AREA-PRESERVING DIFFEOMORPHISMS FROM THE C^1 STANDPOINT

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A . More than thirty years have passed since Newhouse ([23]) published a remarkable dichotomy on C^1 area-preserving diffeomorphisms. Here we revisit some central results on surface conservative C^1-diffeomorphisms by presenting, in particular, a new proof of Newhouse's theorem and also by proving some, although folklore, not yet proved results on this setting. We intend that this exposition can be used by a large audience as an introduction to the concept of dominated splitting and its relevance to the theory of C^1-stability of area-preserving diffeomorphisms.

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1. Introduction 2
1.1. Statement of the main results 5
2. Preliminaries and basic definitions 7
2.1. Charts and neighborhoods 7
2.2. Some elementary linear algebra 8
2.3. Hyperbolicity and dominated splitting 11
3. Two proofs of Theorem A 14
4. Perturbation Lemmas 15
5. Creating elliptic points 18
5.1. Mixing the eigendirections-Part I 18
5.2. Mixing the eigendirections-Part II 21
6. Proof of Theorem B 26
7. More results on area-preserving diffeomorphisms 29
7.1. Robust transitivity 29
7.2. Area-preserving star diffeomorphisms 30

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7.3. Homoclinic tangencies 32
7.4. Lots of Chaos or lack of it? 33
Acknowledgments 34
References 34

1. I

Let M be a compact, connected, boundaryless, Riemannian surface and let ω be an area-form on M. Denote by $\mathrm{Diff}^1_\omega(M)$ the space of diffeomorphisms on M, of class C^1, such that $f_*\omega = \omega$, that is, any Lebesgue measurable subset $\mathcal{M} \subset M$ satisfy $\mathrm{Leb}(\mathcal{M}) = \mathrm{Leb}(f(\mathcal{M}))$, where $\mathrm{Leb}(\cdot)$ denotes the Lebesgue measure induced by the two-form ω. We endow the set $\mathrm{Diff}^1_\omega(M)$ with the Whitney C^1 topology (see Section 2.1). The set $(\mathrm{Diff}^1_\omega(M), C^1)$ is a *Baire space*, hence every intersection of countably many C^1-dense and C^1-open sets is C^1-dense.

These area-preserving (or conservative) diffeomorphisms in surfaces are a traditional object of study from Classical Mechanics, see e.g. [5]. Despite being outside the scope of our text we recall the Kolmogorov, Arnold and Moser (KAM) theorem, see e.g. [36], which gives prevalence of dynamically invariant circles supporting irrational rotations.

The concept of periodic points plays a central role in dynamical systems and so we recall that a point x is said to be *periodic* for the diffeomorphism $f\colon M \to M$ if

$$f^n(x) = x \text{ where } f^n(x) = \overbrace{f \circ f \circ \ldots \circ f}^{n\text{-times}}(x), \text{ for } n \in \mathbb{N},$$

and the least of these positive integers is called the *period* of x. Moreover, it is well known that the knowledge of the behavior of the derivative of f, Df, along periodic orbits gives us a deep understanding of the local dynamics of f.

Given a periodic point x of period n of a diffeomorphism f if the n-iterated tangent map of f at x, denoted by Df^n_x, has its spectrum in $\mathbb{S}^1 \setminus \mathbb{R}$, then x is called *elliptic*. On the other hand if the spectrum does not intersect $\mathbb{S}^1$ then the point x is called *hyperbolic*. We recall that a periodic point is said to be *Lyapunov stable* if the iterates of all nearby points remain bounded for all time. So, KAM's theorem, implies abundance of Lyapunov stable elliptic points.

In spite that KAM theory needs higher order of differentiability of the diffeomorphisms it is our purpose to study systems with only C^1 regularity; which means closeness up to the first derivative.

The aim of this paper is to understand the typical dynamics for the elements $f \in \mathrm{Diff}^1_\omega(M)$. Some property could be considered to be *typical* if it holds for an open and dense subset, or even for some dense subset. However, the notion of *typical* that we are going to use here means that for a *generic* (or *residual*) set some property holds. Let us make this idea more precise; we say that the property $\mathcal{P}$ holds in a C^1-residual set of $\mathrm{Diff}^1_\omega(M)$ if $\mathcal{P}$ contains a G_δ, that is, a countable intersection of C^1-open and C^1-dense sets. In particular, as we already mention, by Baire's theorem (see e.g. [15]), any G_δ is dense in $\mathrm{Diff}^1_\omega(M)$.

Let us display some capital results on C^1-generic conservative diffeomorphisms in surfaces:

(A) every periodic point is hyperbolic or elliptic;
(B) M is the closure of the set of periodic points;
(C) the diffeomorphism is transitive, that is, it has a dense orbit.

The property (A) is a consequence of Thom's transversality theorem and was proved by Robinson [31], actually, this is a C^r-generic property, $r \geq 2$. Property (C) is a corollary of an outstanding theorem by Bonatti and Crovisier [10]. Item (B) is the so-called *general density theorem* proved by Pugh and Robinson (see [30]) and says that for a C^1-generic set $\mathcal{G} \subset \mathrm{Diff}^1_\omega(M)$, we have that the set of periodic points for $f \in \mathcal{G}$ is dense in the *nonwandering set*[1] of f denoted by $\Omega(f)$.

We say that $x \in M$ is an *f-recurrent point* if given any neighborhood U of x, there exists n such that $f^n(x) \in U$. Poincaré's recurrence theorem (see e.g. [21]) states that for $f \in \mathrm{Diff}^1_\omega(M)$ Lebesgue almost every point is recurrent. Hence, we conclude that Lebesgue almost every point is nonwandering and that, in the conservative class, C^1-generically the closure of the set of periodic points is the entire manifold M.

At this point we ask, given a C^1-generic area-preserving diffeomorphism, how often we find elliptic periodic orbits? And hyperbolic ones?

