THE SPECTRA OF LAMPLIGHTER GROUPS AND CAYLEY MACHINES

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Abstract. We calculate the spectra and spectral measures associated to random walks on restricted wreath products G wr $\mathbb{Z}$, with G a finite abelian group, by realizing them as groups generated by automata. This generalizes the work of Grigorchuk and Żuk on the lamplighter group. More generally we calculate the spectra of random walks on groups generated by Cayley machines of finite groups and calculate Kesten-von Neumann-Serre spectral measures of random walks on their Schreier graphs with respect to their parabolic subgroups. Some of these results were obtained by Dicks and Schick via a different method.

1. Introduction

The systematic study of random walks on discrete non-abelian groups began with the seminal work of Kesten [17] and has since become an important area of mathematics; it links such diverse fields as probability theory, group theory, geometry and analysis. There are still not many examples of complete computations of the spectra of Markov operators for simple random walks [17, 21, 3, 11] and even fewer computations of the spectral measure [21, 11].

Grigorchuk and Żuk [11] computed the spectrum and the spectral measure of the Markov operator for the lamplighter group

$$\mathbb{Z}/2\mathbb{Z} \text{ wr } \mathbb{Z} = \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \right) \rtimes \mathbb{Z},$$

by realizing it as the group generated by a two-state automaton; this calculation was with respect to the automaton generators. In particular, the spectral measure was discrete (a previously unseen phenomenon [11, 21]). They used this computation to obtain a negative answer to the strong form

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of the Atiyah conjecture on L^2-Betti numbers [8]. The techniques used were developed earlier by Bartholdi and Grigorchuk [1] to calculate the spectra and spectral measures associated to random walks of automata groups on the Schreier graphs with respect to their parabolic subgroups.

These results were generalized to (restricted) wreath products G wr $\mathbb{Z}$, with G finite, by Dicks and Schick [6]. The generators they considered were inspired by the ones Grigorchuk and Żuk used for $G = \mathbb{Z}/2\mathbb{Z}$, but their method was entirely different, using computations in the group ring.

In [25] the second and third authors, in an attempt to apply Krohn-Rhodes theory [18, 19] to automata groups, studied the groups generated by Cayley machines of finite groups. The automaton used by Grigorchuk and Żuk to generate the lamplighter group was none other than the Cayley machine of $\mathbb{Z}/2\mathbb{Z}$. The second and third authors showed that if G is a finite abelian group, then the group Γ generated by the Cayley machine of G is the wreath product G wr $\mathbb{Z}$. Moreover, the automaton generators for these groups are precisely the generators considered by Dicks and Schick. If G is non-abelian, it transpires [25] that Γ is still a semidirect product of a locally finite group with the integers; we prove in this paper that Γ is not in general a wreath product of a finite group with a torsion-free group.

The results of [25] show that in the case G is abelian, the parabolic subgroups for the action of G wr $\mathbb{Z}$ on the boundary can be chosen to be trivial. We compute in this paper the spectra of the Markov operators associated to the simple random walks on groups generated by Cayley machines, proving that it is always the whole interval $[-1, 1]$. We compute also the so-called Kesten-von Neumann-Serre spectral measure, defined in [13], for the random walks on the Schreier graphs with respect to parabolic subgroups and show that, for the case of G wr $\mathbb{Z}$ with G abelian, this coincides with the spectral measure of the simple random walk. This gives a proof using automata of the results of Dicks and Schick [6] for this case.

To prove this, we first show that the fixed-point set of a transformation of the Cantor set computed by a finite state automaton has measure zero if and only if it is nowhere dense. Hence, if Γ is a group generated by a finite automaton, acting on the boundary of the associated tree, then freeness of the action in the sense of Baire category is equivalent to freeness in the sense of ergodic theory. Moreover, we prove that these conditions are equivalent to that of having the Kesten-von Neumann-Serre (KNS) spectral measure coincide with the Kesten spectral measure of the random walk on the Cayley graph. We use these results to calculate the Ihara zeta functions (in the sense defined by Grigorchuk and Żuk [13]) of the Schreier graphs with respect to parabolic subgroups and hence for the Cayley graph of G wr $\mathbb{Z}$, in the case G is abelian.

The main result is that if $|G| = n$, then the KNS spectral measure for the random walk on the Schreier graph of Γ with respect to a parabolic subgroup is discrete, supported at the points $\lambda_{p,q} = \cos \frac{p}{q}\pi$ with $1 \leq p < q$,

$(p,q)=1$, and the weight at $\lambda_{p,q}$ is $\frac{(n-1)^2}{n^q-1}$; moreover if G is abelian this is the spectral measure for the simple random walk on the Cayley graph of $G \operatorname{wr} \mathbb{Z}$.

Several computations of spectra and Kesten-von Neumann-Serre spectral measures for groups generated by finite state automata have been performed [1, 11, 12]. A novel element of this paper is that, as far as we know, this is the first such computation performed simultaneously for an infinite class of groups generated by automata, with respect to their automaton generators.

Since this paper is intended both for people interested in group theory and for those interested in random walks, we have endeavoured to make it self-contained by explaining as much of the background as possible. Thus the first several sections explain the aspects of random walks and probability theory that we shall need. Then we give a brief introduction to groups generated by finite state automata. Our longest section performs the spectral computations. We end with a proof that the Cayley machines of non-abelian groups are not in general wreath products of finite groups with torsion-free groups.

2. Markov and Hecke operators, spectral measures, random walks and Ihara zeta functions

This section contains the notions that motivate this work and defines the various objects that we shall compute.

2.1. **Markov and Hecke-type operators.** We consider only real Hilbert spaces. Let $X=(V,E)$ be a k-regular (undirected) graph with vertex set V and edge set E. We allow multiple edges and loops. The primary example for us is where Γ is a finitely generated group with symmetric generating set S of size k, $P \leq \Gamma$ is a subgroup and $X=\mathsf{Sch}(\Gamma,P,S)$ is the associated (left) *Schreier* (or coset) graph. The vertices are the left cosets Γ/P. The edge set is $S\times\Gamma/P$. The edge (s,gP) goes from gP to sgP. In particular, the (left) *Cayley graph* of Γ is $\mathsf{Sch}(\Gamma,1,S)$.

For $v\in V$, let $E(v)$ be the set of edges incident with v. For each edge $e\in E(v)$, let $o_v(e)$ denote the vertex at the other end of e from v; for loops $o_v(e)$ is taken to be v. The *random walk* or *Markov operator* [17, 11] on X is the operator $M:\ell^2(V)\to\ell^2(V)$ given by

$$Mf(v)=\frac{1}{k}\sum_{e\in E(v)} f(o_v(e)) \tag{2.1}$$

Here $\ell^2(V)$ is the space of square summable functions from V to $\mathbb{R}$. If $\delta_v : V\to\mathbb{R}$ is the characteristic function of $\{v\}$ for each $v\in V$, and we write the "matrix" for M with respect to the basis $\{\delta_v\}_{v\in V}$, then we obtain the normalized incidence matrix for X. That is, the matrix coefficient $\langle M\delta_{v_1},\delta_{v_2}\rangle$ is the probability that an edge incident on v_1 is also incident on v_2. For the case of a random walk on a Schreier graph $\mathsf{Sch}(\Gamma,P,S)$, one has that $M : \ell^2(\Gamma/P) \to \ell^2(\Gamma/P)$ is given by

$$Mf(gP) = \frac{1}{|S|}\sum_{s\in S} f(sgP).$$

This is a special case of a Hecke-type operator [1, 11].

Let Γ be a (discrete) group with finite symmetric generating set S and $\pi : \Gamma \to \mathcal{B}(\mathfrak{H})$ be a unitary representation of Γ on a Hilbert space $\mathfrak{H}$; here $\mathcal{B}(\mathfrak{H})$ denotes the algebra of bounded linear operators on $\mathfrak{H}$. The associated Hecke-type operator is then $H_\pi : \mathfrak{H} \to \mathfrak{H}$ given by

$$H_\pi = \frac{1}{|S|}\sum_{s\in S} \pi(s).$$

Then H_π is a self-adjoint operator and $\|H_\pi\| \leq 1$ by the triangle inequality.

Suppose now that $P \leq \Gamma$ is a subgroup and let $\lambda_{\Gamma/P} : \Gamma \to \mathcal{B}(\ell^2(\Gamma/P))$ be the (left) quasi-regular representation; so $\lambda_{\Gamma/P}(g)f(hP) = f(g^{-1}hP)$. Then $H_{\lambda_{\Gamma/P}}$ is precisely the Markov operator associated to the random walk on $\mathsf{Sch}(\Gamma, P, S)$. The case $P = 1$ is called the *left regular representation*, denoted λ_Γ.

Another important example is the following. Suppose that Γ acts on a measure space (X, μ) by measure-preserving transformations. Then there is an associated unitary representation $\pi : \Gamma \to \mathcal{B}(L^2(X, \mu))$ (where $L^2(X, \mu)$ is the space of square-integrable functions on X) given by $\pi(g)f(x) = f(g^{-1}x)$. The case of interest to us arises from the action of an automata group on the boundary of a rooted tree, viewed as a measure space with the product (Bernoulli) measure.

2.2. **Kesten spectral measures.** Recall that the *spectrum* $\mathsf{Sp}(T)$ of a bounded operator T on a Hilbert space consists of all real numbers λ such that $T - \lambda$ is not invertible. Then $\mathsf{Sp}(T)$ is a closed subset of the interval $[-\|T\|, \|T\|]$.

We return now to the Markov operator M for a random walk on a k-regular graph X. As in the case of Schreier graphs, one can verify that $\|M\| \leq 1$. Also M is self-adjoint, so it has a spectral decomposition

$$M = \int_{-1}^{1} \lambda \mathrm{d}E(\lambda) \tag{2.2}$$

where E is the spectral measure [22]. That is, E is a projection-valued measure defined on the Borel subsets of $[-1, 1]$, taking values in the projections of $\mathcal{B}(\mathfrak{H})$.

For those unfamiliar with these notions, if X is a finite graph, then

$$E(B) = \sum_{B\cap \mathsf{Sp}(M)} E_\lambda \tag{2.3}$$

where E_λ is the projection to the eigenspace associated with λ and (2.2) is the usual orthogonal decomposition of a symmetric matrix into its eigenspaces.

The matrix μ^X of measures associated with E is given by

$$\mu^X_{v_1,v_2}(B) = \langle E(B)\delta_{v_1}, \delta_{v_2}\rangle. \tag{2.4}$$

Of particular interest are the diagonal entries $\mu^X_v := \mu^X_{v,v}$. These are called the *Kesten* spectral measures associated to the random walk [13]. To explain the significance of these measures, we remind the reader about the moments of a measure. If μ is a (Borel) probability measure on $\mathbb{R}$ (we do not necessarily assume the support of the measure is the whole real line) and $f : \mathbb{R} \to \mathbb{R}$ is a measurable function, then the *expected value* of f, denoted $E[f]$, (or $E_\mu[f]$ if we want to emphasize the measure) is given by

$$E[f] = \int_{\mathbb{R}} f(x)\mathrm{d}\mu(x).$$

The m^{th} *moment* of μ, denoted $\mu^{(m)}$ is $E[x^m]$. The (formal) *moment generating function* is then the power series

$$\sum_{m=0}^{\infty} \frac{1}{m!}\mu^{(m)}t^m \in \mathbb{R}[\![t]\!].$$

This is the Taylor expansion about 0 of the function $M(t) = E[e^{tx}]$ (where the integral is taken over x). Returning to the situation of a simple random walk on a graph X and the spectral measure μ^X_v, denote by $p_m(v)$ the probability of return to v on the m^{th} step of the random walk; then $p_m(v)$ is the m^{th} moment of μ^X_v [17, 11]. In the case of a Schreier graph $\mathsf{Sch}(\Gamma, P, S)$ we shall use the notation $\mu^{\Gamma/P}$ to denote $\mu^{\Gamma/P}_P$.

If the automorphism group of X acts transitively on the vertices (for example this occurs for $\mathsf{Sch}(\Gamma, P, S)$ when P is normal) then the Kesten measures are independent of the chosen vertex; moreover the spectral decomposition (2.2) is determined by the Kesten spectral measure at a single vertex. In particular, this happens for Cayley graphs. In the case of the Cayley graph of Γ, one can alternatively use von Neumann traces. The von Neumann trace on the von Neumann algebra generated by the left regular representation of Γ is given by

$$\mathrm{tr}(T) = \langle T\delta_1, \delta_1\rangle; \tag{2.5}$$

that is, it is the "coefficient" in T of the identity element (this is literally true for elements of finite support, i.e. elements of $\mathbb{R}\Gamma$). Then $\mu^\Gamma = \mathrm{tr}(E(B))$ coincides with the Kesten spectral measure (at, for example, the identity), as one easily sees by comparing (2.4) and (2.5).

2.3. **Ihara zeta functions and Kesten-von Neumann-Serre spectral measures.** The following is from [10, 13]. Recall that if $X = (V, E)$ is a graph, one can define the path metric on V by setting $d(v_1, v_2)$ to be the number of edges in the shortest path connecting v_1 and v_2. Denote by $B_X(v, r)$ the open ball of radius r in V around v.

