}\sum_g \chi_{(2)}(g)^2 \;=\; 3, \quad I_H(6) \;=\; \frac{1}{24}\sum_g \chi_{(3)}(g)^2 \;=\; 6,$$

$$I_H(8) \;=\; \frac{1}{24}\sum_g \chi_{(4)}(g)^2 \;=\; 13\;.$$

The numbers of equivariants of degree 3, 5, 7 are, using (2.6),

$$\begin{aligned} E_H(3) \;&=\; \frac{1}{24}\sum_g \chi_{(2)}(g)\chi(g)^2 \;=\; 3\,,\\ E_H(5) \;&=\; \frac{1}{24}\sum_g \chi_{(3)}(g)\chi_{(2)}(g)\chi(g) \;=\; 9\,,\\ E_H(7) \;&=\; \frac{1}{24}\sum_g \chi_{(4)}(g)\chi_{(3)}(g)\chi(g) \;=\; 21\,. \end{aligned}$$

Note that the group S_4 has another three-dimensional irreducible representation in which the character $\chi_5(g)$ is the same as $\chi(g)$ except for a sign change of the elements (1234) and (12). See Table 1 where $\chi = \chi_4$ and $\chi_5 = \chi_2\chi_4$. Hence by Proposition 6.1, the numbers of invariants and equivariants for Hopf bifurcation in this representation are the same as those given above. Hopf bifurcation in this representation, which arises from the symmetries of rotations of the cube, was investigated by Ashwin and Podvigina [1], who also pointed out this equivalence.

G	D	$I(3)$	$I(4)$	$I(5)$	$E(2)$	$E(3)$	$E(4)$	$E(5)$	$I_H(4)$	$I_H(6)$	$I_H(8)$	$E_H(3)$	$E_H(5)$	$E_H(7)$
S_3	2	1	1	1	1	1	2	2	2	3	5	2	4	7
S_4	3	1	2	1	1	2	2	4	3	6	13	3	9	21
S_5	4	1	2	2	1	2	3	4	3	7	19	3	11	33
S_6	5	1	2	2	1	2	3	5	3	8	24	3	12	41
A_4	3	1	2	1	1	3	3	6	4	10	21	5	16	39
A_5	3	0	1	0	0	1	0	2	2	3	6	2	4	9
A_5	4	1	2	2	1	2	3	4	3	8	24	3	14	48
A_5	5	2	2	4	2	3	8	12	6	24	92	7	46	210
D_4	2	0	2	0	0	2	0	3	3	4	7	3	6	10
D_5	2	0	1	1	0	1	1	1	2	2	3	2	3	4
D_6	2	0	1	0	0	1	0	2	2	3	5	2	4	7

Table 3. *Dimensions of vector spaces of invariants $I(k)$ and equivariants $E(k)$ of degree k for stationary bifurcation, and invariants $I_H(k)$ and equivariants $E_H(k)$ for Hopf bifurcation, for several symmetric, alternating and dihedral groups. D denotes the dimension of the irreducible representation.*

In Table 3 we show the numbers of invariants and equivariants for stationary and Hopf bifurcation for a number of finite groups. These results were obtained by adapting an existing computer program originally written to obtain isotropy subgroups using characters and trace formulas, see Matthews [22]. The program is written in the GAP [11] language, where $\chi_{(k)}$ can be found from χ with a single command and the formulas in (2.1), (2.5) and (2.6) can be implemented as inner products.

For the dihedral groups D_n, for $n = 4, 5, 6$, we consider the standard irreducible representation of dimension 2 where the n-cycle $(12 \ldots n)$ acts as rotation through $2\pi/n$.

For the alternating group A_5, which is isomorphic to the group I of rotations of the icosahedron, there are unique faithful irreducible representations of dimension 3, 4 and 5, up to quasi-equivalence, that is, equivalence composed with an outer automorphism of A_5. From Table 3 we can see that for stationary bifurcation, $E(2) = I(3)$ and $E(3) = I(4)$ in the 3- and 4-dimensional representations but not in the 5-dimensional one. This means that the quadratic and cubic terms are variational in the 3- and 4-dimensional representations but not in the 5-dimensional one. This is consistent with the work of Hoyle [19] who found heteroclinic cycles in the cubic truncation for the 5-dimensional representation.

For the symmetric groups S_n we consider the natural irreducible representation of dimension $n-1$, as in the example above. In this case it is known that for the stationary bifurcation, for $n > 3$ there is one equivariant quadratic and two equivariant cubic terms [7,15]. The dynamics truncated to cubic order is variational, but since $E(4) > I(5)$ for $n = 5, 6$ the quartic equivariants are non-variational. For the case of the Hopf bifurcation with S_n symmetry we see that there are three cubic equivariants for $n = 4, 5, 6$. In fact we can show that the number of cubic equivariants is three for all $n \geqslant 4$; the proof of this will be given in the following section.