Recall that, due to the Hartman-Grobman theorem (see e.g. [31]), we have that hyperbolic periodic points are topological conjugated

[1]Recall that $x \in \Omega(f)$ if for every neighborhood U of x there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U = \emptyset$.

to its derivative, and so its local dynamics is simple. The hyperbolicity reveals stable also for sets (see e.g. [34, Chapter 8]). We say that a surface diffeomorphism is *completely hyperbolic*, or *Anosov*, if there exists $0 < \lambda < 1$ such that, for all $x \in M$, the tangent space decomposes into two one-dimensional subbundles on which the derivative contracts backward by a factor of λ in one subbundle and contracts under positive iterates by a factor of λ in the other direction. These geometric and dynamical properties imply a topological restriction in the manifold; the only surfaces that supports Anosov diffeomorphisms are the tori (see [14]). Another relevant property is that the Anosov diffeomorphisms are open (see [34]), thus the set of Anosov diffeomorphisms in $\mathrm{Diff}^1_\omega(M)$ is also open in $\mathrm{Diff}^1_\omega(M)$.

In the mid-1970's, (see [23]), Newhouse proved a celebrated result on area-preserving diffeomorphisms. He presented a C^1-generic set $\mathfrak{R} \subset \mathrm{Diff}^1_\omega(M)$ such that for any $f \in \mathfrak{R}$ either f is Anosov or else the elliptic points are dense in M. As a corollary of this result and of the aforementioned topological restriction, we obtain that, for example, in any surface aside from the torus, C^1-generic area-preserving diffeomorphisms have dense elliptic orbits.

In this paper we will give a new proof of Newhouse's theorem based in the perturbation techniques *à la Mañé* (see [19, 20]). These perturbations were first developed, in the conservative setting, by Bochi in ([8]) to prove the so-called Bochi-Mañé Theorem (see Theorem F and the references wherein).

Let us stress that, since $\mathrm{Diff}^1_\omega(M)$ is not C^1 dense among the set of C^1 dissipative diffeomorphisms in surfaces, our perturbations are more rigid and some careful is needed to perform them.

In order to obtain Newhouse's dichotomy we apply some perturbation results by Arbieto and Matheus [2] and Arnaud ([4]), jointly with the approach in [1, 7] and by making use of the above-mentioned Bochi-Mañé theorem. Mañé's ideas are an intrinsic part of this exposition and a recurrent influence.

The main dynamical ingredient is to use the absent of a hyperbolic behavior to perturb, in the C^1 topology and along a large period orbit, in order to transform this hyperbolic periodic orbit into an elliptic one with the same period. One crucial fact can be taken in account; we need to take small neighborhoods of the periodic hyperbolic orbit, and that is why we are restricted to the C^1 topology. The C^1 topology allow us to *rescale* the support of the perturbation with no implication to the size of the perturbation (see Lemma 4.1).

However, the attempt to replace the C^1 topology by higher order ones is very difficult because the size of the perturbations increases if we decrease the support of the perturbation. These are the main difficulties which are the base of one of the most challenging problems in the modern theory of dynamical systems; the C^r-*closing lemma* (for $r \geq 2$), see [11] A.1 for details.

We recall that Newhouse's proof of [23, Theorem 1.1] (see Theorem B) uses the concept of *homoclinic point* (see Section 1.1 for the definitions). Actually, in [23, Lemma 4.1], it is proved that a homoclinic tangency $T \in M$ associated to a hyperbolic periodic point for $f \in \text{Diff}^1_\omega(M)$ has a g-elliptic periodic point near T for g C^1-close to f. Then, Newhouse apply [30, 35] and the Birkhoff norm form to perturb $f \in \text{Diff}^1_\omega(M)$ in order to obtain that the homoclinic points of the perturbed diffeomorphism are dense in M. Finally, if the original diffeomorphism f is not Anosov, then there exists g, C^1-close to f, and exhibiting an elliptic orbit passing through any pre-fixed open set $U \subset M$.

1.1. **Statement of the main results.**

We start with Newhouse's dichotomy for area-preserving diffeomorphisms.

Theorem A. *There exists a residual set $\mathfrak{R} \subset \text{Diff}^1_\omega(M)$ such that for $f \in \mathfrak{R}$*

- *either f is Anosov,*
- *or else the elliptic points are dense in M.*

This theorem is a consequence of the following result.

Theorem B. *Given any non Anosov diffeomorphism $f \in \text{Diff}^1_\omega(M)$, $\varepsilon > 0$ and any non empty open subset U of M, then there exists $g \in \text{Diff}^1_\omega(M)$ ε-C^1-close to f and exhibiting an elliptic orbit passing through U.*

Previous theorems were proved by Newhouse (see [23];Theorem 1.1 and Theorem 1.2). Saghin and Xia (see [33, Theorem 2]), proved a general $2n$ symplectic perturbation results which allowed them to obtain the higher dimensional version of Theorem B. Let us stress that the perturbation results used by these authors were already explored by Bochi and Viana in [9] and also that, in [3], Arnaud obtained the four dimensional counterpart of Theorem B. We point out that these results are restricted to the symplectic context , and not to the broader setting of the volume-preserving diffeomorphisms, because the stability of elliptic points (which is false for volume-preserving diffeomorphisms on dimension ≥ 3) plays a crucial role in the arguments.

We say that a diffeomorphism $f\colon M \to M$ is *transitive* if there exists a dense orbit $x \in M$, that is, $\overline{\cup_{n\in\mathbb{N}} f^n(x)} = M$ where $\overline{A}$ stands for the closure of the set A. Moreover, a diffeomorphism $f\colon M \to M$ is said to be C^1-*robustly transitive* (in the conservative class) if it is transitive and every sufficiently C^1-close and conservative one is also transitive. Classical examples are the area-preserving Anosov diffeomorphisms. Actually, in dimension two these are the only examples. In Section 7.1 we will present another proof of Arbieto-Matheus' theorem [2, Theorem 5.1] by making use of a KAM-type theorem.