Fix a positive integer k as before. The space of (isomorphism classes of) pointed k-regular graphs (X, v) becomes an ultrametric compact totally disconnected space [10, 13] by taking

$$d((X_1, v_1), (X_2, v_2)) = \frac{1}{n+1}$$

where n is the largest integer for which $B_{X_1}(v_1, n)$ is pointedly isometric to $B_{X_2}(v_2, n)$.

The case of interest to us is the following: Γ is a discrete group with finite symmetric generating set S and $P \leq \Gamma$. Suppose $P_n \leq \Gamma$, $n \in \mathbb{N}$ are finite index subgroups with $\bigcap P_n = P$ (so P is a closed subgroup in the profinite topology on Γ). Then one easily sees that the graphs $X_n = \mathsf{Sch}(\Gamma, P_n, S)$ converge to the graph $X = \mathsf{Sch}(\Gamma, P, S)$ (see [10, 13]) where the coset of the identity is taken as the base point.

Recall that a sequence of probability measures μ_n on a measure space is said to converge *weakly* to a measure μ if, for all measurable subsets B, $\mu_n(B) \to \mu(B)$ [4]. Suppose (X_n, v_n) is a sequence of pointed finite graphs converging to (X, v), and let $N > 0$ be given. Then $B_X(v, N)$ is pointedly isometric to $B_{X_n}(v_n, N)$ for n sufficiently large. From this it can be deduced using the method of moments that $\mu_{v_n}^{X_n} \to \mu_v^X$ weakly [10, 13].

To motivate the so-called Kesten-von Neumann-Serre spectral measures [13] we recall the definition of the *Ihara zeta function* ζ_X of a finite k-regular graph X [16]. It is the power series

$$\zeta_X(t) = \prod_{[C]} (1 - t^{|C|})^{-1}$$

where $[C]$ runs over the equivalence classes of primitive, cyclically reduced closed paths in X and $|C|$ denotes the length of C. It is known [16, 13] that

$$\ln \zeta_X(t) = \sum_{r=1}^{\infty} \frac{c_r}{r} t^r$$

where c_r is the number of cyclically reduced closed paths of length r in X. We remark that ζ_X can be viewed as a discrete analogue of the Riemann zeta function, and satisfies the Riemann hypothesis if and only if X is a Ramanujan graph [13, 5].

Let M be the Markov operator associated to X. Recalling that kM is the incidence matrix of the graph X, the results of [16] show that

$$\zeta_X(t) = (1 - t^2)^{-\frac{1}{2}(k-2)|E(X)|} \det(1 - tkM + (k-1)t^2)^{-1}$$

where $E(X)$ denotes the edge set of X.

Grigorchuk and Żuk extended this to infinite graphs that are limits of sequences of finite graphs. Let $\{X_n\}$ be a sequence of finite k-regular graphs with associated Markov operators M_n. Denote by V_n the vertex set of X_n. We use $\mathrm{tr}(A)$ to denote the trace of a matrix A. Define

$$\mu_n = \frac{1}{|V_n|}\sum_{v\in V_n}\mu_v^{X_n} = \sum_{\lambda\in\mathsf{Sp}(M_n)}\frac{\mathrm{tr}(E_\lambda)}{|V_n|}\delta_\lambda = \sum_{\lambda\in\mathsf{Sp}(M_n)}\frac{\#_n(\lambda)}{|V_n|}\delta_\lambda \tag{2.6}$$

where $\#_n(\lambda)$ denotes the multiplicity of λ as an eigenvalue of M_n, where E_λ is the projection to the λ-eigenspace of M_n and where δ_λ is the Dirac measure at λ. The second equality of (2.6) follows from (2.3). The probability measure μ_n counts the frequency of the eigenvalues weighted by their multiplicities.

Following Serre [24] (see also [13]) the eigenvalues of the Markov operators M_n are said to be *equidistributed* with respect to a measure μ having support on $[-1, 1]$ if $\mu_n \to \mu$ weakly.

Suppose now that we are in the situation above with $X = \mathsf{Sch}(\Gamma, P, S)$ and $X_n = \mathsf{Sch}(\Gamma, P_n, S)$. It is shown in [13] that the eigenvalues of the M_n are equidistributed with respect to some measure μ, which Grigorchuk and Żuk call the *Kesten-von Neumann-Serre* (KNS) spectral measure of X with respect to the approximating sequence X_n.

Serre [24] proved that the eigenvalues of M_n are equidistributed with respect to some measure if and only if the sequence $\zeta_{X_n}^{\frac{1}{|E(X_n)|}}$ converges in $\mathbb{R}[\![t]\!]$ with the topology of pointwise convergence of the coefficients. Grigorchuk and Żuk then defined the zeta function of X (in the above context) with respect to the approximating sequence X_n to be given by

$$\ln\zeta_X(t) = \lim_{n\to\infty}\frac{1}{|E(X_n)|}\ln\zeta_{X_n}(T).$$

They showed [13] that $\ln\zeta_X(t)$ has radius of convergence at least $\frac{1}{k-1}$ and that, for $|t| < \frac{1}{k-1}$,

$$\ln\zeta_X(t) = -\frac{k-2}{2}\ln(1-t^2) - \int_{-1}^{1}\ln(1 - tk\lambda + (k-1)t^2)\mathrm{d}\mu \tag{2.7}$$

where μ is the KNS spectral measure. This serves as a motivation for computing the measure μ. Conversely, it is shown in [13] that ζ_X determines the moments of μ and hence determines μ itself.

They also define in [13] the zeta function of Γ (with respect to the generators S) by

$$\ln\zeta_\Gamma(t) = -\frac{k-2}{2}\ln(1-t^2) - \mathrm{tr}\ln(1 - tkM + (k-1)t^2)$$

where tr is, as above, the von Neumann trace. Notice that this is the same formula as (2.7), but with μ replaced by the Kesten spectral measure for the random walk on the Cayley graph of Γ.

3. Automata groups and Cayley machines

Recall that a finite (Mealy) automaton [19, 7] $\mathcal{A}$ is a 4-tuple (Q, A, δ, λ) where Q is a finite set of states, A is a finite alphabet, $\delta : Q \times A \to Q$ is the transition function and $\lambda : Q \times A \to A$ is the output function. One writes qa for $\delta(q, a)$ and $q \circ a$ for $\lambda(q, a)$. These functions extend to the free monoid A^* by

$$q(au) = (qa)u \tag{3.1}$$

$$q \circ (au) = (q \circ a)(qa) \circ u. \tag{3.2}$$

We use $\mathcal{A}_q$ to denote the *initial automaton* $\mathcal{A}$ with designated start state q. There is a function $\mathcal{A}_q : A^* \to A^*$ given by $w \mapsto q \circ w$. This function is length preserving and extends continuously [9] to the set of right infinite words A^ω via the formula

$$\mathcal{A}_q(a_0 a_1 \cdots) = \lim_{n\to\infty} \mathcal{A}_q(a_0 \cdots a_n) \tag{3.3}$$

where A^ω is given the product topology, making it homeomorphic to a Cantor set. If, for each q, the state function $\lambda_q : A \to A$ given by $\lambda_q(a) = q \circ a$ is a permutation, then $\mathcal{A}_q$ is an isometry of A^ω for the metric $d(u, v) = 1/(n+1)$ where n is the length of the longest common prefix of u and v [9]. In this case the automaton is called *invertible*. We shall assume here (unless otherwise stated) that all automata are invertible. Let $\Gamma = \mathcal{G}(\mathcal{A})$ be the group generated by the $\mathcal{A}_q$ with $q \in Q$.

If we let T be the Cayley tree of A^*, then Γ acts on the left of T by rooted tree automorphisms of T [9, 2] via the action (3.2). The induced action on the boundary ∂T (the space of infinite directed paths from the root) is just the action (3.3) of Γ on A^ω.

The automorphism group $\mathrm{Aut}(T)$ is the iterated (permutational) wreath product of countably many copies of the left permutation group (S_A, A) [9, 2, 23], where S_A denotes the symmetric group on A. In this paper, our notation will be such that the wreath product of left permutation groups has a natural projection to its leftmost factor; this is in contrast to the case of restricted wreath products of abstract groups where our notation is such that there is a projection to the rightmost factor. For a group $\Gamma = \mathcal{G}(\mathcal{A})$ generated by an automaton over A, one has an embedding

$$(\Gamma, A^\omega) \hookrightarrow (S_{|A|}, A) \wr (\Gamma, A^\omega). \tag{3.4}$$

The maps sends $\mathcal{A}_q$ to the element with wreath product coordinates:

$$\mathcal{A}_q = \lambda_q(\mathcal{A}_{qa_1}, \ldots, \mathcal{A}_{qa_n}) \tag{3.5}$$

where $A = \{a_1, \ldots, a_n\}$. See [9, 2, 25] for more details.

The action of Γ on T is called *spherically transitive* if Γ acts transitively on each level of the tree. Here the k^{th} level of T is the set of all vertices corresponding to words of length k. It will be convenient to denote by T_k the finite rooted tree obtained by pruning the levels after level k. Denote by $\mathrm{St}_\Gamma(k)$ the set of all elements of Γ that fix each vertex of level k; it is a finite index normal subgroup, being the kernel of the projection $\Gamma \to \mathrm{Aut}(T_k)$. Notice that $\{1\} = \bigcap_{k=0}^{\infty} \mathrm{St}_\Gamma(k)$ (and so Γ is residually finite).

Let G be a non-trivial finite group. By the *Cayley machine* $\mathcal{C}(G)$ of G we mean the automaton with state set and alphabet G. Both the transition and the output functions are the multiplication of the group. So in state g_0 on input g the machine goes to state g_0g and outputs g_0g. You can view $\mathcal{C}(G)$ as the Cayley graph of G with respect to the generators G where output is the next state. The state function λ_g is just left translation by g and hence a permutation, so $\mathcal{C}(G)$ is invertible. Cayley machines for semigroups are an important part of Krohn-Rhodes theory [19, 18]. The study of the automata group of the Cayley machine of a finite group was initiated by the second and third authors [25].

An automaton is called a *reset automaton* if, for each $a \in A$, $|Qa| = 1$; that is, each input resets the automaton to a single state. The second and third authors showed that the inverse of a state $\mathcal{C}(G)_g$ is computed by the corresponding state of the reset automaton $\mathcal{A}(G)$ with states G and input alphabet G, where in state g_0 on input g, the automaton goes to state g and outputs $g_0^{-1}g$. Therefore $\mathcal{G}(\mathcal{C}(G)) = \mathcal{G}(\mathcal{A}(G))$. In wreath product coordinates

$$\mathcal{A}(G)_g = g^{-1}(\mathcal{A}_{g_1}, \ldots, \mathcal{A}_{g_n}) \tag{3.6}$$

where $G = \{g_1, \ldots, g_n\}$. Hence, in our situation, there is an embedding

$$\mathcal{G}(\mathcal{A}(G)) \hookrightarrow (G, G) \wr (\mathcal{G}(\mathcal{A}(G)), G^\omega). \tag{3.7}$$

Moreover, the action of $\mathcal{G}(\mathcal{A}(G))$ on the Cayley tree of G^* is spherically transitive [25].

Let $x = \mathcal{A}(G)_1$. Notice that

$$x\mathcal{A}(G)_g^{-1} = x\mathcal{C}(G)_g = g(1, \ldots, 1),$$

so we can identify G with a subgroup of $\mathcal{G}(\mathcal{A}(G))$ via $g \leftrightarrow x\mathcal{A}(G)_g^{-1}$. Let

$$N = \langle x^n G x^{-n} \mid n \in \mathbb{Z} \rangle. \tag{3.8}$$

It is shown in [25] that x has infinite order, N is a locally finite group and $\mathcal{G}(\mathcal{A}(G)) = N \rtimes \langle x \rangle$. If G is abelian, it is shown [25] that

$$\mathcal{G}(\mathcal{A}(G)) = G \text{ wr } \mathbb{Z} = \left(\bigoplus_{\mathbb{Z}} G \right) \rtimes \mathbb{Z}$$

where in the latter semidirect product, $\mathbb{Z}$ acts by the shift. In particular, $\mathcal{G}(\mathcal{A}(\mathbb{Z}/n\mathbb{Z}))$ is the lamplighter-type group $\mathbb{Z}/n\mathbb{Z} \text{ wr } \mathbb{Z}$.

Recall that the *depth* of an element $\gamma \in \Gamma$ is the least integer n (if such exists) so that γ only changes the first n letters of a word. An automorphism of finite depth is often called finitary. An important role is also played by the subgroup

$$N_0 = \langle x^n G x^{-n} \mid n \geq 0 \rangle. \tag{3.9}$$

It is shown in [25] that $x^n g x^{-n}$ has depth $n + 1$ and so N_0 consists of finitary automorphisms. We recall also from [25] that the elements of the form $\mathcal{A}(G)_g \in \Gamma$ with $g \in G$ generate a free subsemigroup of Γ (this holds for a large class of semigroups generated by invertible reset automata [25]).