6.1.2. *Invariants and Equivariants of $S_N \times \mathbf{S}^1$*

We consider the standard action of $S_N \times \mathbf{S}^1$ on $\mathbf{C}^N$ given by

$$\begin{aligned}\sigma(z_1,\ldots,z_N) &= \left(z_{\sigma^{-1}(1)},\ldots,z_{\sigma^{-1}(N)}\right) ,\\ \theta(z_1,\ldots,z_N) &= \left(e^{i\theta}z_1,\ldots,e^{i\theta}z_N\right) ,\end{aligned}$$

for $\sigma \in \mathbf{S}_N$, $\theta \in \mathbf{S}^1$ and $z = (z_1,\ldots,z_N) \in \mathbf{C}^N$. This action is obtained by the complexification of the standard action of S_N on $\mathbf{R}^N$. Observe that, if we denote by

$$\begin{aligned}V_0 &= \{(z,\ldots,z) : \ z \in \mathbf{C}\} ,\\ \mathbf{C}^{N,0} &= \{(z_1,\ldots,z_N) \in \mathbf{C}^N : \ z_1+\cdots+z_N = 0\}\end{aligned}$$

then

$$\mathbf{C}^N = V_0 \oplus \mathbf{C}^{N,0}$$

where V_0 and $\mathbf{C}^{N,0}$ are irreducible subspaces under the $S_N \times \mathbf{S}^1$-action. Moreover, S_N acts trivially on V_0 and S_N-simply on $\mathbf{C}^{N,0}$.

Remark 6.3. It is known that the S_N-equivariant mappings on $\mathbf{C}^N$ have a nice characterisation. Consider the natural inclusion of groups

$$S_{N-1} = \{\sigma \in S_N : \ \sigma(1) = 1\} \subset S_N .$$

A mapping $g : \mathbf{C}^N \to \mathbf{C}^N$ is S_N-equivariant if and only if

$$g(z) = \big(g_1(z), g_1((12)\,z),\ldots,g_1((1N)\,z)\big)^{\mathrm{T}} \tag{6.1}$$

where $g_1 : \mathbf{C}^N \to \mathbf{C}$ is S_{N-1}-invariant in the last $N-1$-coordinates. Combining with the $\mathbf{S}^1$-action, it follows that g is $S_N \times \mathbf{S}^1$-equivariant if and only if in addition, g_1 satisfies

$$g_1(e^{i\theta}z) = e^{i\theta}g_1(z) \tag{6.2}$$

for all $z \in \mathbf{C}^N$ and $\theta \in \mathbf{S}^1$. $\diamond$

In order to study codimension-one Hopf bifurcation with S_N-symmetry, for the above S_N-simple action, one needs (at least) the most general cubic $S_N \times \mathbf{S}^1$-equivariant normal form. We start by calculating the number of cubic $S_N \times \mathbf{S}^1$-equivariants for the action on $\mathbf{C}^N$ and then on $\mathbf{C}^{N,0}$.

Proposition 6.4. *Consider the standard action of $S_N \times \mathbf{S}^1$ on $\mathbf{C}^N$. Then*

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^N}(S_N \times \mathbf{S}^1) = \begin{cases} 10 & \text{if } N = 3,\\ 11 & \text{if } N \geqslant 4.\end{cases}$$

Proof. By Remark 6.3 we have that $\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^N}(S_N \times \mathbf{S}^1)$ is equal to the number of the linearly independent polynomial functions $g_1 : \mathbf{C}^N \to \mathbf{C}$ of degree 3 that are S_{N-1}-invariant and satisfy (6.2). Thus

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^N}(S_N \times \mathbf{S}^1) = \dim_{\mathbf{C}} \mathrm{Fix}\left(S_{N-1}, \left(S^2\left(\mathbf{C}^N\right)\right)^* \otimes_{\mathbf{C}} \overline{\mathbf{C}^N}^*\right) .$$

By the Trace Formula we have

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^N}(S_N \times \mathbf{S}^1) \;=\; \int_{S_{N-1}} \chi_{\mathbf{C}^N}\, \chi_{\mathbf{C}^N,(2)}$$

where $\chi_{\mathbf{C}^N}$, $\chi_{\mathbf{C}^N,(2)}$ are the characters for the representations of S_{N-1} on $\mathbf{C}^N$ and $S^2(\mathbf{C}^N)$, respectively. Let

$$\begin{aligned} V_1 &= \{(z_1, z_2, \ldots, z_2) : \; z_1, z_2 \in \mathbf{C}\}\,, \\ V_2 &= \{(0, z_2, \ldots, z_N) \in \mathbf{C}^N : \; z_2 + \cdots + z_N = 0\}\,. \end{aligned}$$

It follows then that

$$\mathbf{C}^N \;=\; V_1 \oplus V_2$$

and S_{N-1} acts trivially on V_1 and irreducibly on V_2. By the formula for the decomposition of the symmetric tensor power (3.1) and the fact that $S^1(V_i) \cong V_i$ $(i = 1, 2)$, we have the following decomposition into S_N-modules:

$$S^2(\mathbf{C}^N) \;=\; S^2(V_1 \oplus V_2) \;\cong\; S^2(V_1) \oplus (V_1 \otimes V_2) \oplus S^2(V_2)\,.$$

Denote by χ_{V_2} the character of the (irreducible) representation of S_{N-1} on V_2 and $\chi_{V_2,(2)}$ the character of the induced representation of S_{N-1} on $S^2(V_2)$. Note that V_1 is two-dimensional and so $S^2(V_1)$ is three-dimensional. Moreover, S_{N-1} acts trivially on V_1 and $S^2(V_1)$. It follows then that

$$\chi_{\mathbf{C}^N} \;=\; 2 + \chi_{V_2}\,, \qquad \chi_{\mathbf{C}^N,(2)} \;=\; 3 + 2\chi_{V_2} + \chi_{V_2,(2)}\,. \tag{6.3}$$