Theorem C. *If $f \in \mathit{Diff}^{\,1}_{\omega}(M)$ is C^1-robustly transitive, then f is Anosov.*

Let $f \in \mathrm{Diff}^{\,1}_{\omega}(M)$ we say that f is a *conservative star-diffeomorphism* if there exists a neighborhood $\mathcal{V}$ of f in $\mathrm{Diff}^{\,1}_{\omega}(M)$ such that any $g \in \mathcal{V}$, has all the periodic orbits hyperbolic. We denote this set by $\mathcal{F}^1_{\omega}(M)$. We define analogously the set $\mathcal{F}^1(M)$ in the broader set of dissipative diffeomorphisms $\mathrm{Diff}^{\,1}(M)$.

Let $\mathcal{A}^2_{\omega}$ denote the set of conservative Anosov diffeomorphisms on the surface M. Recall that the set $\mathcal{A}^2_{\omega}$ is open in $\mathrm{Diff}^{\,1}_{\omega}(M)$. Moreover, if $A \in \mathcal{A}^2_{\omega}$, then $f \in \mathcal{F}^1_{\omega}(M)$. In the next result we obtain the converse.

Theorem D. *If $f \in \mathcal{F}^1_{\omega}(M)$, then f is Anosov.*

We recall that the *dissipative* version of previous result was proved by Mañé (see [18]), loosely speaking, any $f \in \mathcal{F}^1(M)$ has a hyperbolic-type behavior.

A diffeomorphism is said to be C^1-*structurally stable* if there is a C^1-neighborhood of f on $\mathrm{Diff}^{\,1}_{\omega}(M)$ such that any $g \in \mathrm{Diff}^{\,1}_{\omega}(M)$ in this neighborhood is topologically conjugate to f, i.e., there exists a global homeomorphism h such that $h \circ f = g \circ h$. As we already pointed out the Anosov systems are structurally stable (see [34]), and in Theorem G we will obtain the converse.

Given a periodic hyperbolic orbit $\mathcal{O}$ and $p \in \mathcal{O}$ let W^s_p (respectively W^u_p) denote the stable (respectively unstable) manifold of p that is:

$$W^s_p := \left\{x \in M\colon\ \mathrm{dist}(f^n(x), f^n(p)) \underset{n\to+\infty}{\longrightarrow} 0\right\}$$

and

$$W^u_p := \left\{x \in M\colon\ \mathrm{dist}(f^{-n}(x), f^{-n}(p)) \underset{n\to+\infty}{\longrightarrow} 0\right\}.$$

There exists a very complete theory about these invariant manifolds (see [34]).

We say that $\mathcal{O}$ has a *homoclinic tangency* at $q \neq p$ if:

- $T_q W^s_p \cap T_q W^u_p$ contains a nonzero vector and
- $T_q W^s_p \oplus T_q W^u_p \neq T_q M$.

We say that q is a *transversal homoclinic point* if it is not a homoclinic tangency.

The next result, that will be proved in Section 7.3, is in the spirit of Palis' conjecture ([27]) and with respect to the C^1-topology.

Theorem E. *Any $f \in \mathrm{Diff}^1_\omega(M)$ can be C^1-approximated by another one $g \in \mathrm{Diff}^1_\omega(M)$ satisfying one of the following properties:*

(1) *g is Anosov or else,*
(2) *g has a homoclinic tangency associated to a hyperbolic periodic orbit.*

In Section 2 we set up notation, terminology and standard facts on uniform hyperbolic theory. Section 4 provides a detailed exposition of the perturbations that we will use in order to go on with the main proofs. In Section 6 we present the proof of Theorem B. Theorem A shall be proved in Section 3 assuming Theorem B. In Section 5 we will be concerned with the creation of elliptic periodic orbits by C^1 small perturbations. Finally, in Section 7 we will restrict our attention to some results about robust transitivity, stability, bifurcations on periodic points and some questions about the coexistence of two different definitions of chaos in the C^1 sense (see Theorem H).

2. P

2.1. **Charts and neighborhoods.** By compacity of M we can use Darboux's theorem (see e.g. [5]) and obtain a finite atlas $\mathcal{A} = \{\varphi_i \colon U_i \to \mathbb{R}^2\}$, for $i = 1, \dots, k$ and thus define local coordinates such that the pull-back of the two form ω by φ_i is the canonical area in the plane, i.e., $(\varphi_i)_* \omega = dx \wedge dy$. Note that we can switch the metric associated to the Riemannian structure of M at $x \in M$ by the metric $\|\cdot\| = \|D(\varphi_{i(x)})_x(\cdot)\|$ where $i(x)$ is uniquely defined and associated to each $x \in M$. For this reason we will not use the Riemannian metric *a priori* fixed on M. Denote by $\mathrm{dist}(\cdot,\cdot)$ the distance inherit from the Riemannian structure in M and the pre-fixed charts; that is, given $x, y \in M$ with $y \in U_{i(x)}$, $d(x,y) := \|\varphi_{i(x)}(x) - \varphi_{i(x)}(y)\|$.

We sometimes consider balls in M defined by

$$B(x,r) := \varphi^{-1}_{i(x)}[B(\varphi_{i(x)}(x), r)],$$

where $r > 0$ is chosen to be sufficiently small in order to have each ball contained in the open set U_i for $i = 1, ..., k$.

Given any 1-linear map $A \in \mathcal{L}(\mathbb{R}^2)$ we consider the norm

$$\|A\| := \sup_{v \neq \vec{0}} \frac{\|A \cdot v\|}{\|v\|}. \tag{2.1}$$

This norm will be used to estimate distances between two maps and it will be the one fixed in the preceding paragraph. In the sequel we will also use another norm which will reveal to be useful when dealing with estimates (see Section 2.2).