4. Dynamics, free actions and spectral measures

Let A be a finite alphabet and T be the Cayley tree of A^*. Let ν be the product measure on $\partial T = A^\omega$; so for each $u \in A^*$, the cylinder set uA^ω is given the measure $1/|A|^{|u|}$. Let $\varphi : T \to T$ be a morphism of trees preserving the distance from the root (and hence the levels); equivalently $\varphi : A^* \to A^*$ is a function computed by a possibly infinite automaton [9]; such a map is called in [23] an *elliptic contraction*. We also use $\varphi : \partial T \to \partial T$ to denote the induced morphism on the boundary (which is a contraction [9]). Define

$$\mathrm{Fix}(\varphi) = \{w \in \partial T \mid \varphi(w) = w\}.$$

It is a closed subset of ∂T. Also define, for $k \geq 0$,

$$\mathrm{Fix}_k(\varphi) = \{w \in A^k \mid \varphi(w) = w\}.$$

Proposition 4.1. *Let T be the Cayley tree of A^* and $\varphi : T \to T$ be an elliptic contraction. Then*

$$\nu(\mathrm{Fix}(\varphi)) = \lim_{k\to\infty} \frac{1}{|A|^k} |\mathrm{Fix}_k(\varphi)|. \tag{4.1}$$

In particular, (4.1) *holds for rooted automorphisms of T.* □

Proof. Set $F_k = \bigcup\{wA^\omega \mid w \in \mathrm{Fix}_k(\varphi)\}$. Then $\{F_k\}_{k\geq 0}$ is a decreasing sequence of measurable sets and

$$\mathrm{Fix}(\gamma) = \bigcap_{k=0}^{\infty} F_k.$$

Hence, since ν is a finite measure,

$$\nu(\mathrm{Fix}(\varphi)) = \lim_{k\to\infty} \nu(F_k) = \lim_{k\to\infty} \frac{1}{|A|^k} |\mathrm{Fix}_k(\varphi)|,$$

thereby establishing the proposition. □

Recall that a subset of a topological space is said to be *nowhere dense* if it does not contain a non-empty open subset. In general, measure zero is very far from nowhere dense – for instance the irrational numbers in the interval $[0, 1]$ are nowhere dense but have full measure. However, for the case of a contraction of the Cantor set computed by a finite state automata we can prove the following (somewhat surprising) result, which is inspired by a much more complicated argument from [11] for the case of the lamplighter group.

Theorem 4.2. *Let $\varphi : A^\omega \to A^\omega$ be a function computed by a finite state automaton. Then* $\mathrm{Fix}(\varphi)$ *has measure zero if and only if it is nowhere dense.*

Proof. Since each non-empty open subset of A^ω has positive measure, it suffices to show that if $\mathrm{Fix}(\varphi)$ is nowhere dense, then it has measure zero. Let $\mathcal{A} = (Q, A, \delta, \lambda)$ be a finite state automaton such that $\varphi = \mathcal{A}_{q_0}$, $q_0 \in Q$. Let Q' be the set of $q \in Q$ such that $\mathcal{A}_q$ is not the identity. The hypothesis on φ then implies that if $u \in A^*$ is fixed by φ, then $q_0 \cdot u \in Q'$ – if this were not the case φ would fix uA^ω. Since Q is finite, we can find an integer $p > 0$ such that, for each $q \in Q'$, $\mathcal{A}_q$ does not fix some element of A^p. Let $n = |A|$. We claim:

$$|\mathrm{Fix}_{pk}(\varphi)| \le (n^p - 1)^k \tag{4.2}$$

for all $k \ge 0$. We proceed by induction. The result is clear for $k = 0$. Suppose that (4.2) holds for $k-1$, with $k \ge 1$, and consider the n^{pk} words of length pk. By the inductive hypothesis, at most $(n^p-1)^{k-1}$ of the possible prefixes of length $p(k-1)$ of such words are fixed by φ. Now let $u \in A^{p(k-1)}$ be such a prefix that is fixed by φ. Then $q_0 \cdot u \in Q'$ by the hypothesis on φ. By choice of p, $\mathcal{A}_{q_0 \cdot u}$ does not fix all words of length p, so there are at most $n^p - 1$ words of length p that are fixed by $\mathcal{A}_{q_0\cdot u}$. Hence, there are at most $(n^p-1)^{k-1} \cdot (n^p - 1) = (n^p-1)^k$ words from A^{pk} that are fixed by φ, establishing (4.2). Thus

$$\frac{1}{n^{pk}} |\mathrm{Fix}_{pk}(\varphi)| \le \left(1 - \frac{1}{n^p}\right)^k .$$

Since the right hand side of the above equation tends to 0 as $k \to \infty$, the theorem follows by an application of Proposition 4.1. □

Let Γ be a group acting spherically transitively by rooted automorphisms of T. Then Γ acts ergodically on the measure space $(\partial T, \nu)$ by measure preserving transformations and topologically transitively on ∂T by isometries [11, 9]. Let $\pi : \Gamma \to \mathcal{B}(L^2(\partial T, \nu))$ be the associated representation, S be a finite symmetric generating set for Γ and H_π be the associated Hecke-type operator. Let $\pi_k : \Gamma \to \mathcal{B}(\ell^2(A^k))$ be the permutation representation of Γ on the k^{th} level of the tree. Identifying the elements of the A^k with the characteristic functions of the associated cylinder sets, π_k can be viewed as a subrepresentation of π and π_{k-1} as subrepresentation of π_k. If, for $k \ge 1$, π'_k is the orthogonal complement of π_{k-1} in π_k and $\pi'_0 = \pi_0$, then [1] $\pi = \bigoplus_{k=0}^\infty \pi'_k$ from which one obtains [1]

$$\mathsf{Sp}(H_\pi) = \overline{\bigcup_{n\ge 0} \mathsf{Sp}(H_{\pi_k})} \tag{4.3}$$

Recall [14] that if α is a finite dimensional representation of Γ, then the associated character χ_α is given by $\chi_\alpha(\gamma) = \mathrm{tr}(\alpha(\gamma))$. If α is a permutation representation, then χ_α is the *fixed-point* character; it simply counts the number of fixed-points of each element of Γ. It then seems natural to define the *fixed-point character* of π by

$$\chi_\pi(\gamma) = \nu(\mathrm{Fix}(\gamma)).$$

An immediate consequence of Proposition 4.1 is

$$\chi_\pi(\gamma) = \lim_{k\to\infty} \frac{1}{n^k}\chi_{\pi_k}(\gamma). \tag{4.4}$$

An action of a group Γ on a measure space is said to be *free in the sense of ergodic theory* if, for all $1 \neq \gamma \in \Gamma$, $\mathrm{Fix}(\gamma)$ has measure zero. We say that the action is *free in the sense of Baire category* if, for all $1 \neq \gamma \in \Gamma$, $\mathrm{Fix}(\gamma)$ is nowhere dense. Notice that in either of these two cases there is an infinite path $w \in \partial T$ with trivial stabilizer. Indeed, $\bigcup_{1\neq\gamma\in\Gamma}\mathrm{Fix}(\gamma)$ cannot be all of ∂T: in the first case this set has measure zero; in the latter it is a countable union of closed nowhere dense sets and hence cannot be all of ∂T by the Baire category theorem. Theorem 4.2 then has the following interpretation.

Theorem 4.3. *Let Γ be a group acting on the Cayley tree T of A^* (with A finite) by rooted automorphisms computed by finite state automata. Then the following are equivalent:*

(1) *The action of Γ on $(\partial T, \nu)$ is free in the sense of ergodic theory;*
(2) *The action of Γ on ∂T is free in the sense of Baire category;*

□

In light of the above result, it seems natural to say that the action of a group Γ generated by a finite state automaton is *free* if the equivalent conditions of Theorem 4.3 hold. The only non-trivial examples of free actions of automata groups in the literature, so far as we know, are the Cayley machines of finite abelian groups [25], including the automaton of the lamplighter group considered in [11]. For Cayley machines of non-abelian groups, fixed-point sets can have non-empty interior [25]. It would be interesting to find more examples.

Returning to the spectra of Hecke operators, let us fix $w \in \partial T$ and set w_k to be the prefix of w of length k. Let $P = \mathrm{St}_\Gamma(w)$ and $P_k = \mathrm{St}_\Gamma(w_k)$. The subgroup P is called a *parabolic subgroup* [9]. Then, for all $k \geq 0$, P_k is of finite index in Γ and $P = \bigcap_k P_k$. Moreover, since Γ acts spherically transitively, π_k is equivalent to the quasi-regular representation λ_{Γ/P_k} and $H_{\pi_k} = M_k$ the Markov operator on $\mathsf{Sch}(\Gamma, P_k, S)$; if M is the Markov operator on $\mathsf{Sch}(\Gamma, P, S)$, then $H_{\lambda_{\Gamma/P}} = M$. If, in addition, Γ is amenable, then [1][Theorem 3.6] shows that

$$\mathsf{Sp}(H_\pi) = \overline{\bigcup_{n\geq 0}\mathsf{Sp}(H_{\pi_k})} = \mathsf{Sp}(H_{\lambda_{\Gamma/P}}) \subseteq \mathsf{Sp}(H_{\lambda_\Gamma}) \subseteq [-1,1] \tag{4.5}$$

We now relate the fixed-point character to the moments of the KNS spectral measure μ associated to $\mathsf{Sch}(\Gamma, P, S)$ with respect to the approximating sequence $\mathsf{Sch}(\Gamma, P_k, S)$. If $w \in S^*$, we use $[w]$ to denote the image of w in G. Notice that, for $m \geq 0$,

$$M^m = \frac{1}{|S|^m}\sum_{w\in S^m}[w]. \tag{4.6}$$

The moments of μ_k (see definition (2.6)) are then given by:

$$\mu_k^{(m)} = E_{\mu_k}[\lambda^m] = \sum_{\lambda \in \mathsf{Sp}(M_k)} \lambda^m \frac{\#(\lambda)}{n^k} = \frac{1}{n^k}\mathrm{tr}(M_k^m) = \frac{1}{|S|^m}\sum_{w\in S^m}\frac{1}{n^k}\chi_{\pi_k}([w])$$

where the last equality holds from (4.6) and the fact that $M_k = \pi_k(M)$. Notice $0 \le \mu_k^{(m)} \le 1$, indeed it is the average over all $w \in S^m$ of the probability of $\pi_k([w])$ fixing a vertex on the k^{th} level. Thus the moment generating function of μ_k is analytic on $\mathbb{R}$ for all $k \geq 0$. Since the μ_k converge weakly to μ, it follows that, for each $m \geq 0$, $\mu_k^{(m)} \to \mu^{(m)}$ [4]. We are then led by (4.4) to the following formula for the moments of μ:

$$\mu^{(m)} = \frac{1}{|S|^m}\sum_{w\in S^m}\chi_\pi([w]). \tag{4.7}$$

Thus the m^{th} moment of μ is the average over all $w \in S^m$ of the probability of $\pi([w])$ fixing an infinite path, whence $0 \le \mu^{(m)} \le 1$. This implies that the moment generating function of μ is analytic on $\mathbb{R}$.

Let us contrast this with the situation for the Kesten spectral measure μ^Γ for the random walk on the Cayley graph of Γ. As mentioned earlier, the moments correspond to return probabilities. More precisely, if the Markov operator M on $\ell^2(\Gamma)$ has spectral decomposition

$$M = \int_{-1}^{1} \lambda \mathrm{d}E(\lambda)$$

with (projection-valued) spectral measure E, then

$$\mu^\Gamma(B) = \langle \int_B \mathrm{d}E(\lambda)\delta_1, \delta_1\rangle$$

for B a Borel subset of $[-1,1]$. So the m^{th} moment is given by

$$\begin{aligned}(\mu^\Gamma)^{(m)} &= \int_{-1}^{1}\lambda^m \mathrm{d}\mu^\Gamma = \langle \int_{-1}^{1}\lambda^m \mathrm{d}E(\lambda)\delta_1, \delta_1\rangle \\ &= \langle M^m\delta_1, \delta_1\rangle = \mathrm{tr}(M^m)\end{aligned}$$

where the last trace is the von Neumann trace, c.f. (2.5). Since $M^m \in \mathbb{R}\Gamma$, this is just the coefficient of 1. Notice that the return probability $p_m(1)$ is just the fraction of words in S^m representing the identity 1 of Γ. It follows from (4.6) that $(\mu^\Gamma)^{(m)} = p_m(1)$. From this, we may easily deduce that the moment generating function of μ^Γ is analytic on all of $\mathbb{R}$.

Now we are in a position to compare $\mu^{(m)}$ with $(\mu^\Gamma)^{(m)}$. Recalling that $0 \le \chi_\pi(\gamma) \le 1$ and $\chi_\pi(1) = 1$, it is clear that the right hand side of (4.7) is $p_m(1)$ precisely if, for each $w \in S^m$ such that $[w] \neq 1$, $\chi_\pi([w]) = 0$. In other words, the average probability that a word in $w \in S^*$ of length m fixes an infinite path will be the same as the probability that w represents the identity if and only if for all $w \in A^*$, either w represents the identity, or w almost surely does not fix any infinite path. Recall [4, Theorem 30.1] that if two probability measures on $[-1,1]$ have the same moment generating function, and this function is analytic on a neighbourhood of 0, then the measures are the same. Recalling that $\chi_\pi(\gamma) = \nu(\mathrm{Fix}(\gamma))$, we may summarize the previous discussion in the following theorem.