Now

$$\begin{aligned} \chi_{\mathbf{C}^N}\, \chi_{\mathbf{C}^N,(2)} &= \big(2 + \chi_{V_2}\big)\big(3 + 2\chi_{V_2} + \chi_{V_2,(2)}\big) \\ &= 6 + 7\chi_{V_2} + 2\chi_{V_2,(2)} + 2\chi^2_{V_2} + \chi_{V_2}\, \chi_{V_2,(2)}\,. \end{aligned}$$

Since, $2\,\chi_{V_2,(2)}(g) = \chi_{V_2}(g^2) + \chi_{V_2}(g)^2$ for all $g \in S_{N-1}$ and $\int_{S_{N-1}} \chi_{V_2}(g^2) = 1$, it follows that

$$\begin{aligned} \int_{S_{N-1}} \chi_{\mathbf{C}^N}\, \chi_{\mathbf{C}^N,(2)} &= \int_{S_{N-1}} \big(6 + 7\chi_{V_2} + 3\chi^2_{V_2} + \chi_{V_2}(g^2) + \chi_{V_2}\, \chi_{V_2,(2)}\big) \\ &= 6\langle \mathbf{1}, \mathbf{1}\rangle + 7\langle \chi_{V_2}, \mathbf{1}\rangle + 3\langle \chi_{V_2}, \chi_{V_2}\rangle + \int_{S_{N-1}} \chi_{V_2}(g^2) + \langle \chi_{V_2}, \chi_{V_2,(2)}\rangle \\ &= 10 + \langle \chi_{V_2}, \chi_{V_2,(2)}\rangle\,. \end{aligned}$$

Here $\mathbf{1}$ denotes the trivial irreducible character of S_{N-1}. Observe that

$$\langle \chi_{V_2}, \chi_{V_2,(2)}\rangle \;=\; \int_{S_{N-1}} \chi_{V_2}\, \chi_{V_2,(2)}$$

and so by Theorem 2.1, $\langle \chi_{V_2}, \chi_{V_2,(2)}\rangle$ is equal to the number of quadratic equivariants for the standard irreducible representation of S_{N-1} of dimension $N-2$. Thus if $N \geq 4$ we have $\langle \chi_{V_2}, \chi_{V_2,(2)}\rangle = 1$. It follows then that

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^N}\big(S_N \times \mathbf{S}^1\big) \;=\; \int_{S_{N-1}} \chi_{\mathbf{C}^N}\, \chi_{\mathbf{C}^N,(2)} \;=\; 11$$

for $N \geq 4$. Finally, if $N = 3$, it follows that $V_2 = \{(0, z_2, -z_2) : \ z_2 \in \mathbf{C}\}$ is one-dimensional and so the action of $\mathbf{S}_2$ on V_2 is irreducible and nontrivial, and trivial on $S^2(V_2)$. Thus in this case we have $\langle \chi_{V_2}, \chi_{V_2,(2)} \rangle = 0$ and so

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^3} \left(\mathbf{S}_3 \times \mathbf{S}^1\right) \;=\; \int_{\mathbf{S}_2} \chi_{\mathbf{C}^3}\, \chi_{\mathbf{C}^3,(2)} \;=\; 10 + \langle \chi_{V_2}, \chi_{V_2,(2)} \rangle \;=\; 10\,.$$

□

Theorem 6.5. *Consider the action of $S_N \times \mathbf{S}^1$ on $\mathbf{C}^{N,0}$ obtained by restriction of the standard action on $\mathbf{C}^N$. Then*

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^{N,0}}(S_N \times \mathbf{S}^1) \;=\; \begin{cases} 2 & \text{if } N = 3, \\ 3 & \text{if } N \geqslant 4. \end{cases}$$

Proof. By definition

$$\begin{aligned} \dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^N} \left(S_N \times \mathbf{S}^1\right) \;&=\; \dim_{\mathbf{C}} \mathrm{Fix}\Big(S_N, \left(S^2(\mathbf{C}^N)\right)^* \otimes \left(S^1(\overline{\mathbf{C}^N})\right)^* \otimes \mathbf{C}^N\Big) \\ &=\; \dim_{\mathbf{C}} \mathrm{Fix}\Big(S_N, \left(S^2(V_0 \oplus \mathbf{C}^{N,0})\right)^* \otimes \left(S^1(\overline{V_0 \oplus \mathbf{C}^{N,0}})\right)^* \otimes \left(V_0 \oplus \mathbf{C}^{N,0}\right)\Big) \end{aligned}$$

By the formula for the decomposition of the symmetric tensor power (3.1) and the fact that $S^1(V) \cong V$, we have the following decomposition into S_N-modules:

$$S^2(V_0 \oplus \mathbf{C}^{N,0}) \;\cong\; S^2(V_0) \oplus \left(V_0 \otimes S^1(\mathbf{C}^{N,0})\right) \oplus S^2(\mathbf{C}^{N,0})\,.$$