As a consequence, everytime we compute distances between two maps we use Darboux's theorem to translate the scenario to $\mathbb{R}^2$. So let us define properly the distance we are going to consider. Given $f \in \mathrm{Diff}^1_\omega(M)$, a finite atlas $\{\varphi_i\}_{i \in F}$, compact sets $K_i \subset U_i$ such that $f(K_i) \subset U_i$ for all $i \in F$ and $\varepsilon > 0$, we say that $U(f, \varphi, K_i, \varepsilon)$ is an ε-C^1 basic neighborhood of f in the *Whitney C^1-topology* if it is formed by those maps $g \in \mathrm{Diff}^1_\omega(M)$ such that:

- $g(K_i) \subset U_i$ and
- (C^0-closeness) $\sup_{x \in \varphi_i(K_i)} \{\|\varphi_{i(f(x))} f \varphi^{-1}_{i(x)}(x) - \varphi_{i(g(x))} g \varphi^{-1}_{i(x)}(x)\|\} < \varepsilon$ and
- (C^1-closeness) $\sup_{x \in \varphi_i(K_i)} \{\|D(\varphi_{i(f(x))} f \varphi^{-1}_{i(x)})(x) - D(\varphi_{i(g(x))} g \varphi^{-1}_{i(x)})(x)\|\} < \varepsilon$.

In this way we obtain what we shall call the ε-C^1-neighborhood of f and we denote it by $\mathcal{N}^\omega_\varepsilon(f)$.

2.2. Some elementary linear algebra.

2.2.1. *The linear group $SL(2, \mathbb{R})$, angles and eigenvalues.* We say that a 2×2 matrix A belongs to $SL(2, \mathbb{R})$ if $\det(A) = 1$. Moreover, if the eigenvalues of $A \in SL(2, \mathbb{R})$ are real and distinct we say that the matrix is *hyperbolic*. A matrix $A \in SL(2, \mathbb{R})$ is *elliptic* if the eigenvalues are different complex conjugates. Finally, we call $A \in SL(2, \mathbb{R})$ *parabolic* if it is not hyperbolic neither elliptic. It is easy to see that stability (with respect to the norm defined in (2.1)), within these three classes of matrices, holds both for hyperbolic and elliptic matrices whilst the parabolic ones are unstable.

Given a hyperbolic matrix $A \in SL(2, \mathbb{R})$, let $\sigma > 1$ be the upper eigenvalue and $\theta > 0$ be the angle between its eigenspaces. We define the function

$$\eta_\theta(\sigma) = \|A - Id\|, \tag{2.2}$$

where Id denotes the identity in $\mathbb{R}^2$. Of course that $\eta_\theta(\cdot)\colon]1,+\infty[\to]0,+\infty[$ defined by $\sigma \mapsto \eta_\theta(\sigma)$ is a strictly increasing diffeomorphism. On the other hand $\eta_{(\cdot)}(\sigma)\colon]0,\pi/2[\to]\sigma,+\infty[$ defined by $\theta \mapsto \eta_\theta(\sigma)$ is a strictly decreasing diffeomorphism.

2.2.2. *A new norm.* Let be given $x, y \in M$, a linear map $A\colon T_xM \to T_yM$ and two invariant 1-dimensional splittings $E_x^1 \oplus E_x^2 = T_xM$ and $E_y^1 \oplus E_y^2 = T_yM$ that is $A(E_x^i) = E_y^i$ for $i = 1, 2$. We define four linear actions as:

$$a_{11}\colon E_x^1 \to E_y^1,\ a_{12}\colon E_x^2 \to E_y^1,\ a_{21}\colon E_x^1 \to E_y^2 \text{ and } a_{22}\colon E_x^2 \to E_y^2,$$

and let $v = v_1 + v_2$ where $v_i \in E_x^i$ for $i = 1, 2$. Let

$$A \cdot v = (a_{11} + a_{21})v_1 + (a_{12} + a_{22})v_2. \tag{2.3}$$

The linear map A can be represented by the matrix

$$\widetilde{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.4}$$

related to these previous splittings. We define a new norm by

$$\lfloor\!\lfloor A \rfloor\!\rfloor = \max\{|a_{11}|, |a_{12}|, |a_{21}|, |a_{22}|\},$$

and we call it the *norm of the maximum*.

Example 2.1. *Let us consider a linear map in the plane represented by a conservative hyperbolic matrix (in the canonical base of* $\mathbb{R}^2$*),*

$$A = \begin{pmatrix} 2 & 1000 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

This matrix has eigendirections associated to the vectors $b = \{(1, 0), (-2000, 3)\}$ *(associated to eigenvalues* 2 *and* $1/2$*). We observe that the angle* θ *between the eigendirections is close to zero. If we consider the diagonalized matrix with respect to the base of eigenvectors, then we get the matrix*

$$\widetilde{A} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

When we compute the norm of A*, related to the usual metric in* $\mathbb{R}^2$ *we get* $\|A\| = 1000$*, on the other hand the norm of the maximum of* $\widetilde{A}$ *is* 2*. In Lemma 2.2 we will obtain a relation between these quantities, namely that* $\|A\| \leq 4\lfloor\!\lfloor \widetilde{A} \rfloor\!\rfloor \sin^{-1}\theta$*. In fact, in this example*

$$\theta = \arccos\left(\frac{(1, 0) \cdot (-2000, 3)}{\|(1, 0)\|\|(-2000, 3)\|}\right) \approx 0.0015,$$

and we get an estimate since $1000 \leq 8 \sin^{-1}(0.0015) \approx 5334$*.*

Let us now show how we can relate the usual norm and the norm of the maximum.