Theorem 4.4. *Let Γ be a group acting spherically transitively on the Cayley tree of A^* with finite symmetric generating set S. Let $w \in A^\omega$ be an infinite path and, for all $k \geq 0$, let w_k be the prefix of w of length k. Set $P = \mathrm{St}_\Gamma(w)$ and $P_k = \mathrm{St}_\Gamma(w_k)$ and let X and X_k be the respective Schreier graphs with respect to a finite generating set S. Then the following are equivalent:*

1. *The KNS spectral measure for X, with respect to the approximating sequence $\{X_k\}$, coincides with the Kesten spectral measure for the simple random walk on the Cayley graph of Γ*
2. *The action of Γ is free in the sense of ergodic theory.*

□

An immediate consequence of Theorems 4.3 and 4.4 is:

Corollary 4.5. *Let Γ be an automata group acting freely on the boundary of T. Choose $w \in \partial T$ with trivial stabilizer and define P_k, $k \geq 0$, as per Theorem 4.4. Then ζ_Γ coincides with the limit zeta function from the approximating sequence $\mathsf{Sch}(\Gamma, P_k, S)$.* □

5. Calculation of the spectral measures for Cayley machines

Fix for this section a non-trivial finite group $G = \{g_1, \dots, g_n\}$. As standing notation we set $\Gamma = \mathcal{G}(\mathcal{A}(G))$. It is locally finite-by-infinite cyclic [25] and hence amenable so (4.5) applies to computing the spectrum. In the case that G is abelian, the results of [25] show the action of Γ is free.

Let us establish some notation. We use $\overline{g_i}$ as a shorthand notation for the element $\mathcal{A}_{g_i} \in \Gamma$. Let

$$S = \{\overline{g_1}, \dots \overline{g_n}, \overline{g_1}^{-1}, \dots \overline{g_n}^{-1}\}; \tag{5.1}$$

so $|S| = 2n$. We fix a parabolic subgroup P. If G is abelian, we choose P to be trivial. If G is non-abelian, we can choose P to be a locally finite group [25]. Let w be an infinite path with stabilizer P and for $k \geq 0$ let w_k denote the prefix of w of length k. Let $P_k \leq \Gamma$ be the stabilizer of w_k. Now P_k has index n^k in Γ (by spherical transitivity) and $P = \bigcap P_k$. We shall show that all the spectra considered in equation (4.5) are the entire interval $[-1,1]$. We shall also calculate the KNS spectral measure associated to the Schreier graph $X = \mathsf{Sch}(\Gamma, P, S)$ with respect to the approximating sequence of graphs $X_k = \mathsf{Sch}(\Gamma, P_k, S)$, in particular obtaining the Kesten spectral measure for Γ in the case G is abelian.

5.1. **Operator recursion, wreath products and the monomial representation.** We begin by recalling some standard facts about the matrix representation of wreath products of permutations groups [14, 19]. Let (H, X) and (K, Y) be left permutation groups and let $(W, X \times Y) = (H, X) \wr (K, Y)$.

The associated *monomial representation* [14, 19] is described as follows. Let $(h, f) \in W = H \times K^X$. The *monomial matrix* for (h, f) is obtained from the $|X| \times |X|$ permutation matrix for h by replacing the 1 in column $i \in X$ by $f(i)$. The action on $X \times Y$ is recovered by considering column vectors of size $|X|$ with entries in Y. To linearize this representation, one has to replace column vectors of size $|X|$ with entries in Y by size $|X|$ block column vectors with entries size $|Y|$ column vectors. Then the matrix representation associated to $(W, X \times Y)$ takes the $|X| \times |X|$ monomial matrix corresponding to an element (h, f) and replaces each entry from $k \in K$ with the $|Y| \times |Y|$ permutation matrix associated to k from the permutation representation (K, Y) and replaces the zeroes by the $|Y| \times |Y|$ zero matrix. Thus the matrices for $(W, X \times Y)$ are block monomial.

Observe that the wreath product coordinates (3.6), restricted to T_{k+1}, show

$$(\pi_{k+1}(\Gamma), G^{k+1}) \leq (G, G) \wr (\pi_k(\Gamma), G^k).$$

On the automata generators, the embedding is given by

$$\pi_{k+1}(\overline{g}_i) \longmapsto g_i^{-1}(\pi_k(\overline{g}_1), \ldots, \pi_k(\overline{g}_n)) \tag{5.2}$$

This lets us construct inductively the matrices for the representations π_k using the monomial representation; this procedure is called *operator recursion* [11, 1]. For $i = 1, \ldots, n$, set $\pi_0(\overline{g}_i) = 1$. Then, for $k \geq 0$, the matrix for $\pi_{k+1}(\overline{g}_i)$ is obtained by taking the permutation matrix from the left regular representation of G corresponding to g_i^{-1} and replacing the 1 in column j by $\pi_k(\overline{g}_j)$ and the zeroes by the $n^k \times n^k$ zero matrix. So $\pi_{k+1}(\overline{g}_i)$ is an $n \times n$ block monomial matrix with blocks of size n^k. There is exactly one non-zero block in column j, namely in row l, where $g_i^{-1} g_j = g_l$; this block is $\pi_k(\overline{g}_j)$. Since all these matrices are permutation matrices, the inverse of a matrix $\pi_{k+1}(\overline{g}_i)$ is simply the transpose $\pi_{k+1}(\overline{g}_i)^T$.

For example, if $G = \mathbb{Z}/2\mathbb{Z} = \{a, b\}$ where a is the identity and b the non-trivial element, then (c.f. [11])

$$\pi_{k+1}(\overline{a}) = \begin{pmatrix} \pi_k(\overline{a}) & 0 \\ 0 & \pi_k(\overline{b}) \end{pmatrix}, \quad \pi_{k+1}(\overline{b}) = \begin{pmatrix} 0 & \pi_k(\overline{b}) \\ \pi_k(\overline{a}) & 0 \end{pmatrix}.$$

If $G = \mathbb{Z}/3\mathbb{Z} = \{a, b, c\}$ where a is the identity, then

$$\pi_{k+1}(\overline{a}) = \begin{pmatrix} \pi_k(\overline{a}) & 0 & 0 \\ 0 & \pi_k(\overline{b}) & 0 \\ 0 & 0 & \pi_k(\overline{c}) \end{pmatrix}, \ \pi_{k+1}(\overline{b}) = \begin{pmatrix} 0 & \pi_k(\overline{b}) & 0 \\ 0 & 0 & \pi_k(\overline{c}) \\ \pi_k(\overline{a}) & 0 & 0 \end{pmatrix}$$

$$\pi_{k+1}(\overline{c}) = \begin{pmatrix} 0 & 0 & \pi_k(\overline{c}) \\ \pi_k(\overline{a}) & 0 & 0 \\ 0 & \pi_k(\overline{b}) & 0 \end{pmatrix}.$$

Given square matrices A and B we write $A \otimes B$ for their tensor product, given in block matrix form:

$$A \otimes B = [A_{ij} B]_{i,j=1}^n.$$

Let T be the $n \times n$ matrix defined by

$$T_{ij} = \begin{cases} 1 \text{ if } i \neq j \\ 0 \text{ if } i = j \end{cases}$$

and set

$$S_0 = n - 1, \quad S_k = T \otimes I_{n^k}, \; k > 0.$$

Notice that T is the sum of the permutation matrices corresponding to the non-identity elements of G under the left regular representation; indeed for each $i \neq j$ there is a unique non-identity permutation of G taking i to j.

We shall need the following lemma later.

Lemma 5.1. *For all $k \in \mathbb{N}$, we have*

$$\sum_{i,j=1, i \neq j}^{n} \pi_k(\overline{g}_i)\pi_k(\overline{g}_j)^T = nS_k$$

Proof. For $k = 0$ this is clear, since, for each $i = 1, \ldots, n$, we add up 1 exactly $n - 1$ times. For $k > 0$, observe that (5.2) easily implies that in wreath product coordinates

$$\pi_k(\overline{g}_i)\pi_k(\overline{g}_j)^{-1} = g_i^{-1}g_j(1, \ldots, 1).$$

So in matrix form, we obtain $A \otimes I_{n^k}$ where A is the matrix for $g_i^{-1}g_j$ in the regular representation. If we fix i and sum over $j \neq i$ we get the sum of all permutation matrices from the regular representation of G except the identity, that is T, tensored with I_{n^k}. But this is precisely S_k. Since we have n choices for i, we obtain the formula of the lemma. □

5.2. **Calculation of the characteristic polynomial.** It will be convenient to not have to always divide by $2n$ so set $A_k = 2nM_k$; A_k is the incidence matrix of the Schreier graph $\mathsf{Sch}(\Gamma, P_k, S)$. More explicitly,

$$A_k = \sum_{i=1}^{n} (\pi_k(\overline{g}_i) + \pi_k(\overline{g}_i)^T).$$

Our first objective is to calculate the spectrum of the matrix A_k. To this end, define a function of two variables by

$$\Phi_k(\lambda, \mu) = |A_k - \lambda I_{n^k} - \mu S_k|$$

so that $\Phi_k(\lambda, 0)$ is the characteristic polynomial of A_k. Our objective, then, is to find the roots of $\Phi_k(\lambda, 0)$. To facilitate this, we shall obtain a recursive formula for Φ_{k+1} in terms of Φ_k. Of course

$$\Phi_0(\lambda, \mu) = 2n - \lambda - (n - 1)\mu \tag{5.3}$$

The additional term μS_k serves as a garbage collecting term; it arises when one tries to express the characteristic polynomial of A_{k+1} in terms of the characteristic polynomial of A_k.

The matrices $\pi_k(\overline{g}_i)$ and $\pi_k(\overline{g}_i)^T$, $i = 1, \ldots, n$, are block monomial matrices coming from the image of g_i^{-1}, respectively, g_i, under the left regular representation of G. Since the sum of the permutation matrices from the left regular representation of G is the matrix of all ones (see the discussion above concerning T, but now add in the identity matrix), we see that

$$A_{k+1} = \begin{pmatrix} \pi_k(\overline{g}_1) + \pi_k(\overline{g}_1)^T & \pi_k(\overline{g}_2) + \pi_k(\overline{g}_1)^T & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_1)^T \\ \pi_k(\overline{g}_1) + \pi_k(\overline{g}_2)^T & \pi_k(\overline{g}_2) + \pi_k(\overline{g}_2)^T & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_2)^T \\ \vdots & \vdots & \ddots & \vdots \\ \pi_k(\overline{g}_1) + \pi_k(\overline{g}_n)^T & \pi_k(\overline{g}_2) + \pi_k(\overline{g}_n)^T & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T \end{pmatrix}$$

and so $A_{k+1} - \lambda I_{n^{k+1}} - \mu S_{k+1}$ is given by

$$\begin{pmatrix} \pi_k(\overline{g}_1) + \pi_k(\overline{g}_1)^T - \lambda I & \pi_k(\overline{g}_2) + \pi_k(\overline{g}_1)^T - \mu I & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_1)^T - \mu I \\ \pi_k(\overline{g}_1) + \pi_k(\overline{g}_2)^T - \mu I & \pi_k(\overline{g}_2) + \pi_k(\overline{g}_2)^T - \lambda I & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_2)^T - \mu I \\ \vdots & \vdots & \ddots & \vdots \\ \pi_k(\overline{g}_1) + \pi_k(\overline{g}_n)^T - \mu I & \pi_k(\overline{g}_2) + \pi_k(\overline{g}_n)^T - \mu I & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T - \lambda I \end{pmatrix}$$

where I denotes the $n^k \times n^k$ identity matrix.

We now apply some row and column operations at the block level, designed to simplify the computation of the determinant. Applying the operation $C_i \mapsto C_i - C_n$, for $i = 1, \dots, n-1$, yields the matrix

$$\begin{pmatrix} \pi_k(\overline{g}_1) - \pi_k(\overline{g}_n) - (\lambda-\mu)I & \pi_k(\overline{g}_2) - \pi_k(\overline{g}_n) & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_1)^T - \mu I \\ \pi_k(\overline{g}_1) - \pi_k(\overline{g}_n) & \pi_k(\overline{g}_2) + \pi_k(\overline{g}_n) - (\lambda-\mu)I & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_2)^T - \mu I \\ \vdots & \vdots & \ddots & \vdots \\ \pi_k(\overline{g}_1) - \pi_k(\overline{g}_n) + (\lambda-\mu)I & \pi_k(\overline{g}_2) - \pi_k(\overline{g}_n) + (\lambda-\mu)I & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T - \lambda I \end{pmatrix}$$

Applying the operation $R_i \mapsto R_i - R_n$, for $i = 1, \dots, n-1$, we obtain the matrix $(*)$:

$$\begin{pmatrix} -2(\lambda-\mu)I & -(\lambda-\mu)I & \dots & \pi_k(\overline{g}_1)^T - \pi_k(\overline{g}_n)^T + (\lambda-\mu)I \\ -(\lambda-\mu)I & -2(\lambda-\mu)I & \dots & \pi_k(\overline{g}_2)^T - \pi_k(\overline{g}_n)^T + (\lambda-\mu)I \\ \vdots & \vdots & \ddots & \vdots \\ -(\lambda-\mu)I & -(\lambda-\mu)I & \dots & \pi_k(\overline{g}_{n-1})^T - \pi_k(\overline{g}_n)^T + (\lambda-\mu)I \\ \pi_k(\overline{g}_1) - \pi_k(\overline{g}_n) + (\lambda-\mu)I & \pi_k(\overline{g}_2) - \pi_k(\overline{g}_n) + (\lambda-\mu)I & \dots & \pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T - \lambda I \end{pmatrix}$$

To calculate the determinant of this matrix, we need some technical results.