Note that V_0 is one-dimensional and the action of S_N on V_0 and $S^2(V_0)$ is trivial. Denote by $\chi_{\mathbf{C}^{N,0}}$ and $\chi_{\mathbf{C}^{N,0},(2)}$ the characters of the representations of S_N on $\mathbf{C}^{N,0}$ and $S^2(\mathbf{C}^{N,0})$, respectively. By the Trace Formula we have

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^N}(S_N \times \mathbf{S}^1) \;=\; \int_{S_N} (\mathbf{1} + \chi_{\mathbf{C}^{N,0}} + \chi_{\mathbf{C}^{N,0},(2)})(\mathbf{1} + \chi_{\mathbf{C}^{N,0}})^2\,.$$

Now we compute

$$\begin{aligned} &\int_{S_N} (\mathbf{1} + \chi_{\mathbf{C}^{N,0}} + \chi_{\mathbf{C}^{N,0},(2)})(\mathbf{1} + \chi_{\mathbf{C}^{N,0}})^2 \\ &\qquad =\; \langle \mathbf{1}, \mathbf{1} \rangle + 3\langle \chi_{\mathbf{C}^{N,0}}, \mathbf{1} \rangle + 3\langle \chi_{\mathbf{C}^{N,0}}, \chi_{\mathbf{C}^{N,0}} \rangle + \langle \chi_{\mathbf{C}^{N,0},(2)}, \mathbf{1} \rangle \\ &\qquad\quad + 2\int_{S_N} \chi_{\mathbf{C}^{N,0},(2)}\, \chi_{\mathbf{C}^{N,0}} + \int_{S_N} \chi^3_{\mathbf{C}^{N,0}} + \int_{S_N} \chi_{\mathbf{C}^{N,0},(2)} \chi^2_{\mathbf{C}^{N,0}} \\ &\qquad =\; 7 + \int_{S_N} \chi^3_{\mathbf{C}^{N,0}} + \int_{S_N} \chi_{\mathbf{C}^{N,0},(2)}\, \chi^2_{\mathbf{C}^{N,0}}\,. \end{aligned}$$

Here we used the orthogonality of irreducible characters, the fact that $\chi_{\mathbf{C}^{N,0}}$ is realised over $\mathbf{R}$ and so $\langle \chi_{\mathbf{C}^{N,0},(2)}, \mathbf{1} \rangle = 1$ and $\int_{S_N} \chi_{\mathbf{C}^{N,0},(2)} \chi_{\mathbf{C}^{N,0}}$ is the number of quadratic S_N-equivariants which is 1 if $N \geq 3$ (see [7, 15]). By formula (2.6) of Theorem 2.2

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^{N,0}}(S_N \times \mathbf{S}^1) \;=\; \int_{S_N} \chi_{\mathbf{C}^{N,0},(2)} \chi^2_{\mathbf{C}^{N,0}}\,.$$

On the other hand, $\chi^3_{\mathbf{C}^N} = (\mathbf{1} + \chi_{\mathbf{C}^{N,0}})^3$ and hence

$$\begin{aligned}\int_{S_N} \chi^3_{\mathbf{C}^N} &= \int_{S_N} (\mathbf{1} + \chi_{\mathbf{C}^{N,0}})^3 \\ &= \langle \mathbf{1}, \mathbf{1}\rangle + 3\langle \chi_{\mathbf{C}^{N,0}}, \mathbf{1}\rangle + 3\langle \chi_{\mathbf{C}^{N,0}}, \chi_{\mathbf{C}^{N,0}}\rangle + \int_{S_N} \chi^3_{\mathbf{C}^{N,0}} \\ &= 4 + \int_{S_N} \chi^3_{\mathbf{C}^{N,0}}\,.\end{aligned}$$

Therefore

$$\dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^N}(S_N \times \mathbf{S}^1) - \dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{\mathbf{C}^{N,0}}(S_N \times \mathbf{S}^1) = \int_{S_N} \chi^3_{\mathbf{C}^N} + 3\,.$$

We finish the proof by computing $\int_{S_N} \chi^3_{\mathbf{C}^N}$. By the Trace Formula

$$\int_{S_N} \chi^3_{\mathbf{C}^N} = \dim_{\mathbf{C}} \mathrm{Fix}\left(S_N, \left(S^1(\mathbf{C}^N) \otimes S^1(\overline{\mathbf{C}^N})\right)^* \otimes \mathbf{C}^N\right)\,.$$

That is, $\int_{S_N} \chi^3_{\mathbf{C}^N}$ is equal to the number of quadratic S_N-equivariants g on $\mathbf{C}^N$ satisfying $\theta g(z) = g(z)$ for all $\theta \in \mathbf{S}^1$ and $z \in \mathbf{C}^N$. By Remark 6.3 we have that $g : \mathbf{C}^N \to \mathbf{C}^N$ is S_N-equivariant and satisfies $\theta g(z) = g(z)$ if and only if $g(z) = (g_1(z), g_1((12)\,z), \cdots, g_1((1N)\,z))^{\mathrm{T}}$ where $g_1 : \mathbf{C}^N \to \mathbf{C}$ is S_{N-1}-invariant in the last $N-1$ coordinates and $g_1(\theta z) = g_1(z)$ for all $\theta \in \mathbf{S}^1$ and $z \in \mathbf{C}^N$. Then using the same notation as in Proposition 6.4 (recall (6.3)), we have

$$\begin{aligned}\int_{S_N} \chi^3_{\mathbf{C}^N} &= \dim_{\mathbf{C}} \mathrm{Fix}\left(S_{N-1}, (S^1(\mathbf{C}^N))^* \otimes (S^1(\overline{\mathbf{C}^N}))^*\right) \\ &= \int_{S_{N-1}} (2 + \chi_{V_2})^2 = 5\,.\end{aligned}$$

Now the result follows from Proposition 6.4. □

Remark 6.6. See Rodrigues [25] for the explicit calculation of cubic equivariants under the above actions of $S_N \times \mathbf{S}^1$.