Lemma 2.2. *Given $A \in \mathcal{L}(\mathbb{R}^2)$ as above, if $\measuredangle(E^1_\sigma, E^2_\sigma) > \theta$ for $\sigma = x, y$, then A satisfies:*

(1) $\|A\| \leq 4 \sin^{-1}\theta \llbracket A \rrbracket$.
(2) $\llbracket A \rrbracket \leq \sin^{-1}\theta \|A\|$.

Proof. We follow [9, Lemma 4.5]. Let $v = v_1 + v_2$ where $v_i \in E^i_x$ for $i = 1, 2$. Using elementary geometry it is easy to see that

$$\|v_i\| \leq \|v\| \sin^{-1}\theta, \text{ for } i = 1, 2.$$

Hence, using (2.3) and the preceding inequality

$$\begin{aligned}
\|A \cdot v\| &\leq \|a_{11}v_1\| + \|a_{11}v_2\| + \|a_{22}v_1\| + \|a_{22}v_2\| \\
&= |a_{11}|\|v_1\| + |a_{11}|\|v_2\| + |a_{22}|\|v_1\| + |a_{22}|\|v_2\| \\
&\leq 4\llbracket A \rrbracket \|v\| \sin^{-1}\theta.
\end{aligned}$$

Therefore, by definition (2.1) we obtain (1).

Given $v_1 \in E^1_x$ using (2.3) we write $A \cdot v_1 = a_{11}v_1 + a_{21}v_1 \in E^1_y \oplus E^2_y$ and so,

- $|a_{11}|\|v_1\| = \|a_{11}v_1\| \leq \|A \cdot v_1\| \sin^{-1}\theta \leq \|A\|\|v_1\| \sin^{-1}\theta$,
- $|a_{21}|\|v_1\| = \|a_{21}v_1\| \leq \|A \cdot v_1\| \sin^{-1}\theta \leq \|A\|\|v_1\| \sin^{-1}\theta$,

Analogously, given $v_2 \in E^2_x$ we write $A \cdot v_2 = a_{12}v_2 + a_{22}v_2 \in E^1_y \oplus E^2_y$ and so, $|a_{12}|\|v_2\| \leq \|A\|\|v_2\| \sin^{-1}\theta$ and $|a_{22}|\|v_2\| \leq \|A\|\|v_2\| \sin^{-1}\theta$ and therefore (2) follows directly. $\square$

Finally, we present a simple lemma that will not be needed until Section 5.

Lemma 2.3. *(*[8, Lemma 3.9]*) Given $\theta > 0$, there exists $c > 1$ such that for any linear map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\|A \cdot s\|.\|A \cdot u\|^{-1} > c$, where u, s are unit vectors, we can find a nonzero vector v such that $\measuredangle(v, u) < \theta$ and $\measuredangle(A \cdot v, A \cdot s) < \theta$.*

2.2.3. *Orthogonal decompositions.* Sometimes we need to consider orthogonal decompositions in order to proceed with the estimates in a more treatable way. Consider the same map $A\colon T_xM \to T_yM$ as before and two new orthogonal decompositions $E^1_x \oplus (E^1_x)^\perp$ and $E^1_y \oplus (E^2_y)^\perp$ of T_xM and T_yM respectively. Denote by θ_x (resp. θ_y) the angle between E^1_x and E^2_x (resp. E^1_y and E^2_y). Identify, using a rotation, the directions E^1_x and E^1_y with the direction $\mathbb{R}(1, 0)$, the direction E^2_x

with $\mathbb{R}(\cos\theta_x, \sin\theta_x)$ and the direction E^2_y with $\mathbb{R}(\cos\theta_y, \sin\theta_y)$. The $SL(2,\mathbb{R})$ matrix

$$\Psi_x := \begin{pmatrix} \sin^{-1}\theta_x & \cos\theta_x \\ 0 & \sin\theta_x \end{pmatrix},$$

maps E^1_x into E^1_x and $(E^1_x)^\perp$ into E^2_x, thus performs a conservative change from the decomposition $E^1_x \oplus (E^1_x)^\perp$ into $E^1_x \oplus E^2_x$. In the same way we define the matrix

$$\Psi_y := \begin{pmatrix} \sin^{-1}\theta_y & \cos\theta_y \\ 0 & \sin\theta_y \end{pmatrix},$$

mapping E^1_y into E^1_y and $(E^1_y)^\perp$ into E^2_y.

We now represent the linear action A in a new coordinate system by

$$A^\perp := \Psi_y^{-1} \circ A \circ \Psi_x. \tag{2.5}$$

We point out that everytime we perform these change of coordinates we can keep track of the constants of estimation using the following inequality:

$$\|A^\perp\| \leq \frac{\|A\|}{(\sin\theta_x \sin\theta_y)}. \tag{2.6}$$

In conclusion, if the angle is bounded from bellow from zero, then it is possible to control the norm and thus to use this orthogonal splitting (see hypothesis (1) of Lemma 5.3).

2.3. **Hyperbolicity and dominated splitting.** Given a diffeomorphism f, a compact f-invariant set $\Lambda \subset M$ is said to be *hyperbolic* if there is $m \in \mathbb{N}$ such that, for every $x \in \Lambda$, there is a Df-invariant continuous splitting $T_xM = E^u_x \oplus E^s_x$ such that we have:

(1) $\|Df^m_x|_{E^s_x}\| \leq \frac{1}{2}$ and
(2) $\|(Df^m_x)^{-1}|_{E^u_x}\| \leq \frac{1}{2}$.