Lemma 5.2. *Suppose that we have a block matrix*

$$A = \begin{pmatrix} A_{11} & 0 & \dots & 0 & A_{1n} \\ 0 & A_{22} & 0 & \dots & A_{2n} \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & A_{n-1,n-1} & A_{n-1,n} \\ A_{n1} & A_{n2} & \dots & A_{n,n-1} & A_{nn} \end{pmatrix}$$

where the A_{ij} are square matrices of the same size. Moreover, suppose that $A_{11}, A_{22}, \dots, A_{n-1,n-1}$ commute with all the other matrices. Then

$$|A| = \left| A_{11} \cdots A_{nn} - \sum_{i=1}^{n-1} A_{11} \cdots \widehat{A_{ii}} \cdots A_{n-1,n-1} A_{ni} A_{in} \right| \tag{5.4}$$

where $\widehat{A_{ii}}$ *means omit* A_{ii}.

Proof. Since the invertible matrices are dense in the space of matrices, we may assume without loss of generality that A_{ii} is invertible, $i = 1, \ldots, n-1$. Let I be the identity matrix of the same size as the A_{ij}. Then one verifies directly that

$$A = \begin{pmatrix} A_{11} & 0 & \ldots & 0 & 0 \\ 0 & A_{22} & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & A_{n-1,n-1} & 0 \\ A_{n1} & A_{n2} & \ldots & A_{n,n-1} & I \end{pmatrix} \times$$

$$\begin{pmatrix} I & 0 & \ldots & 0 & A_{11}^{-1}A_{1n} \\ 0 & I & 0 & \vdots & A_{22}^{-1}A_{2n} \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & I & A_{n-1,n-1}^{-1}A_{n-1,n} \\ 0 & 0 & \ldots & 0 & A_{nn} - \sum_{i=1}^{n-1} A_{ni}A_{ii}^{-1}A_{in} \end{pmatrix}$$

Using that the determinant of a block upper (lower) triangular matrix is the product of the determinant of the diagonal blocks and that A_{ii}, $i = 1, \ldots, n-1$, commutes with the remaining matrices gives (5.4). □

Corollary 5.3. *Consider a block matrix*

$$M = \begin{pmatrix} 2A & A & \ldots & A & B_{1n} \\ A & 2A & \ldots & A & B_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ A & A & \ldots & 2A & B_{n-1,n} \\ B_{n1} & B_{n2} & \ldots & B_{n,n-1} & B_{nn} \end{pmatrix}$$

where A is a square matrix and $B_{1n}, \ldots, B_{nn}, B_{n1}, \ldots, B_{nn}$ are square matrices of the same size as A, commuting with A. Then

$$|M| = \left| A^{n-2} \left(nAB_{nn} - (n-1) \sum_{i-1}^{n-1} B_{ni}B_{in} + \sum_{i \neq j}^{n-1} B_{ni}B_{jn} \right) \right|.$$

Proof. We proceed by applying the following elementary row and column operations to the rows and columns of *blocks* in M.

(i) $C_i \mapsto C_i - C_{i+1}$ for $i = 1, \ldots, n-2$;
(ii) $R_i \mapsto \sum_{j=1}^{i} R_j$ for $i = n-1, \ldots, 2$;
(iii) $C_{n-1} \mapsto C_{n-1} - \sum_{j=1}^{n-2} jC_j$.

These operations leave the determinant unchanged and it is easy to verify that they result in the matrix:

$$\begin{pmatrix} A & 0 & \dots & 0 & 0 & B_{1n} \\ 0 & A & \dots & 0 & 0 & B_{1n}+B_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A & 0 & \sum_{j=1}^{n-2} B_{jn} \\ 0 & 0 & \dots & 0 & nA & \sum_{j=1}^{n-1} B_{jn} \\ B_{n1}-B_{n2} & B_{n2}-B_{n3} & \dots & B_{n,n-2}-B_{n,n-1} & (n-1)B_{n,n-1}-\sum_{j=1}^{n-2} B_{nj} & B_{nn} \end{pmatrix}$$

Since this matrix has the same determinant as M, applying Lemma 5.2 gives us that

$$\begin{aligned} |M| = nA^{n-1}B_{nn} - nA^{n-2}\sum_{i=1}^{n-2}\left([B_{ni}-B_{n,i+1}]\sum_{j=1}^{i} B_{jn}\right) \\ - A^{n-2}\left((n-1)B_{n,n-1}-\sum_{i=1}^{n-2}B_{ni}\right)\left(\sum_{j=1}^{n-1}B_{jn}\right). \end{aligned} \tag{5.5}$$

By telescoping we obtain

$$\sum_{i=1}^{n-2}\left([B_{ni}-B_{n,i+1}]\sum_{j=1}^{i} B_{jn}\right) = \sum_{i=1}^{n-2} B_{ni}B_{in} - B_{n,n-1}\sum_{i=1}^{n-2} B_{in}. \tag{5.6}$$

Substituting (5.6) into (5.5) gives

$$\begin{aligned} |M| &= A^{n-2}\left(nAB_{nn} - n\sum_{i=1}^{n-2}B_{ni}B_{in} + nB_{n,n-1}\sum_{i=1}^{n-2}B_{in}\right. \\ &\qquad \left. -(n-1)B_{n,n-1}\sum_{j=1}^{n-1}B_{jn} + \sum_{i=1}^{n-2}\sum_{j=1}^{n-1}B_{ni}B_{jn}\right) \\ &= A^{n-2}\left(nAB_{nn} - n\sum_{i=1}^{n-2}B_{ni}B_{in} + B_{n,n-1}\sum_{i=1}^{n-2}B_{in}\right. \\ &\qquad \left. -(n-1)B_{n,n-1}B_{n-1,n} + \sum_{i=1}^{n-2}\sum_{j=1}^{n-1}B_{ni}B_{jn}\right) \\ &= A^{n-2}\left(nAB_{nn} - n\sum_{i=1}^{n-1}B_{ni}B_{in} + \sum_{i=1,j=1}^{n-1}B_{ni}B_{jn}\right) \\ &= A^{n-2}\left(nAB_{nn} - (n-1)\sum_{i=1}^{n-1}B_{ni}B_{in} + \sum_{i\neq j}^{n-1}B_{ni}B_{jn}\right) \end{aligned}$$

as desired. □

Corollary 5.3 gives us a method of calculating the determinant of $(*)$. To apply it we set

$$\begin{aligned} A &= (\mu - \lambda)I \\ B_{ni} &= \pi_k(\overline{g}_i) - \pi_k(\overline{g}_n) - (\mu - \lambda)I, \ i = 1, \ldots, n-1 \\ B_{jn} &= \pi_k(\overline{g}_j)^T - \pi_k(\overline{g}_n)^T - (\mu - \lambda)I, \ j = 1, \ldots, n-1 \\ B_{nn} &= \pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T - \lambda I \end{aligned}$$

From this we obtain for $i, j = 1, \ldots, n-1$

$$\begin{aligned} B_{ni}B_{jn} &= \pi_k(\overline{g}_i)\pi_k(\overline{g}_j)^T - \pi_k(\overline{g}_i)\pi_k(\overline{g}_n)^T - \pi_k(\overline{g}_n)\pi_k(\overline{g}_j)^T + I \\ &\quad + (\mu - \lambda)^2 I - (\mu - \lambda)\left(\pi_k(\overline{g}_i) + \pi_k(\overline{g}_j)^T - \pi_k(\overline{g}_n) - \pi_k(\overline{g}_n)^T\right). \end{aligned}$$

Notice that if $i = j$, then the first term becomes I.

Substituting the above values into the formula from Corollary 5.3 gives

$$\begin{aligned} \Phi_{k+1}(\lambda, \mu) &= (\mu - \lambda)^{(n-2)n^k} |n(\mu - \lambda)(\pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T - \lambda I) \\ &\quad - (n-1)\sum_{i=1}^{n-1}\left[2I - \pi_k(\overline{g}_i)\pi_k(\overline{g}_n)^T - \pi_k(\overline{g}_n)\pi_k(\overline{g}_i)^T + (\mu - \lambda)^2 I\right. \\ &\qquad \left. -(\mu - \lambda)\left(\pi_k(\overline{g}_i) + \pi_k(\overline{g}_i)^T - \pi_k(\overline{g}_n) - \pi_k(\overline{g}_n)^T\right)\right] \\ &\quad + \sum_{i \neq j}^{n-1}[\pi_k(\overline{g}_i)\pi_k(\overline{g}_j)^T - \pi_k(\overline{g}_i)\pi_k(\overline{g}_n)^T - \pi_k(\overline{g}_n)\pi_k(\overline{g}_j)^T + I \\ &\qquad + (\mu - \lambda)^2 I - (\mu - \lambda)\left(\pi_k(\overline{g}_i) + \pi_k(\overline{g}_j)^T - \pi_k(\overline{g}_n) - \pi_k(\overline{g}_n)^T\right)]| \\ &= (\mu - \lambda)^{(n-2)n^k} |n(\mu - \lambda)(\pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T - \lambda I) \\ &\quad - 2(n-1)^2 I + (n-1)\sum_{i=1}^{n-1}(\pi_k(\overline{g}_i)\pi_k(\overline{g}_n)^T + \pi_k(\overline{g}_n)\pi_k(\overline{g}_i)^T) \\ &\quad - (n-1)^2(\mu - \lambda)^2 I + (n-1)(\mu - \lambda)\sum_{i=1}^{n-1}(\pi_k(\overline{g}_i) + \pi_k(\overline{g}_i)^T) \\ &\quad - (n-1)^2(\mu - \lambda)(\pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T) + \sum_{i \neq j}^{n-1}\pi_k(\overline{g}_i)\pi_k(\overline{g}_j)^T \\ &\quad - (n-2)\sum_{i=1}^{n-1}(\pi_k(\overline{g}_i)\pi_k(\overline{g}_n)^T + \pi_k(\overline{g}_n)\pi_k(\overline{g}_i)^T) \\ &\quad + (n-1)(n-2)I + (n-1)(n-2)(\mu - \lambda)^2 I \\ &\quad - (n-2)(\mu - \lambda)\sum_{i=1}^{n-1}(\pi_k(\overline{g}_i) + \pi_k(\overline{g}_i)^T) \\ &\quad + (n-1)(n-2)(\mu - \lambda)(\pi_k(\overline{g}_n) + \pi_k(\overline{g}_n)^T| \end{aligned}$$

$$= (\mu - \lambda)^{(n-2)n^k} \times$$
$$\left|(\mu - \lambda)(A_k - \lambda I - (n-1)\mu I) - n(n-1)I + \sum_{i \neq j}^{n} \pi_k(\overline{g}_i)\pi_k(\overline{g}_j)^T\right|$$

Applying Lemma 5.1 we obtain

$$\begin{aligned}
\Phi_{k+1}(\lambda, \mu) & \\
&= (\mu - \lambda)^{(n-2)n^k} |(\mu - \lambda)(A_k - \lambda I - (n-1)\mu I) \\
&\qquad - n(n-1)I + nS_k| \\
&= (\mu - \lambda)^{(n-1)n^k} \left| A_k - \left(\lambda + (n-1)\mu + \frac{n(n-1)}{\mu - \lambda}\right) I \right. \\
&\qquad \left. + \frac{n}{\mu - \lambda} S_k \right| \qquad (5.7) \\
&= (\mu - \lambda)^{(n-1)n^k} \Phi_k \left(\lambda + (n-1)\mu + \frac{n(n-1)}{\mu - \lambda}, \quad -\frac{n}{\mu - \lambda}\right) \\
&= (\mu - \lambda)^{(n-1)n^k} \times \\
&\qquad \Phi_k \left(\frac{-\lambda^2 + (n-1)\mu^2 + (2-n)\lambda\mu + n(n-1)}{\mu - \lambda}, -\frac{n}{\mu - \lambda}\right)
\end{aligned}$$

Next, we seek to solve the recursion and obtain an explicit formula.