6.2. Continuous Groups

6.2.1. *Character Formulas for* $\mathbf{O}(3)$ *and* $\mathbf{SO}(3)$

We apply our results to the calculation of the dimensions of the spaces of invariants and equivariants for the groups $\mathbf{O}(3)$ and $\mathbf{SO}(3)$. Bifurcation with these symmetry groups is a topic of considerable interest [4, 6, 18, 20, 26].

We first recall some facts about these groups and their representations, see [16, XIII, §7] for details. For each $l \geqslant 0$ there is only one (absolutely) irreducible representation of $\mathbf{SO}(3)$ of dimension $2l+1$ denoted by V_l. Each of these spaces carry two representations of $\mathbf{O}(3)$ called *plus* and *minus* representations: on the first one $-I$ acts trivially and on the second $-I$ acts non-trivially. In applications, the usual way that $\mathbf{O}(3)$ acts is induced from the natural action on $\mathbf{R}^3$. This leads to the representation *plus* for l even and *minus* for l odd which is called *natural representation* of $\mathbf{O}(3)$ on V_l.

The character afforded by the irreducible representation of $\mathbf{SO}(3)$ on V_l is given by

$$\chi_l(R_\theta) = \sum_{m=-l}^{l} e^{im\theta} = 1 + 2\sum_{m=1}^{l}\cos(m\theta) = \frac{\cos(l\theta) - \cos((l+1)\theta)}{1-\cos(\theta)} \tag{6.4}$$

where $\theta \in [0,\pi]$ parametrises the conjugacy classes of $\mathbf{SO}(3)$ and represents the rotation R_θ. The Haar integral of a class function f on $\mathbf{SO}(3)$ is (see Wigner [29, p. 156])

$$\frac{1}{\pi}\int_0^\pi f(R_\theta)(1-\cos\theta)\,\mathrm{d}\theta\ . \tag{6.5}$$

The conjugacy classes of $\mathbf{O}(3)$ are also parametrised by $\theta \in [0,\pi]$, however there are two classes for each θ: one class is represented by the rotation R_θ and the other is represented by $-I \circ R_\theta = -R_\theta$. In this case the Haar integral of a class function f on $\mathbf{O}(3)$ is

$$\frac{1}{2\pi}\int_0^\pi [f(R_\theta) + f(-R_\theta)](1-\cos\theta)\,\mathrm{d}\theta\ . \tag{6.6}$$

Observe that for any representation of $\mathbf{O}(3)$ on V_l we have that

$$\begin{aligned} \dim_{\mathbf{R}} \mathcal{P}^{2k}_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1) &= \dim_{\mathbf{R}} \mathcal{P}^{2k}_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{SO}(3)\times\mathbf{S}^1) \\ \dim_{\mathbf{C}} \vec{\mathcal{P}}^{2k+1}_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1) &= \dim_{\mathbf{C}} \vec{\mathcal{P}}^{2k+1}_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{SO}(3)\times\mathbf{S}^1)\ . \end{aligned} \tag{6.7}$$

To see this note the following. Let χ_l be the character of an irreducible representation V_l of $\mathbf{O}(3)$. If $-I$ acts trivially on V_l and $k \geq 1$, we have

$$\chi_{l,(k)}(R_\theta) \ = \ \chi_{l,(k)}(-R_\theta)$$

and so

$$\int_{\mathbf{O}(3)} \chi^2_{l,(k)} \ = \ \int_{\mathbf{SO}(3)} \chi^2_{l,(k)}, \quad \int_{\mathbf{O}(3)} \chi_{l,(k+1)}\,\chi_{l,(k)}\,\chi \ = \ \int_{\mathbf{SO}(3)} \chi_{l,(k+1)}\,\chi_{l,(k)}\,\chi.$$

By Theorem 2.2 we have the equalities (6.7).

Now, if $-I$ acts non-trivially on V_l, then the function $\lambda : \mathbf{O}(3) \to \mathbf{R}$ defined by

$$\lambda(g) \ = \ \begin{cases} 1 & \text{if} \quad g \in \mathbf{SO}(3), \\ -1 & \text{if} \quad g \in \mathbf{O}(3) \setminus \mathbf{SO}(3), \end{cases}$$

is a linear character of $\mathbf{O}(3)$. Moreover, we have that $\lambda\chi_l$ is an irreducible character of $\mathbf{O}(3)$ where $-I$ acts trivially on V_l. Also, $\lambda\chi_l(g) = \chi_l(g)$ for $g \in \mathbf{SO}(3)$. By Proposition 6.1 and the above observation we have the equalities (6.7).