There are several ways to weaken the definition of uniform hyperbolicity. Here we use the one introduced independently by Mañé ([17, 18]), Liao ([16]) and Pliss ([28]) around the 1970's when motivated by the desire to prove the stability conjecture ([26]). Given $m \in \mathbb{N}$, a compact f-invariant set $\Lambda \subset M$ is said to have an *m-dominated splitting* if there is, over Λ, a Df-invariant continuous splitting $TM = E^u \oplus E^s$ such that for all $x \in \Lambda$ we have:

$$\|Df^m_x|_{E^s_x}\|.\|Df^m_x|_{E^u_x}\|^{-1} \leq \frac{1}{2}. \tag{2.7}$$

It is worth pointing out that both subbundles may expand. However, E^u expands more than E^s. If both subbundles contract, E^u is less contracting than E^s. Like in the uniform hyperbolicity, the angle between the subbundles is uniformly bounded away from zero. This follows because the splitting is continuous and the base set is compact. Moreover, the dominated splitting extends to the closure of Λ. See [11] for the complete proofs of these properties.

Example 2.4. *For $\mu > 1$ let us define*

$$A := \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \text{ and } B := \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$

The matrices A and B are not hyperbolic. However, A has an m-dominated splitting $E^u = \mathbb{R}(0,1)$ and $E^s = \mathbb{R}(1,0)$, and B has also an m-dominated splitting $E^u = \mathbb{R}(1,0)$ and $E^s = \mathbb{R}(0,1)$, where $m \geq \frac{\log 2}{\log \mu}$. It is immediate that μ close to 1 implies m very large.

Given $p \in Per(f)$, if p is hyperbolic and E^u_x and E^s_x are the Df-invariant subbundles, then the real numbers

- $\lambda_u(p) := \lim\limits_{n\to\pm\infty} \frac{1}{n} \log \|Df^n_p|_{E^u_x}\|$ and
- $\lambda_s(p) := \lim\limits_{n\to\pm\infty} \frac{1}{n} \log \|Df^n_p|_{E^s_x}\| < \lambda_u(p)$,

are called the *upper Lyapunov exponent* and the *lower Lyapunov exponent* respectively. By the celebrated Oseledet's theorem ([24]) (see [29] for a proof on dimension two) these numbers exist for Lebesgue almost every point in M and not necessarily a periodic point.

A central result about the Lyapunov exponents of C^1-generic conservative surface diffeomorphisms is the following result of Bochi based on a conjecture of Mañé.

Theorem F. *(Bochi-Mañé* [8, 19, 20]*) There exists a C^1-generic subset $\mathcal{R}$ of $Diff^1_\omega(M)$ such that if $f \in \mathcal{R}$, then f is Anosov or else Lebesgue almost every point in M has zero Lyapunov exponents.*

As we will see, this result will play an important role in the proof of our results.

As a consequence of Oseledets' theorem we obtain the equality,

$$\lambda_u(p) + \lambda_s(p) = \lim_{n\to\pm\infty} \frac{1}{n} \log |\det Df^n_p|. \tag{2.8}$$

Then, by the area-preserving property, $|\det Df^n_p| = 1$ for every p, and so we obtain that $\lambda_u(p) = -\lambda_s(p)$. Therefore, if $\lambda_u(p) = 0$, then the spectrum of Df_p^τ lies in $\mathbb{S}^1$, where τ denotes the period of p. Otherwise, the real eigenvalues $\sigma^{\pm 1}$ of the map Df_p^τ, satisfy

$$e^{\lambda_u(p)\tau} = |\sigma| > 1 > |\sigma^{-1}| = e^{-\lambda_u(p)\tau}.$$

Let $Per_{\text{hyp}}(f)$ denote the subset of all hyperbolic periodic points in $Per(f)$. Note that if $x \in Per_{\text{hyp}}(f)$, then x has a dominated splitting, but in general we have that $m(x)$ is unbounded. Also, the weak hyperbolic behavior relates with the splitting angle being close to zero.

Since M is compact and the hyperbolic splitting varies continuously, given a uniformly hyperbolic invariant set $\Lambda \subset \overline{Per_{\text{hyp}}(f)}$, the splitting angle between E^u and E^s, denoted by $\sphericalangle(E^u, E^s)$, is bounded away from zero over Λ.

Given $f \in \text{Diff}_\omega^{\,1}(M)$, we define

$$\Delta_m(f) := \left\{ x \in Per_{\text{hyp}}(f) \colon \ \|Df_x^m|_{E_x^s}\|.\|Df_x^m|_{E_x^u}\|^{-1} \geq \frac{1}{2} \right\} ,$$

and

$$\Lambda_m(f) := \left\{ x \in Per_{\text{hyp}}(f) \colon \|Df_{f^n(x)}^m|_{E_x^s}\|.\|Df_{f^n(x)}^m|_{E_x^u}\|^{-1} \leq \frac{1}{2} \ \text{ for all } n \in \mathbb{N} \right\}.$$

Since $\overline{\Lambda_m(f)}$ has m-dominated splitting, and M is a surface, then, by the area-preserving property, $\overline{\Lambda_m(f)}$ is a hyperbolic set (see Lemma 2.5 below). Of course that we have

$$Per_{\text{hyp}}(f) = \Lambda_m(f) \dot{\bigcup} \left(\underset{n \in \mathbb{N}}{\cup} f^n(\Delta_m(f)) \right).$$

The following simple lemma, which only holds because M is a surface, with be useful in the sequel.

Lemma 2.5. *Let f be an area-preserving diffeomorphism and $\Lambda \subset M$ a compact f-invariant set. If Λ has a dominated splitting, then this splitting is hyperbolic.*

Proof. Since f admits a dominated splitting over Λ one gets that there exists $m \in \mathbb{N}$ such that

$$\Delta(x, m) := \|Df_x^m|_{E_x^s}\| \|Df_{f^m(x)}^{-m}|_{E_{f^m(x)}^u}\| \leq \frac{1}{2}, \ \forall x \in \Lambda,$$

where E^s and E^u are Df-invariant and one-dimensional.