5.3. **Calculation of the eigenvalues.** Given $\lambda, \mu \in \mathbb{C}$, we write

$$\lambda' = \frac{-\lambda^2 + (n-1)\mu^2 + (2-n)\lambda\mu + n(n-1)}{\mu - \lambda}$$

and

$$\mu' = -\frac{n}{\mu - \lambda}$$

We define a sequence of functions $F_k(\lambda, \mu)$ inductively by

$$F_1(\lambda, \mu) = \mu - \lambda$$

and for $k \geq 1$

$$F_{k+1}(\lambda, \mu) = F_k(\lambda', \mu').$$

Lemma 5.4. *Define a sequence by* $(\lambda_1, \mu_1) = (\lambda, \mu)$ *and* $(\lambda_{k+1}, \mu_{k+1}) = (\lambda'_k, \mu'_k)$ *for* $k \geq 1$*. Then for any* $k \geq 1$ *we have*

(i) $\lambda_{k+1} + (n-1)\mu_{k+1} = \lambda_k + (n-1)\mu_k$;
(ii) $\mu_{k+1} - \lambda_{k+1} = -(\lambda + (n-1)\mu) - \frac{n^2}{\mu_k - \lambda_k}$;
(iii) $F_k(\lambda, \mu) = \mu_k - \lambda_k$; *and*
(iv) $F_{k+1}(\lambda, \mu) = -(\lambda + (n-1)\mu) - \frac{n^2}{F_k(\lambda, \mu)}$.

Proof. It is a straightforward computation to verify claims (i) and (ii) using the definitions of λ_{k+1} and μ_{k+1}, while claims (iii) and (iv) are immediate consequences of (ii), together with the definitions. □

We remark that the rational function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(\lambda, \mu) = (\lambda', \mu')$ is integrable in the sense of [13]. Namely, if $\psi : \mathbb{R}^2 \to \mathbb{R}$ is given by $\psi(\lambda, \mu) = \lambda + (n-1)\mu$ and $\alpha : \mathbb{R} \to \mathbb{R}$ is the identity, then the previous lemma implies $\alpha\psi = \psi f$.

Lemma 5.5. *For all μ, λ and all $k \geq 0$*

$$\Phi_k(\lambda, \mu) = (2n - \lambda - (n-1)\mu) \prod_{i=1}^{k} (F_i(\lambda, \mu))^{(n-1)n^{k-i}}$$

Proof. For $k = 0$, this is clear from (5.3). Now suppose the lemma holds for $k \geq 0$. Then by (5.7),

$$\begin{aligned}
\Phi_{k+1}(\lambda, \mu) &= (\mu - \lambda)^{(n-1)n^k} \Phi_k(\lambda', \mu') \\
&= (\mu - \lambda)^{(n-1)n^k} (2n - \lambda' - (n-1)\mu') \prod_{i=1}^{k} (F_i(\lambda', \mu'))^{(n-1)n^{k-i}} \\
&= (\mu - \lambda)^{(n-1)n^k} (2n - \lambda - (n-1)\mu) \prod_{i=1}^{k} (F_{i+1}(\lambda, \mu))^{(n-1)n^{k-i}} \\
&\qquad \text{(by Lemma 5.4(i))} \\
&= (2n - \lambda - (n-1)\mu)(F_1(\lambda, \mu))^{(n-1)n^k} \prod_{i=2}^{k+1} (F_i(\lambda, \mu))^{(n-1)n^{k+1-i}}
\end{aligned}$$

establishing the lemma. □

We now want to express each F_k as a rational function P_k/Q_k so that we can compute our determinant. We define inductively polynomials

$$\begin{aligned}
P_1(\lambda, \mu) &= \mu - \lambda, & P_{k+1}(\lambda, \mu) &= -(\lambda + (n-1)\mu)P_k(\lambda, \mu) - n^2 Q_k(\lambda, \mu) \\
Q_1(\lambda, \mu) &= 1, & Q_{k+1}(\lambda, \mu) &= P_k(\lambda, \mu).
\end{aligned}$$

Lemma 5.6. *For $k \geq 1$ we have*

$$F_k(\lambda, \mu) = \frac{P_k(\lambda, \mu)}{Q_k(\lambda, \mu)}.$$

Proof. For $k = 1$ the result is clear. Now let $k \geq 1$ and assume by induction that

$$F_k(\lambda, \mu) = \frac{P_k(\lambda, \mu)}{Q_k(\lambda, \mu)}.$$

Then Lemma 5.4(iv) tells us that

$$F_{k+1}(\lambda, \mu) = -(\lambda + (n-1)\mu) - \frac{n^2}{F_k(\lambda, \mu)}$$

so we obtain

$$F_{k+1}(\lambda, \mu) = -(\lambda + (n-1)\mu) - \frac{n^2 Q_k(\lambda, \mu)}{P_k(\lambda, \mu)}$$

$$= \frac{-(\lambda + (n-1)\mu)P_k(\lambda,\mu) - n^2 Q_k(\lambda,\mu)}{P_k(\lambda,\mu)}$$
$$= \frac{P_{k+1}(\lambda,\mu)}{Q_{k+1}(\lambda,\mu)}.$$

as required. □

We are primarily interested in the case $\mu = 0$, so set $P_k(\lambda) = P_k(\lambda, 0)$ and $Q_k(\lambda) = Q_k(\lambda, 0)$. Now $P_k(\lambda)$ and $Q_k(\lambda)$ satisfy:

$$\begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix} = \begin{pmatrix} -\lambda & -n^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{k-1}(\lambda) \\ Q_{k-1}(\lambda) \end{pmatrix} = \begin{pmatrix} -\lambda & -n^2 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Calculating the eigenvalues and eigenvectors and then diagonalizing, we obtain:

$$\begin{pmatrix} -\lambda & -n^2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{2n^2}{-\lambda+\sqrt{\lambda^2-4n^2}} & \frac{2n^2}{-\lambda-\sqrt{\lambda^2-4n^2}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2n^2}{-\lambda+\sqrt{\lambda^2-4n^2}} & 0 \\ 0 & \frac{2n^2}{-\lambda-\sqrt{\lambda^2-4n^2}} \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{-1}{\sqrt{\lambda^2-4n^2}} & \frac{1}{2} - \frac{\lambda}{2\sqrt{\lambda^2-4n^2}} \\ \frac{1}{\sqrt{\lambda^2-4n^2}} & \frac{1}{2} + \frac{\lambda}{2\sqrt{\lambda^2-4n^2}} \end{pmatrix} \tag{5.8}$$

where the last matrix on the right hand side is the inverse of the first.

We now make a change of variables by setting $\lambda = 2n \cos z$ for $z \in [0, \pi]$. This change of variables gives a bijection between $[0, \pi]$ and $[-2n, 2n]$. Since $\|A_k\| \leq 2n$, all our eigenvalues belong to $[-2n, 2n]$ and so we can use this change of variables to compute the eigenvalues. Then (5.8) becomes

$$\begin{pmatrix} -2n\cos z & -n^2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{n}{-\cos z + i \sin z} & \frac{n}{-\cos z - i \sin z} \\ 1 & 1 \end{pmatrix} \times$$
$$\begin{pmatrix} \frac{n}{-\cos z + i \sin z} & 0 \\ 0 & \frac{n}{-\cos z - i \sin z} \end{pmatrix} \begin{pmatrix} \frac{-1}{2ni \sin z} & \frac{1}{2} - \frac{\cos z}{2i \sin z} \\ \frac{1}{2ni \sin z} & \frac{1}{2} + \frac{\cos z}{2i \sin z} \end{pmatrix}.$$

Applying this decomposition to solve the above recursion we obtain

$$\begin{pmatrix} P_k(2n\cos z) \\ Q_k(2n\cos z) \end{pmatrix} = \begin{pmatrix} \frac{-n}{e^{-iz}} & \frac{-n}{e^{iz}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{-n}{e^{-iz}}\right)^k & 0 \\ 0 & \left(\frac{-n}{e^{iz}}\right)^k \end{pmatrix} \times$$
$$\begin{pmatrix} \frac{-1}{2ni \sin z} & \frac{1}{2} - \frac{\cos z}{2i \sin z} \\ \frac{1}{2ni \sin z} & \frac{1}{2} + \frac{\cos z}{2i \sin z} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{1}{2ni \sin z} \left((-ne^{iz})^{k+1} - (-ne^{-iz})^{k+1} \right) \\ -\frac{1}{2ni \sin z} \left((-ne^{iz})^{k} - (-ne^{-iz})^{k} \right) \end{pmatrix}.$$

Thus, we have

$$F_k(\lambda, 0) = \frac{P_k(\lambda)}{Q_k(\lambda)}$$

$$= -n\left(\frac{e^{iz(k+1)} - e^{-iz(k+1)}}{e^{izk} - e^{-izk}}\right)$$
$$= -n\left(\frac{\sin(z(k+1))}{\sin(zk)}\right).$$

It now follows from Lemma 5.5 that

$$\begin{aligned}\Phi_k(\lambda, 0) &= \Phi_k(2n\cos z, 0)\\ &= (2n - 2n\cos z)\prod_{j=1}^{k}\left[-n\left(\frac{\sin(z(j+1))}{\sin(zj)}\right)\right]^{(n-1)n^{k-j}}\\ &= 2n(1-\cos z)(-n)^{(n-1)n^{\frac{(k-1)k}{2}}}\left(\frac{1}{\sin z}\right)^{(n-1)n^{k-1}} \\ &\quad \left(\prod_{j=2}^{k}(\sin(zj))^{(n-1)^2 n^{k-j}}\right)(\sin(z(k+1)))^{n-1}\end{aligned} \tag{5.9}$$

Thus, $\Phi_k(2n\cos z, 0) = 0$ if and only if either $\cos z = 1$ or $\sin(zj) = 0$ for some $j \in \{2, \ldots, k+1\}$. That is, if and only if either $z = 0$ or $zj = l\pi$ for some $j \in \{2, \ldots, k+1\}$ and integer l. That is, if and only if $z = 0$ or $z = \frac{l}{j}\pi$ with $j \in \{2, \ldots k+1\}$ and $1 \leq l \leq j$. The corresponding values of λ give the set of eigenvalues:

$$\{2n\} \cup \left\{2n\cos\frac{p}{q}\pi \mid q \in \{2, \ldots, k+1\}, 1 \leq p < q\right\}.$$

Notice that $-2n$ is not an eigenvalue because the factor $\left(\frac{1}{\sin z}\right)^{(c-1)n^{k-1}}$ compensates exactly for the remaining factors.

Our next objective is to determine the multiplicities of the eigenvalues. First we determine the multiplicity of $2n$ as an eigenvalue. It is a basic result in Perron-Frobenius theory [20, 5] that, for a connected $2n$-regular graph, the multiplicity of $2n$ as an eigenvalue of the incidence matrix is 1. Hence, $2n$ has multiplicity 1 as an eigenvalue of A_k.

The multiplicities of the remaining eigenvalues can be computed from the formula (5.9). Suppose p and q are such that $q \in \{2, \ldots, k+1\}$, $1 \leq p < q$ and $(p, q) = 1$; we wish to calculate the multiplicity of $2n\cos\frac{p}{q}\pi$. If $q = k+1$, then only the last term of (5.9) contributes to the multiplicity, so we have multiplicity $n-1$. Suppose now $q \in \{2, \ldots, k\}$. Let $j \in \{2, \ldots, k+1\}$. Then we have $\sin\frac{pj}{q}\pi = 0$ if and only if $q \mid pj$, that is, if and only if $q \mid j$. Thus, setting $[r]$ to be the integer part of a real number r and $\chi_{\mathrm{Div}(k+1)}$ to be the characteristic function for the set of divisors of $k+1$, we obtain that the eigenvalue $2n\cos\frac{p}{q}\pi$ has multiplicity:

$$(n-1)^2\sum_{i=1}^{[\frac{k}{q}]} n^{k-qi} + (n-1)\chi_{\mathrm{Div}(k+1)}(q)$$

$$= n^k(n-1)^2\left(\frac{1-n^{-q([\frac{k}{q}]+1)}}{1-n^{-q}}-1\right)+(n-1)\chi_{\mathrm{Div}(k+1)}(q).$$

Summing up, we have proven the following:

Theorem 5.7. *The spectrum of the Markov operator M_k from the random walk on $\mathsf{Sch}(\Gamma, P_k, S)$ is:*

$$\{1\}\cup\left\{\cos\frac{p}{q}\pi \mid q\in\{2,\dots,k+1\}, 1\le p<q\right\}.$$

The eigenvalue 1 has multiplicity 1. For $1\le p<q$ with p and q coprime, the multiplicity is

$$\#\left(\cos\frac{p}{q}\pi\right)=\begin{cases} n-1 & \text{if } q=k+1\\ n^k(n-1)^2\left(\frac{1-n^{-q([\frac{k}{q}]+1)}}{1-n^{-q}}-1\right) & \text{else.}\\ \qquad +(n-1)\chi_{\mathrm{Div}(k+1)}(q) & \end{cases}$$

□

It is interesting to note that this result only depends on the size n of our finite group G and not on the structure of G. At this point, it would be easy to compute the Ihara zeta function of X_k. We instead wait to compute the zeta function for X.

Let ϕ be the Euler totient function, so that $\phi(q)$ denotes the number of positive integers less than or equal to q and coprime to q . Using Theorem 5.7 we obtain a proof of a classical result from number theory [15, Theorem 309]. A probabilistic interpretation can be given to this result from our computation of the KNS spectral measure; see Proposition 5.9 in the next subsection.

