A general index integral of the product of Meijer's G- functions

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The purpose of this note is to give a formal proof of the following formula

$$\frac{1}{2\pi^2}\int_{-\infty}^{\infty} te^{2\pi t} G^{\,q_1-m_1, p_1-n_1+2}_{\,p_1+2, q_1}\left(a\left|\begin{matrix} it,\ -it,\ (-\alpha_{p_1}^{n_1+1}),\ (-\alpha_{n_1}) \\ (-\beta_{q_1}^{m_1+1}),\ (-\beta_{m_1})\end{matrix}\right.\right)$$

$$\times G^{\,m_1+m_2, n_1+n_2+2}_{\,p_1+p_2+2, q_1+q_2}\left(b\left|\begin{matrix} 1+it,\ 1-it,\ (c_{p_1+p_2}) \\ (d_{q_1+q_2})\end{matrix}\right.\right) dt$$

$$= a^{-1}\, G^{\,m_2, n_2}_{\,p_2, q_2}\left(\frac{b}{a}\left|\begin{matrix} (\gamma_{p_2}) \\ (\delta_{q_2})\end{matrix}\right.\right), \qquad (1)$$

which is associated with three Meijer's G-functions of different parameters. We note, that integral (1) contains most of known integrals with respect to parameters of hypergeometric functions, which can be found in [1], [2]. Here as usual, $0 \le m_i \le q_i,\ 0 \le n_i \le p_i,\ i = 1,2,\ a \neq 0,\ b$ are parameters and vectors of other parameters are defined on a usual way:

$$(-\alpha_{p_1}^{n_1+1}) = (-\alpha_{n_1+1}, \dots, -\alpha_{p_1}),$$

$$(-\alpha_{n_1}) = (-\alpha_1, \dots, -\alpha_{n_1}),$$

$$(-\beta_{q_1}^{m_1+1}) = (-\beta_{m_1+1}, \dots, -\beta_{q_1}),$$

$$(-\beta_{m_1}) = (-\beta_1, \dots, -\alpha_{m_1}),$$

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$$(c_{p_1+p_2}) = (\alpha_1, \alpha_2, \ldots, \alpha_{n_1}, \gamma_1, \ldots, \gamma_{n_2}, \alpha_{n_1+1}, \ldots, \alpha_{p_1}, \gamma_{n_2+1}, \ldots, \gamma_{p_2}),$$
$$(d_{q_1+q_2}) = (\beta_1, \beta_2, \ldots, \beta_{m_1}, \delta_1, \ldots, \delta_{m_2}, \beta_{m_1+1}, \ldots, \beta_{q_1}, \delta_{m_2+1}, \ldots, \delta_{q_2}),$$
$$(\gamma_{p_2}) = (\gamma_1, \ldots, \gamma_{p_2}),$$
$$(\delta_{q_2}) = (\delta_1, \ldots, \delta_{q_2}).$$

We also note, that all parameters satisfy the corresponding convergence conditions, which can be done accordingly.

In order to prove (1), we denote its left hand-side as $I(a,b)$ and by using the evenness with respect to t of the Meijer G-functions under the integral sign, we write $I(a,b)$ in the form of the inverse Wimp - Yakubovich transform [5, Ch. 7], [4] namely

$$I(a,b) = \frac{1}{\pi^2}\int_0^\infty t\sinh(2\pi t)$$
$$\times G\,{}^{q_1-m_1,\, p_1-n_1+2}_{p_1+2,\, q_1}\left(a \,\middle|\, \begin{matrix} it,\ -it,\ (-\alpha_{p_1}^{n_1+1}),\ (-\alpha_{n_1}) \\ (-\beta_{q_1}^{m_1+1}),\ (-\beta_{m_1}) \end{matrix}\right)$$
$$\times G\,{}^{m_1+m_2,\, n_1+n_2+2}_{p_1+p_2+2,\, q_1+q_2}\left(b \,\middle|\, \begin{matrix} 1+it,\ 1-it,\ (c_{p_1+p_2}) \\ (d_{q_1+q_2}) \end{matrix}\right) dt. \tag{2}$$

Hence the direct Wimp-Yakubovich transform [5] should give reciprocally

$$G\,{}^{m_1+m_2,\, n_1+n_2+2}_{p_1+p_2+2,\, q_1+q_2}\left(b \,\middle|\, \begin{matrix} 1+it,\ 1-it,\ (c_{p_1+p_2}) \\ (d_{q_1+q_2}) \end{matrix}\right)$$
$$= \int_0^\infty I(x,b)\, G\,{}^{m_1,\, n_1+2}_{p_1+2,\, q_1}\left(x \,\middle|\, \begin{matrix} 1+it,\ 1-it,\ (\alpha_{p_1}) \\ (\beta_{q_1}) \end{matrix}\right) dx, \tag{3}$$

and this will be the case if and only if

$$I(a,b) = a^{-1}\, G\,{}^{m_2,\, n_2}_{p_2,\, q_2}\left(\frac{b}{a} \,\middle|\, \begin{matrix} (\gamma_{p_2}) \\ (\delta_{q_2}) \end{matrix}\right). \tag{4}$$

In fact, substituting the right-hand side of (4) into (3), we treat the corresponding integral of the product of G-functions by the Mellin-Parseval equality [2], [5], [6]. Hence with the definition of the Meijer G-function we derive

$$\int_0^\infty G\,{}^{m_2,\, n_2}_{p_2,\, q_2}\left(\frac{b}{x} \,\middle|\, \begin{matrix} (\gamma_{p_2}) \\ (\delta_{q_2}) \end{matrix}\right) G\,{}^{m_1,\, n_1+2}_{p_1+2,\, q_1}\left(x \,\middle|\, \begin{matrix} 1+it,\ 1-it,\ (\alpha_{p_1}) \\ (\beta_{q_1}) \end{matrix}\right) \frac{dx}{x}$$

$$
= \frac{1}{2\pi i} \int_L \frac{\Gamma(it-s)\Gamma(-it-s)\prod_{j=1}^{m_1}\Gamma(\beta_j+s)\prod_{j=1}^{n_1}\Gamma(1-\alpha_j-s)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-\beta_j-s)\prod_{j=n_1+1}^{p_1}\Gamma(\alpha_j+s)}
$$

$$
\times \frac{\prod_{j=1}^{m_2}\Gamma(\delta_j+s)\prod_{j=1}^{n_2}\Gamma(1-\gamma_j-s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-\delta_j-s)\prod_{j=n_2+1}^{p_2}\Gamma(\gamma_j+s)}\ b^{-s}ds
$$

$$
= G\, \begin{matrix} m_1+m_2, n_1+n_2+2 \\ p_1+p_2+2, q_1+q_2 \end{matrix} \left(b \middle| \begin{matrix} 1+it,\ 1-it,\ (c_{p_1+p_2}) \\ (d_{q_1+q_2}) \end{matrix} \right),
$$

where the corresponding parameters are defined above. Thus we have proved (3) and consequently, the value of the integral (1).

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