For any $i \in \mathbb{N}$ we have $\Delta(x, im) \leq 1/2^i$. For every $n \in \mathbb{R}$ we may write $n = im + r$, for $0 \leq r < m$, and since $\|Df^r\|$ is bounded, say by L, take $C = 2^{\frac{r}{m}} L^2$ and $\sigma = 2^{-\frac{1}{m}}$ to get $\Delta(x, n) \leq C\sigma^n$, for every $x \in \Lambda$ and $n \in \mathbb{N}$.

Denote by α_n the angle between $E^s_{f^n(x)}$ e $E^u_{f^n(x)}$. We already know, by domination, that this angle is bounded bellow from zero, say by β.

Since f is area-preserving and the subbundles are both one-dimensional we have that

$$\sin\alpha_0 = \|Df^n_x|_{E^s_x}\|\|Df^n_x|_{E^u_x}\| \sin\alpha_n.$$

So

$$\|Df^n_x|_{E^s_x}\|^2 = \frac{\sin\alpha_0}{\sin\alpha_n}\Delta(x,n) \leq \Delta(x, im + r)\sin^{-1}\beta \leq \sigma^n C \sin^{-1}\beta.$$

Analogously we get

$$\|Df^{-n}_x|_{E^u_x}\|^2 = \frac{\sin\alpha_n}{\sin\alpha_0}\Delta(x,n) \leq \Delta(x, im + r)\sin^{-1}\beta \leq \sigma^t C \sin^{-1}\beta.$$

These two inequalities show that Λ is hyperbolic for Df completing the proof of the lemma.

□

3. T T A

First proof of Theorem A: Assuming Theorem B we give now the proof of Theorem A and we postpone the proof of Theorem B to Section 6. Recall that $\mathcal{A}^2_\omega \subset \mathrm{Diff}^1_\omega(M)$ denotes the open set of Anosov area-preserving diffeomorphisms and let $\overline{\mathcal{A}^2_\omega}$ be its C^1-closure. We define the open set $\mathcal{N} := \mathrm{Diff}^1_\omega(M) \setminus \overline{\mathcal{A}^2_\omega}$. Consider the C^1-topology in $\mathrm{Diff}^1_\omega(M)$, the topology inherited by the Riemannian metric in M, $\mathrm{dist}(\cdot,\cdot)$, and the usual euclidean distance in $\mathbb{R}$. Let $\mathcal{H}$ be the subset of $\mathrm{Diff}^1_\omega(M) \times M \times \mathbb{R}^+$ of all triples (f, x, ε) such that f has a closed elliptic orbit going through the ball $B(x,\varepsilon) \subset M$. Finally, we endow $\mathcal{H}$ with the product topology. Since M is two-dimensional we get that the elliptic orbits are stable concluding that $\mathcal{H}$ is open.

Given any open set $\mathcal{U} \subseteq \mathcal{N}$ consider the following (also open) set

$$\mathcal{H}(\mathcal{U}, x, \varepsilon) := \{\, g \in \mathcal{U} \;:\; (g, x, \varepsilon) \in \mathcal{H} \,\}.$$

It follows directly from Theorem B that if we take $\varepsilon > 0$, $x \in M$ and an open set $\mathcal{U} \subseteq \mathcal{N}$, then $\mathcal{H}(\mathcal{U}, x, \varepsilon)$ is an open and dense subset of $\mathcal{U}$.

Using the smooth charts $\varphi_i \colon U_i \to \mathbb{R}^2$ for $i = 1, ..., k$ we take k dense sequences in $\varphi_i(U_i) \subset \mathbb{R}^2$ and so, using φ_i^{-1}, we define $\{x_n\}_n$ to be a dense sequence in M. Let $\{\varepsilon_n\}_n > 0$ be a sequence converging to zero. Defining recursively

$$\mathcal{U}_0 = \mathcal{N} \qquad \text{and} \qquad \mathcal{U}_{n+1} = \mathcal{H}(\mathcal{U}_n, x_n, \varepsilon_n) \quad \text{for } n \geq 1,$$

the residual set $\mathcal{R} = \cap_{n=1}^{\infty} \mathcal{U}_n$ is such that for all $g \in \mathcal{R}$, the elliptic closed orbits of g are dense in M. Then $\mathfrak{R} = \mathcal{A}^2_\omega \cup \mathcal{R}$ is the residual subset of $\text{Diff}^1_\omega(M)$, announced in Theorem A.

Second proof of Theorem A: We could also obtain another proof of Theorem A from Theorem B by using the elegant arguments explored in [23]. Denote by $\mathcal{E}_N(f)$ the set of elliptic periodic points (of the diffeomorphism f) of period less than N. Consider now the function

$$\begin{array}{rccc} P_N: & \text{Diff}^1_\omega(M) & \longrightarrow & \mathfrak{M} \\ & f & \longmapsto & \mathcal{E}_N(f), \end{array}$$

where $\text{Diff}^1_\omega(M)$ is endowed with the C^1-topology and $\mathfrak{M}$ denotes the set of all closed subsets of M endowed with the Hausdorff metric. By the stability of the elliptic periodic points it follows that P_N is a continuous function. Hence we obtain that $P = \sup_{N \in \mathbb{N}}\{P_N\}$ is a lower semi-continuous function (see [15]). Actually, we have

$$\begin{array}{rccc} P: & \text{Diff}^1_\omega(M) & \longrightarrow & \mathfrak{M} \\ & f & \longmapsto & \overline{\mathcal{E}(f)}, \end{array}$$

where $\overline{\mathcal{E}(f)}$ denotes the closure of the set of the elliptic periodic points of f.

Using [31, Proposition 26] we obtain that there exists a residual $\mathcal{R} \subset \text{Diff}^1_\omega(M)$ formed by continuity points of P.