Corollary 5.8. *Let $n\ge 2$ be an integer. Then*

$$(n-1)^2\sum_{q=2}^{\infty}\frac{\phi(q)}{n^q-1}=1.$$

Proof. The operator M_k, being symmetric, has n^k eigenvalues with multiplicity. Using Theorem 5.7 and observing that the multiplicity of $\cos\frac{p}{q}\pi$, where $1\le p<q$, $(p,q)=1$, depends only on q, we obtain

$$n^k = 1+n^k(n-1)^2\sum_{q=2}^{k}\phi(q)\left(\frac{1-n^{-q([\frac{k}{q}]+1)}}{1-n^{-q}}-1\right)+(n-1)\sum_{q\mid k+1, q\neq 1}\phi(q).$$

Dividing both sides by n^k gives:

$$1=\frac{1}{n^k}+(n-1)^2\sum_{q=2}^{k}\phi(q)\left(\frac{1-n^{-q([\frac{k}{q}]+1)}}{1-n^{-q}}-1\right)+\frac{(n-1)}{n^k}\sum_{q\mid k+1, q\neq 1}\phi(q).$$

Since $\phi(q) \leq q$, and so

$$\sum_{q|k+1, q\neq 1} \phi(q) \leq \frac{(k+1)(k+2)}{2} - 1,$$

we see, by taking the limit as $k \to \infty$, that

$$\begin{aligned} 1 &= \lim_{k\to\infty} (n-1)^2 \sum_{q=2}^{k} \phi(q) \left(\frac{1 - n^{-q([\frac{k}{q}]+1)}}{1-n^{-q}} - 1 \right) \\ &= \lim_{k\to\infty} (n-1)^2 \sum_{q=2}^{k} \phi(q) \left(\frac{n^q - n^{-q[\frac{k}{q}]}}{n^q - 1} - 1 \right) \\ &= \lim_{k\to\infty} (n-1)^2 \left[\sum_{q=2}^{k} \frac{\phi(q)}{n^q - 1} - \sum_{q=2}^{k} \phi(q) \left(\frac{n^{-q[\frac{k}{q}]}}{n^q - 1} \right) \right]. \end{aligned}$$

Thus, to establish the result, we need only to show that the last term vanishes as $k \to \infty$. But

$$\begin{aligned} \sum_{q=2}^{k} \phi(q) \left(\frac{n^{-q[\frac{k}{q}]}}{n^q - 1} \right) &\leq k \sum_{q=2}^{k} \left(\frac{n^{-q(\frac{k}{q}-1)}}{n^q - 1} \right) = k \sum_{q=2}^{k} \frac{1}{n^k - n^{k-q}} \\ &\leq k(k-1) \frac{1}{n^{k-2}(n^2-1)} \longrightarrow 0 \text{ as } k \to \infty. \end{aligned}$$

□

5.4. **Calculating the KNS measure.** If ψ is a probability measure defined on the Borel subsets of $\mathbb{R}$, then the associated (cumulative) *distribution function* [4] $F_\psi : \mathbb{R} \to [0, 1]$ is given by

$$F_\psi(x) = \psi((-\infty, x]).$$

One has that F_ψ is non-decreasing, right continuous and

$$\lim_{x\to-\infty} F_\psi(x) = 0, \quad \lim_{x\to\infty} F_\psi(x) = 1. \tag{5.10}$$

Conversely, if $F : \mathbb{R} \to [0, 1]$ is a non-decreasing, right continuous function satisfying (5.10), then there is a unique probability measure ψ_F on the Borel sets of $\mathbb{R}$ such that $\psi_F((a, b]) = F(b) - F(a)$ [4]. The function F can have at most countabley many discontinuity points [4]. If the sum of the jumps of these points is 1, then ψ_F will be a discrete measure supported at the discontinuity points of F with the weight at a discontinuity point equal to the amount of the jump.

Suppose $\{\psi_k\}$ is a sequence of probability measures on $\mathbb{R}$. Then $\psi_k \to \psi$ weakly if and only if $F_{\psi_n}(x) \to F_\psi(x)$ at each point of continuity of F_ψ [4].

Let μ_k be the measure associated to M_k as per (2.6). The KNS spectral measure μ is the weak limit of μ_k. We shall compute it by computing its distribution function. To ease notation, we shall perform a change of variables. Let $f : [0,1] \to [-1,1]$ be given by $f(x) = \cos \pi x$. Let us define the measures σ_k, $k \geq 0$, and σ on the Borel sets of $[0,1]$ by

$$\sigma_k(B) = \mu_k(f(B)), \quad \sigma(B) = \mu(f(B)).$$

Since f is a homeomorphism it follows that σ and μ determine each other and that $\sigma_k \to \sigma$ weakly.

To calculate F_σ, we define a one-parameter family of Euler ϕ-functions $\{\phi_x : \mathbb{N} \to \mathbb{N}\}_{x \in [0,1]}$, by

$$\phi_x(q) = \left| \left\{ p \in \mathbb{N} \mid (p,q) = 1 \text{ and } \frac{p}{q} \leq x \right\} \right|.$$

Then, $\phi_0(q) = 0$, $\phi(q_1) = \phi(q)$ and $\phi_x(q)$ is non-decreasing as a function of x for fixed $q \in \mathbb{N}$.

Proposition 5.9. *For all $x \in [0,1]$,*

$$\lim_{k \to \infty} F_{\sigma_k}(x) = (n-1)^2 \sum_{q=2}^{\infty} \frac{\phi_x(q)}{n^q - 1}.$$

Proof. This proof is exactly like the proof of Corollary 5.8, only in the right hand side of the various equations, the role of $\phi(q)$ is taken by $\phi_x(q)$, while in the left hand side the role of 1 is taken by $F_{\sigma_k}(x)$ before taking limits. The same estimates apply since $\phi_x(q) \leq \phi(q)$. □

Set $F = \lim F_{\sigma_k}$. We know that $F = F_\sigma$ since $\sigma_k \to \sigma$ weakly, but we prefer to verify directly that F is indeed a distribution function, thereby giving a direct proof, independent of [13], that the sequence of measures σ_k has a weak limit.

Proposition 5.10. *F is a probability distribution function defining a discrete measure σ supported on the rational points of the interval $(0,1)$. More precisely,*

$$\sigma = (n-1)^2 \sum_{q=2}^{\infty} \left(\sum_{1 \leq p < q, \ (p,q)=1} \frac{1}{n^q - 1} \delta_{\frac{p}{q}} \right) \tag{5.11}$$

where $\delta_{\frac{p}{q}}$ is a Dirac measure.

Proof. Since $\phi_x(q)$ is non-decreasing as a function of x (for q fixed), F is clearly non-decreasing. By Corollary 5.8, $F(1) = 1$, while clearly $F(0) = 0$. Now we show right continuity. It is immediate from Proposition 5.9 that if $\frac{p}{q} \in (0,1)$ is a rational point, then

$$\lim_{x \to \frac{p}{q}-} \left(F\left(\frac{p}{q}\right) - F(x) \right) = (n-1)^2 \frac{1}{n^q - 1} \tag{5.12}$$

Hence, by Corollary 5.8, the sum of jumps at the rational points is 1. It follows that $F(x)$ is continuous at irrational points and the jump at a rational point $\frac{p}{q}$ is $\frac{(n-1)^2}{n^q-1}$. From this, (5.11) is immediate. □

Changing variables, observing that the set $\{\cos\frac{p}{q}\pi \mid 1 \le p < q\}$ is dense in $[-1,1]$ and using the freeness of the action in the case G is abelian (in conjunction with Theorem 4.4, we obtain our main result.

Theorem 5.11. *Let G be a non-trivial finite group. Then the KNS spectral measure μ for the Schreier graph of $\mathcal{G}(\mathcal{C}(G))$ with respect to a parabolic subgroup P and generators (5.1) is a discrete measure given by*

$$\mu = (n-1)^2 \sum_{q=2}^{\infty} \left(\sum_{1\le p<q,(p,q)=1} \frac{1}{n^q - 1} \delta_{\cos\frac{p}{q}\pi} \right) \tag{5.13}$$

The following equalities of spectra of Hecke operators hold:

$$[-1,1] = \mathsf{Sp}(H_\pi) = \mathsf{Sp}(H_{\lambda_{\mathcal{G}(\mathcal{C}(G))/P}}) = \mathsf{Sp}(H_{\lambda_{\mathcal{G}(\mathcal{C}(G))}}),$$

so the Markov operator for the simple random walk on the Cayley graph of $\mathcal{G}(\mathcal{C}(G))$ has spectrum $[-1,1]$.

In the case G is an abelian group, $\mathcal{G}(\mathcal{C}(G)) = G \operatorname{wr} \mathbb{Z}$ and (5.13) gives the Kesten spectral measure of the Markov operator for the simple random walk on the Cayley graph of $G \operatorname{wr} \mathbb{Z}$ with respect to the automaton generators.

For the case G is abelian, the results for the Markov operator were obtained in a different way by Dicks and Schick [6]. We can now calculate the zeta function using Theorem 5.11 and (2.7).

Corollary 5.12. *Let G be a non-trivial finite group and P be a parabolic subgroup of $\mathcal{G}(\mathcal{C}(G))$. Let $X = \mathsf{Sch}(\mathcal{G}(\mathcal{C}(G)), P, S)$, S as per (5.1). Then*

$$\zeta_X(t) = (1-t^2)^{-(n-1)} \times \prod_{q=2}^{\infty} \left[\prod_{1\le p<q,\ (p,q)=1} \left(1 - 2nt\cos\frac{p}{q}\pi + (2n-1)t^2\right)^{-(n-1)^2\frac{1}{n^q-1}} \right].$$

This product converges for $|t| < \frac{1}{2n-1}$. Moreover, if G is abelian, then $\zeta_X = \zeta_{G\mathrm{wr}\mathbb{Z}}$. □

6. The structure of Cayley machines of non-abelian groups

In [25], the second and third authors showed that for finite *abelian* groups G, the automata group $\mathcal{G}(\mathcal{C}(G))$ is isomorphic to the wreath product $G \operatorname{wr} \mathbb{Z}$. In this section, we consider the case in which G is not abelian, showing that in most cases, the group $\mathcal{G}(\mathcal{C}(G))$ cannot be expressed as a wreath product of any finite group with any torsion-free group. The following simple proposition allows us to consider separately two different cases.

Proposition 6.1. *Let G be a finite group. Then either:*

(i) *G has a non-central element of odd order; or*
(ii) *G is the direct product of a 2-group and an abelian group.*

Proof. Suppose (i) does not hold, that is, that all odd order elements of G are central. We claim first that $G/Z(G)$ is a 2-group. Indeed, suppose not. Then some coset $gZ(G) \in G/Z(G)$ has order an odd prime q. It follows that $g^n \in Z(G)$ if and only if q divides n, and in particular that q divides the order of g. Suppose the order of g is $q^i r$ where $q \nmid r$. Then g^r has order q^i but is not contained in $Z(G)$, which contradicts the supposition that all odd order elements of G are central. Hence, $G/Z(G)$ is a 2-group.

In particular, G is a central extension of a nilpotent group, and so is nilpotent. Hence, G is a direct product of its Sylow subgroups. But for odd primes p, the p-Sylow subgroups are central by assumption and so in particular must be abelian. Thus G is a direct product of a 2-group and an abelian group. □

We shall show that in the first case, our group $\mathcal{G}(\mathcal{C}(G))$ cannot be a wreath product of a finite group with a torsion-free group. The second case is slightly more involved and we can only handle the case where the 2-group component is not nilpotent of class 2. In this case we again show that $\mathcal{G}(\mathcal{C}(G))$ cannot embed in a wreath product of a finite group with a torsion-free group.

Fix now a finite group G. We consider the free monoid G^* over the elements of G and write elements as bracketed, comma-separated sequences, to avoid confusion with the multiplication in G. Set $x = \mathcal{C}(G)_1^{-1} \in \Gamma$. Then we recall from [25] that

$$\begin{aligned} x(g_0, g_1, \ldots, g_n) &= (g_0, g_0^{-1}g_1, g_1^{-1}g_2, \ldots, g_{n-1}^{-1}g_n) \\ x^{-1}(g_0, g_1, \ldots, g_n) &= (g_0, g_0g_1, \ldots, g_0g_1\cdots g_n) \\ x\mathcal{C}(G)_g(g_0, g_1, \ldots, g_n) &= (gg_0, g_1, \ldots, g_n) \end{aligned} \tag{6.1}$$

for every $g \in G$. It follows that the elements of the form $x\mathcal{C}(G)_g$ form a subgroup of $\mathcal{G}(\mathcal{C}(G))$ isomorphic to G. For notational convenience, we identify g with $x\mathcal{C}(G)_g$ and view G as embedded in $\mathcal{G}(\mathcal{C}(G))$. It is shown in [25] that the element $x^n g x^{-n}$ has depth exactly $n+1$.