Remark 6.7. Sattinger [26] proved that for $\mathbf{SO}(3)$-symmetric steady-state bifurcations posed on an absolutely irreducible space V_l, the quadratic terms vanish for odd l, and possess a gradient structure for even l. The gradient structure for the cubic truncation for $\mathbf{O}(3)$-symmetric steady-state bifurcation on V_l, for any $l \geq 1$ was proved by Michel. See Chossat *et al.* [6]. This can be proven in the following way. It is shown by Chossat and Lauterbach [4] that for $\mathbf{O}(3)$-symmetric steady-state bifurcation the number of cubic equivariants $E(3)$ is equal to $1 + [l/3]$. Using (6.4) and (6.5) we obtain the following expressions for $l \geq 1$:

$$\begin{aligned}\int_{\mathbf{SO}(3)} \chi_l^4(R_\theta) &= \frac{1}{\pi}\int_0^\pi \left[\sum_{m=-l}^{l} e^{im\theta}\right]^3 (\cos(l\theta) - \cos((l+1)\theta))\,\mathrm{d}\theta \\ &= 2l+1\end{aligned} \tag{6.8a}$$

$$\begin{aligned}\int_{\mathbf{SO}(3)} \chi_l^2(R_{2\theta}) &= \frac{1}{\pi}\int_0^\pi \left[\sum_{m=-l}^{l} e^{i2m\theta}\right]^2 \left(1 - \frac{e^{i\theta}}{2} - \frac{e^{-i\theta}}{2}\right) \mathrm{d}\theta \\ &= 2l+1\end{aligned} \tag{6.8b}$$

$$\int_{\mathbf{SO}(3)} \chi_l(R_{4\theta}) = \frac{1}{\pi}\int_0^\pi \sum_{m=-l}^{l} e^{i4m\theta} \left(1 - \frac{e^{i\theta}}{2} - \frac{e^{-i\theta}}{2}\right) \mathrm{d}\theta = 1 \tag{6.8c}$$

$$\begin{aligned}&\int_{\mathbf{SO}(3)} \chi_l^2(R_\theta)\chi_l(R_{2\theta}) = \\ &\quad = \frac{1}{\pi}\int_0^\pi \sum_{m=-l}^{l} e^{im\theta} \sum_{n=-l}^{l} e^{i2n\theta} (\cos(l\theta) - \cos((l+1)\theta))\,\mathrm{d}\theta = 1\end{aligned} \tag{6.8d}$$

$$\begin{aligned}\int_{\mathbf{SO}(3)} \chi_l(R_\theta)\chi_l(R_{3\theta}) &= \frac{1}{\pi}\int_0^\pi \sum_{m=-l}^{l} e^{i3m\theta} (\cos(l\theta) - \cos((l+1)\theta))\,\mathrm{d}\theta \\ &= 1 - l + 3\left[\frac{l}{3}\right]\end{aligned} \tag{6.8e}$$

It follows then by Theorem 2.1 that

$$I(4) = \int_{\mathbf{O}(3)} \chi_{l,(4)} = \int_{\mathbf{SO}(3)} \chi_{l,(4)} = 1 + \left[\frac{l}{3}\right].$$

Similarly, we can use formulas in (6.8) to verify that $E(3) = 1 + [l/3]$. That is, $E(3) = I(4) = 1 + [l/3]$. Thus cubic $\mathbf{O}(3)$-equivariants also have a gradient structure. $\diamond$

Now using our results we get a similar property for the case of Hopf bifurcation with $\mathbf{O}(3)$ symmetry.

Proposition 6.8. *Let $\mathbf{O}(3)$ act irreducibly on V_l and denote by χ_l the corresponding character. Then*

$$\dim_{\mathbf{R}} \mathcal{P}^4_{V_l \otimes_{\mathbf{R}} \mathbf{C}}(\mathbf{O}(3) \times \mathbf{S}^1) = \dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{V_l \otimes_{\mathbf{R}} \mathbf{C}}(\mathbf{O}(3) \times \mathbf{S}^1) = l + 1\,.$$

Proof. Direct computations using formulas in (6.8) give the results. We include an alternative proof using orthogonality of the characters of $\mathbf{O}(3)$ and some well known decomposition formulas for the symmetric and alternating tensor square of the representations V_l of $\mathbf{O}(3)$.
To prove the first equality, let

$$E_H(3) \;=\; \dim_{\mathbf{C}} \vec{\mathcal{P}}^3_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1)\,, \qquad I_H(4) \;=\; \dim_{\mathbf{R}} \mathcal{P}^4_{V_l\otimes_{\mathbf{R}}\mathbf{C}}(\mathbf{O}(3)\times\mathbf{S}^1)\,.$$

By Theorem 2.2 we have

$$E_H(3)-I_H(4) \;=\; \int_{\mathbf{O}(3)} \left(\chi_{l,(2)}\chi_l^2-\chi_{l,(2)}^2\right) \;=\; \langle\chi_{l,(2)},\chi_{l,[2]}\rangle$$

where $\chi_{l,[2]}$ denotes the character of $\mathbf{O}(3)$ on the antisymmetric tensor square $A^2(V_l)$ of V_l (and as before $\chi_{l,(2)}$ is the character of $\mathbf{O}(3)$ on the symmetric tensor square $S^2(V_l)$ of V_l). The relation between $\chi_{l,[2]}$ and $\chi_{l,(2)}$ is given by formula (3.8)(d) and takes the form

$$\chi_l^2 \;=\; \chi_{l,(2)}+\chi_{l,[2]}\ .$$

As

$$S^2(V_l) \;=\; \bigoplus_{a=0}^{l} V_{2l-2a} \;=\; V_{2l}\oplus V_{2l-2}\oplus\cdots\oplus V_0$$