There is an infinite sequence of words that will play an important technical role in what follows. Let $X = \{t_0, t_1, t_2, \ldots\}$ be a countably infinite set, which we view as a set of variables. In what follows, we take the viewpoint that X^* consists of terms. If $w \in X^*$, We sometimes write $w(t_0, \ldots, t_n)$ if $w \in \{t_0, \ldots, t_n\}^*$. If $m_0, \ldots, m_n$ are elements of a monoid M, we write $w(m_0, \ldots, m_n)$ to denote the element of M obtained by substituting m_i for t_i. Our sequence $\{w_n\}$ is defined recursively as follows:

- $w_0(t_0) = t_0$ and, for $n \geq 0$,
- $w_{n+1}(t_0, t_1, \ldots, t_n, t_{n+1}) = w_n(t_0, t_0t_1, \ldots, t_0t_1\cdots t_n)t_0t_1\cdots t_{n+1}$

Notice that w_n has content $\{t_0, \ldots, t_n\}$. The first four terms of $\{w_n\}$ are: t_0, $t_0t_0t_1$, $t_0t_0t_0t_1t_0t_1t_2$ and $t_0t_0t_0t_0t_1t_0t_0t_1t_0t_1t_2t_0t_1t_2t_3$. Sometimes it will be convenient to set w_{-1} to be the empty word ε, as the recursion still applies if we follow the usual conventions regarding empty variable sets.

If $w \in X^*$, we denote by $|w|_{t_i}$ the number of occurrences of t_i in w.

Lemma 6.2. *For $0 \leq i \leq n$, $|w_n|_{t_i} = 2^{n-i}$.*

Proof. The proof proceeds by induction on n. For $n = 0$, $w_0 = t_0$ and so the lemma holds for this case. Suppose that the lemma holds for $n \geq 0$. Then

$$w_{n+1}(t_0, \ldots, t_{n+1}) = w_n(t_0, t_0t_1, \ldots, t_0 \cdots t_n)t_0 \cdots t_{n+1}.$$

First we consider $1 \leq i \leq n$. In $w_n(t_0, t_0t_1, \ldots, t_0 \cdots t_n)$, there is one occurrence of t_i for each occurrence of t_j, $i \leq j \leq n$, in $w_n(t_0, \ldots, t_n)$. So by induction, we obtain

$$|w_{n+1}|_{t_i} = 1 + \sum_{j=i}^{n} 2^{n-j} = 2^{n+1-i}.$$

Clearly $|w_{n+1}|_{t_{n+1}} = 1 = 2^0$. This completes the induction, thereby establishing the lemma. □

The next lemma connects the sequence $\{w_n\}$ to our automata groups.

Lemma 6.3. *Let $g \in G$. Then the last entry of $x^n g x^{-n}(g_0, \ldots, g_n)$ is $(g^{(-1)^n})^{w_{n-1}(g_0,\ldots,g_{n-1})} g_n$.*

Proof. The proof is by induction on n. For $n = 0$, $g(g_0) = (gg_0)$, while $(g^{(-1)^0})^\varepsilon g_0 = gg_0$ (recall that $w_{-1} = \varepsilon$). Let us assume, by way of induction, that the lemma holds for $n \geq 0$. Then

$$x^{n+1}gx^{-(n+1)}(g_0, \ldots, g_{n+1}) = xx^n gx^{-n}(g_0, g_0g_1, \ldots, g_0 \cdots g_{n+1}) \tag{6.2}$$

$$= x(u_0, \ldots, u_n, g_0 \cdots g_{n+1}) \tag{6.3}$$

for certain $u_i \in G$ (since $x^n g x^{-n}$ has depth $n+1$). Moreover, we know by induction that

$$u_n = (g^{(-1)^n})^{w_{n-1}(g_0, g_0g_1, \ldots, g_0 \cdots g_{n-1})} g_0 \cdots g_n.$$

Hence, the last entry in (6.3) is

$$\begin{aligned} u_n^{-1} g_0 \cdots g_{n+1} &= (g_0 \cdots g_n)^{-1} (g^{(-1)^{n+1}})^{w_{n-1}(g_0, g_0g_1, \ldots, g_0 \cdots g_{n-1})} (g_0 \cdots g_n) g_{n+1} \\ &= (g^{(-1)^{n+1}})^{w_{n-1}(g_0, g_0g_1, \ldots, g_0 \cdots g_{n-1}) g_0 \cdots g_n} g_{n+1} \\ &= (g^{(-1)^{n+1}})^{w_n(g_0, \ldots, g_n)} g_{n+1}, \end{aligned}$$

as required. □

Our key obstruction to embedding in wreath products is presented by the following observation.

Lemma 6.4. *Let $A = G \operatorname{wr} H$ with G a finite group and H a torsion-free group. Then the set of torsion elements of A is the subgroup $N = \oplus_H G$. Hence every conjugacy class of N is finite.*

Proof. Since $A = (\oplus_H G) \rtimes H$ and H is torsion-free, the torsion elements of A are exactly the elements of the subgroup $N = \oplus_H G$. Since in the group $\oplus_H G$, conjugate elements have the same support and the direct sum only contains elements of finite support, the conjugacy classes of N are finite. □

Theorem 6.5. *Let G be a finite group with a non-central element of odd order. Then $\mathcal{G}(\mathcal{C}(G))$ does not embed in the wreath product of a finite group with a torsion-free group.*

Proof. Let $g \in G$ be a non-central element of odd order. Let $h \in G$ be an element of minimal order amongst those elements that do not commute with g. Let p be a prime factor of the order of h. Then h^p has order less than that of h, and so commutes with g. Let $v = gh^{-1}$.

For each $n \in \mathbb{N}$, we consider the element

$$\gamma_n = (x^{p^n} h^{-1} x^{-p^n})^{-1} v (x^{p^n} h x^{-p^n}) \in \mathcal{G}(\mathcal{C}(G)).$$

Each such element is a conjugate of the torsion element v by another torsion element; see (3.9). Our objective is to show that the γ_n are all distinct. By Lemma 6.4, this cannot happen in a wreath product of a finite group with a torsion-free group, so it will follow that $\mathcal{G}(\mathcal{C}(G))$ cannot embed in such a wreath product.

To this end, we consider the action of γ_n on the word $(1, 1, \ldots, 1) \in G^{p^n+1}$ and in particular compute the last letter. Our goal is to show that the action is non-trivial on the last letter. Since γ_n has depth at most $p^n + 1$, it will then follow that γ_n has depth exactly $p^n + 1$ and so the various γ_n are all distinct. Using (6.1) and Lemma 6.3,

$$\begin{aligned}
\gamma_n(1, \ldots, 1) &= x^{p^n} h^{-1} x^{-p^n} v x^{p^n} (h, 1, \ldots, 1) \\
&= x^{p^n} h^{-1} x^{-p^n} \left(g, h^{-\binom{p^n}{1}}, h^{\binom{p^n}{2}}, \ldots, h^{(-1)^{p^n}\binom{p^n}{p^n}} \right) \\
&= \left(\ldots, ((h^{-1})^{(-1)^{p^n}})^{w} h^{(-1)^{p^n}} \right)
\end{aligned}$$

where

$$w = w_{p^n-1} \left(g, h^{-\binom{p^n}{1}}, h^{\binom{p^n}{2}}, \ldots, h^{(-1)^{p^n-1}\binom{p^n}{p^n-1}} \right) \in G.$$

Now it is well-known that p divides $\binom{p^n}{i}$ for every prime p, every $n \geq 1$ and every $1 \leq i \leq p^n - 1$. In view of the fact that h^p commutes with g, we see using Lemma 6.2 that

$$w = g^{2^{p^n-1}} h^s, \text{ where } s = \sum_{i=1}^{p^n-1} (-1)^i \binom{p^n}{i} 2^{p^n-1-i}.$$

Now

$$\begin{aligned} &((h^{-1})^{(-1)^{p^n}})^w h^{(-1)^{p^n}} = 1 \\ \iff & h^{-s} g^{-2^{p^n-1}} h g^{2^{p^n-1}} h^s = h \\ \iff & g^{-2^{p^n-1}} h g^{2^{p^n-1}} = h \\ \iff & h g^{2^{p^n-1}} = g^{2^{p^n-1}} h \end{aligned} \tag{6.4}$$

But g has odd order, so there is an integer k such that $(g^{2^{p^n-1}})^k = g$. Since g does not commute with h, we conclude $g^{2^{p^n-1}}$ does not commute with h. Therefore, none of the equalities in (6.4) hold and so γ_n acts non-trivially on the last letter of $(1, \ldots, 1) \in G^{p^n+1}$, finishing the proof. ☐

We now consider the case in which G contains a 2-group, which is not nilpotent of class 2.

Theorem 6.6. *Let G be a finite group containing a 2-subgroup that is not nilpotent of class 2. Then $\mathcal{G}(\mathcal{C}(G))$ does not embed in a wreath product of a finite group with a torsion-free group.*

Proof. As in the previous proof, it will suffice to show that some torsion element in $\mathcal{G}(\mathcal{C}(G))$ has infinitely many distinct conjugates by other torsion elements.

Let K be a 2-group in G that is not nilpotent of class 2. Let $g, f, h \in K$ be such that $h^{-1}fhf^{-1}$ does not commute with g. Set

$$\gamma_n = x^n g x^{-n} h x^n g^{-1} x^{-n}.$$

We claim that it suffices to show that the last entry of

$$x^n g x^{-n}(1, \ldots, 1, f, 1, 1)$$

differs from the last entry of

$$x^n g x^{-n}(h, \ldots, 1, f, 1, 1)$$

for infinitely many positive integers n. (Here each of the strings we are acting on has length $n+1$). Indeed, for each n such that this is true, we see that

$$\begin{aligned} \gamma_n[x^n g x^{-n}(1, 1, \ldots, 1, f, 1, 1)] &= x^n g x^{-n} h(1, 1, \ldots, 1, f, 1, 1) \\ &= x^n g x^{-n}(h, 1, \ldots, 1, f, 1, 1) \end{aligned}$$

differs from

$$x^n g x^{-n}(1, 1, \ldots, 1, f, 1, 1)$$

in position $n+1$. Thus, the element γ_n acts non-trivially on the $n+1$ level; we know that γ_n can have depth at most $n+1$, so it must have depth exactly $n+1$. So if we have infinitely many ns for which this is the case, we have conjugates of a given torsion element with arbitrarily large depth, so there must be infinitely many of them.

By Lemma 6.3, we have that the last entry of

$$x^n g x^{-n}(1,\dots,1,f,1,1) \text{ is } (g^{(-1)^n})^{w_{n-1}(1,\dots,1,f,1)}$$

and the last entry of

$$x^n g x^{-n}(h,1,\dots,1,f,1,1) \text{ is } (g^{(-1)^n})^{w_{n-1}(h,\dots,1,f,1)}.$$

By Lemma 6.2 $|w_{n-1}|_{t_{n-2}} = 2$. Hence

$$w_{n-1}(1,\dots,1,f,1) = f^2.$$

Let $k = w_{n-1}(h,\dots,1,f,1) \in G$. Then it will suffice to show that for infinitely many n,

$$f^{-2}gf^2 \neq k^{-1}gk,$$

that is, that for infinitely many n, g does not commute with kf^{-2}.

Now from the definition of w_{n-1}, we have

$$\begin{aligned} w_{n-1}(t_0,1,\dots,1,t_{n-2},1) &= w_{n-2}(t_0,t_0,\dots,t_0,t_0t_{n-2})t_0t_{n-2} \\ &= w_{n-3}(t_0,{t_0}^2,\dots,t_0^{n-2})t_0^{n-1}t_{n-2}t_0t_{n-2}. \end{aligned}$$

Observing that $w_{n-3}(t_0,{t_0}^2,\dots,t_0^{n-2})$ must be a power of t_0 and recalling that $|w_{n-1}|_{t_0} = 2^{n-1}$ by Lemma 6.2, we obtain

$$w_{n-1}(t_0,1,\dots,1,t_{n-2},1) = t_0^{2^{n-1}-1}t_{n-2}t_0t_{n-2}$$

Substituting h for t_0 and f for t_{n-2}, we obtain $k = h^{2^{n-1}-1}fhf$, whence

$$kf^{-2} = h^{2^{n-1}-1}fhf^{-1}.$$

But h is in the 2-group K, and so for infinitely many n, we will have $h^{2^{n-1}} = 1$. But then $kf^{-2} = h^{-1}fhf^{-1}$, which by assumption does not commute with g, as required. □

The question of whether $\mathcal{G}(\mathcal{C}(G))$ is a wreath product of a finite group and a torsion-free group remains open for a small class of groups, namely for those groups G that are direct products of an abelian group with a 2-group of nilpotency class 2. Examples include D_4 and the 8-element quaternion group.

Another interesting question is that of whether, for non-abelian G, the group $\mathcal{G}(\mathcal{C}(G))$ has bounded torsion. We conjecture that this is never the case.

If, as we suspect, the group $\mathcal{G}(\mathcal{C}(G))$ is not isomorphic to G wr $\mathbb{Z}$ for any non-abelian G, then one might ask whether this wreath product arises as an automata group at all and, if so, what conditions can be placed upon the automaton. In particular, can a reset automaton be found?

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