(see for example [10, page 159]), it follows then that

$$\chi_{l,(2)} \;=\; \chi_{2l}+\chi_{2l-2}+\cdots+\chi_0\ .$$

Also,

$$A^2(V_l) \;=\; V_{2l-1}\oplus V_{2l-3}\oplus\cdots\oplus V_1$$

(see for example [10, page 160]), and so

$$\chi_{l,[2]} \;=\; \chi_{2l-1}+\chi_{2l-3}+\cdots+\chi_1\ .$$

Therefore $\langle\chi_{l,(2)},\chi_{l,[2]}\rangle \;=\; 0$.
To prove the second equality observe that by Theorem 2.2 we have

$$E_H(3) \;=\; \int_{\mathbf{O}(3)} \chi_{l,(2)}\chi_l^2 \;=\; \langle\chi_{l,(2)},\chi_l^2\rangle\,.$$

As χ_l^2 is the character of the $\mathbf{O}(3)$-module $V_l\otimes V_l$ and

$$V_l\otimes V_l \;=\; V_0\oplus V_1\oplus\cdots\oplus V_{2l}$$

(see for example [27, p. 138, Lemma 5.20]), we obtain

$$\chi_l^2 \;=\; \chi_0+\chi_l+\cdots+\chi_{2l}$$

and so

$$\langle\chi_{l,(2)},\chi_l^2\rangle \;=\; l+1\ .$$

□

l	$I(2)$	$I^+(3)$	$E^+(2)$	$E(3)$	$I_H(2)$	$I_H(4)$	$I_H(6)$	$E_H(3)$	$E_H(5)$
1	1	0	0	1	1	2	2	2	3
2	1	1	1	1	1	3	5	3	9
3	1	0	0	2	1	4	10	4	21
4	1	1	1	2	1	5	17	5	40
5	1	0	0	2	1	6	28	6	69
6	1	1	1	3	1	7	43	7	110
7	1	0	0	3	1	8	62	8	164
8	1	1	1	3	1	9	87	9	234
9	1	0	0	4	1	10	118	10	322
10	1	1	1	4	1	11	155	11	429

Table 4. *Dimensions of vector spaces of invariants $I(k)$ and equivariants $E(k)$ of degree k for stationary bifurcation, and invariants $I_H(k)$ and equivariants $E_H(k)$ for Hopf bifurcation, for the group $\mathbf{O}(3)$. For the plus representation of $\mathbf{O}(3)$ we denote those by $I^+(k)$ and $E^+(k)$ and omit the values for the minus representation if they are zero.*

In Table 4 we show the numbers of invariants and equivariants for stationary and Hopf bifurcation with $\mathbf{O}(3)$-symmetry for $l = 1, \ldots, 10$. For Hopf bifurcation the values are the same for the plus and minus representations of $\mathbf{O}(3)$ on V_l. For steady-state bifurcation the values for the two representations of $\mathbf{O}(3)$ on V_l differ for the number of cubic invariants and quadratic equivariants. For the plus representation we have the values denoted by $I^+(3)$ and $E^+(2)$. For the minus representation these are zero.

Remark 6.9. As shown above, $E_H(3) = I_H(4) = l + 1$, so the cubic equivariants for Hopf bifurcation can be written as gradients of the quartic invariants. However, in the case of Hopf bifurcation, this does not constrain the dynamics in the way that it does for stationary bifurcation. Note that $E_H(5)$ and $I_H(6)$ increase very rapidly with l; this increase appears to be of order l^3 for large l. ◇

6.2.2. *Hilbert-Poincaré Series of* $\mathbf{O}(3)$ *and* $\mathbf{SO}(3)$

We show how it is possible to explicitly evaluate the Hilbert-Poincaré series for the groups $\mathbf{SO}(3)$ and $\mathbf{O}(3)$ for small values of l, for both stationary bifurcation (see [5, page 176]) and Hopf bifurcation.

The action of a rotation through an angle θ is given by a diagonal matrix with entries $e^{im\theta}$, for $m = -l \ldots l$. Since all rotations through an angle θ are conjugate, they all make the same contribution to the Haar integral, and so

$$\Phi_{\mathbf{SO}(3)}(t) = \frac{1}{\pi}\int_0^\pi \frac{(1-\cos\theta)\,d\theta}{\prod_{m=-l}^{l}(1-te^{im\theta})}, \quad \Psi_{\mathbf{SO}(3)}(t) = \frac{1}{\pi}\int_0^\pi \frac{\cos l\theta - \cos(l+1)\theta\,d\theta}{\prod_{m=-l}^{l}(1-te^{im\theta})}.$$

For the group $\mathbf{O}(3)$, the formulas are the same as for $\mathbf{SO}(3)$ in the '+' representation, while for the '−' representation it follows from (6.6) that

$$\Phi^-_{\mathbf{O}(3)}(t) = \frac{\Phi_{\mathbf{SO}(3)}(t) + \Phi_{\mathbf{SO}(3)}(-t)}{2}, \quad \Psi^-_{\mathbf{O}(3)}(t) = \frac{\Psi_{\mathbf{SO}(3)}(t) - \Psi_{\mathbf{SO}(3)}(-t)}{2}.$$

The integrals can be evaluated by making the substitution $u = e^{i\theta}$ and using complex variable methods, as in Example 2.7. For $l = 1$ the results are

$$\Phi^{l=1}_{\mathbf{SO}(3)}(t) = \frac{1}{1-t^2}, \quad \Psi^{l=1}_{\mathbf{SO}(3)}(t) = \frac{t}{1-t^2},$$

and since these functions are even and odd respectively, the result is the same for $\mathbf{O}(3)$ in either the '+' or '−' representation. For $l = 2$,

$$\Phi^{l=2}_{\mathbf{SO}(3)}(t) = \frac{1}{(1-t^2)(1-t^3)}, \quad \Psi^{l=2}_{\mathbf{SO}(3)}(t) = \frac{t}{(1-t)(1-t^3)},$$

and for $l = 3$,

$$\begin{aligned} \Phi^{l=3}_{\mathbf{SO}(3)}(t) &= \frac{1+t^{15}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})}, \\ \Psi^{l=3}_{\mathbf{SO}(3)}(t) &= \frac{t-t^8}{(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)}. \end{aligned}$$

In the case of Hopf bifurcation, we need to carry out two integrals, firstly an integral over $\mathbf{SO}(3)$ to find $\Phi_{\mathbf{SO}(3)}(z,\overline{z})$ using Theorem 2.4, and secondly over $\mathbf{S}^1$ to apply Theorem 2.6. In this case $\Phi_{\mathbf{O}(3)\times\mathbf{S}^1} = \Phi_{\mathbf{SO}(3)\times\mathbf{S}^1}$ for both representations, as discussed in the preceding section. For $l = 1$,

$$\begin{aligned} \Phi^{l=1}_{\mathbf{SO}(3)}(z,\overline{z}) &= \frac{1}{\pi}\int_0^{\pi} \frac{(1-\cos\theta)\,d\theta}{(1-z)(1-ze^{i\theta})(1-ze^{-i\theta})(1-\overline{z})(1-\overline{z}e^{-i\theta})(1-\overline{z}e^{i\theta})} \\ &= \frac{1}{(1-z^2)(1-\overline{z}^2)(1-z\overline{z})}. \end{aligned}$$

This result was also obtained by Forger [9] for the closely related group $\mathbf{SU}(2)$. To impose $\mathbf{S}^1$-invariance we carry out the second integral

$$\begin{aligned} \Phi^{l=1}_{\mathbf{SO}(3)\times\mathbf{S}^1}(z,\overline{z}) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1-z^2e^{2i\theta})(1-\overline{z}^2e^{-2i\theta})(1-z\overline{z})} \\ &= \frac{1}{(1-z\overline{z})(1-z^2\overline{z}^2)}. \end{aligned}$$

A similar calculation shows that the generating functions for equivariants for $l = 1$ are

$$\Psi^{l=1}_{\mathbf{SO}(3)}(z,\overline{z}) = \frac{z+\overline{z}+z\overline{z}}{(1-z^2)(1-\overline{z}^2)(1-z\overline{z})}, \quad \Psi^{l=1}_{\mathbf{SO}(3)\times\mathbf{S}^1}(z,\overline{z}) = \frac{z}{(1-z\overline{z})^2}.$$

For $l = 2$ the results of the calculations are

$$\begin{aligned} \Phi^{l=2}_{\mathbf{SO}(3)\times\mathbf{S}^1}(z,\overline{z}) &= \frac{1+3z^4\overline{z}^4+2z^5\overline{z}^5+3z^6\overline{z}^6+z^{10}\overline{z}^{10}}{(1-z\overline{z})(1-z^2\overline{z}^2)^2(1-z^3\overline{z}^3)^2(1-z^4\overline{z}^4)} \\ &= 1+z\overline{z}+3z^2\overline{z}^2+5z^3\overline{z}^3\cdots \end{aligned}$$

$$\begin{aligned}\Psi^{l=2}_{\mathbf{SO}(3)\times\mathbf{S}^1}(z,\overline{z}) &= \\ &= \frac{z(1+z\overline{z}+3z^2\overline{z}^2+5z^3\overline{z}^3+5z^4\overline{z}^4+5z^5\overline{z}^5+3z^6\overline{z}^6+z^7\overline{z}^7+z^8\overline{z}^8)}{(1-z\overline{z})^2(1-z^2\overline{z}^2)(1-z^3\overline{z}^3)^2(1-z^4\overline{z}^4)} \\ &= z(1+3z\overline{z}+9z^2\overline{z}^2+23z^3\overline{z}^3\cdots)\end{aligned}$$

From these generating functions we can deduce that for the $l=2$ Hopf bifurcation there is one quadratic invariant and three quartic invariants, and that there are three cubic equivariants and nine quintic equivariants. These results are in agreement with those of Iooss and Rossi [20] and Haaf *et al.* [18] who obtained the invariants and equivariants by direct computation, using two different methods. In principle these generating functions can be obtained for any value of l, but the formulas rapidly become cumbersome as l increases.

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