Therefore, if $f \in \mathcal{R}$ is not Anosov, it is an immediate consequence of Theorem B that the elliptic points are dense in M and Theorem A is proved.

4. P L

In order to achieve our goal we will need to perform some perturbations of the tangent map. One of the main perturbation tools will induce rotations in the tangent bundle and so the next basic lemma will be very useful. We emphasize that a more or less general result will be stated (see Theorem 4.3 and Remark 4.5), however the advantage of presenting the proof of Lemma 4.1 lies in the fact that it sheds some light in the nice properties of the C^1 topology and for this reason we decide to state and prove it nevertheless.

Lemma 4.1. *If $f \in \text{Diff}^1_\omega(M)$ and $\varepsilon > 0$, there exists $\beta_0 > 0$ such that for any $x \in M$, $r \in (0,1)$ and $\beta < \beta_0$ there exists $g \in \mathcal{N}_\varepsilon^\omega(f)$ such that (in local charts):*

a) $Dg_x = Df_x \cdot R_\beta(x)$, *where* $R_\beta(x)$ *denotes the rotation of angle* β *centered in* x, *and*
b) $g = f$ *outside the ball* $B(x, r)$.

Proof. We first prove the result for $r = 1$. Using the aforementioned charts (see Section 2.1) we assume $x = \vec{0}$. Let $\alpha\colon [0,\infty) \to [0,\infty)$ be a C^∞ function such that $\alpha(t) = 0$ for $t \geq 1$, $\alpha(t) = 1$ for $0 \leq t \leq 1/2$ and $|\alpha'| \leq 2$. Take $y \in B(0,1)$ and $\beta > 0$. Let $h_\beta(y) = R_{\alpha(\|y\|)\beta}(\vec{0})$. We define $g_\beta = f \circ h_\beta$. Computing the derivative of g_β at y we obtain that $(Dg_\beta)_y = Df_{h_\beta(y)} \cdot (Dh_\beta)_y$. Therefore:

- if $y \in B(x, 1/2)$, then $(Dg_\beta)_y = Df_{h_\beta(y)} \cdot R_\beta(x)$.
  In particular $(Dg_\beta)_x = Df_{h_\beta(x)} \cdot R_\beta(x) = Df_x \cdot R_\beta(x)$ which gives item a).
- If y lies outside $B(x, 1)$, then $g_\beta = f$ and we get item b).

Since $\det(Dg_\beta)_x = 1$ for all x, our final goal is to to prove that g_β is ε-C^1-close to f. The C^0-closeness is obvious. Let us prove that $(Dh_\beta)_y$ is C^0-close to the identity. In local coordinates we can write:

$$h_\beta(y) = (\cos(\phi_y)y_1 - \sin(\phi_y))y_2, \sin(\phi_y)y_1 + \cos(\phi_y)y_2),$$

where $\phi_y = \alpha(\|y\|)\beta$ and $y = (y_1, y_2)$. Taking derivatives we obtain:

$$(Dh_\beta)_y = A_y + \begin{pmatrix} \cos(\phi_y) & -\sin\phi_y \\ \sin\phi_y & \cos(\phi_y) \end{pmatrix},$$

where

$$A_y = \begin{pmatrix} -\frac{\partial\phi_y}{\partial y_1}\sin(\phi_y)y_1 - \frac{\partial\phi_y}{\partial y_1}\cos(\phi_y)y_2 & -\frac{\partial\phi_y}{\partial y_2}\sin(\phi_y)y_1 - \frac{\partial\phi_y}{\partial y_2}\cos(\phi_y)y_2 \\ \frac{\partial\phi_y}{\partial y_1}\cos(\phi_y)y_1 - \frac{\partial\phi_y}{\partial y_1}\sin(\phi_y)y_2 & \frac{\partial\phi_y}{\partial y_2}\cos(\phi_y)y_1 - \frac{\partial\phi_y}{\partial y_2}\sin(\phi_y)y_2 \end{pmatrix}.$$

It is clear that, if β is chosen to be small, then $(Dh_\beta)_y - A_y$ is arbitrarily close to the identity.

We just have to prove that, for a suitable β, A_y is close to the null matrix. For that we first compute the gradient of ϕ_y.

$$\nabla\phi_y = \left(\frac{\partial\phi_y}{\partial y_1}, \frac{\partial\phi_y}{\partial y_2}\right) = \alpha'(\|y\|)\beta\|y\|^{-1}(y_1, y_2).$$

Recall that $|\alpha'| \leq 2$, $\left|\frac{\partial\phi_y}{\partial y_i}y_i\right| = |\alpha'(\|y\|)\beta\|y\|^{-1}y_i| \leq$ and $\|y\|^{-1}y_i \leq 1$. Hence, we obtain that $\left|\frac{\partial\phi_y}{\partial y_i}\right| \leq 2\beta$.

Therefore, given $\varepsilon > 0$, there exists $\beta_0 > 0$ and $g \in \mathrm{Diff}^1_\omega(M)$ such that $g \in \mathcal{N}^\omega_\varepsilon(f)$ and $g = g_{\beta_0}$ satisfies a) and b). Finally, for $r \in (0, 1]$, we consider the r-homothethy and $h = h_{\beta_0}$ associated to $r = 1$ established above. We define the new h_r and g_r (associated to r) by $rh(y/r)$ and $f \circ h_r$ respectively. Clearly $D(rh(y/r)) = Dh(y/r)$ which is C^0-close to the identity and the lemma is proved. $\square$

Remark 4.2. A slight change in the proof of Lemma 4.1 allow us to obtain a version where, in a), we switch from $Dg_x = Df_x \cdot R_\beta(x)$ to $Dg_x = R_\beta(x) \cdot Df_x$. The details are left to the reader.

In [2] it was proved a weak pasting lemma for diffeomorphisms which, in rough terms, allow us to replace the area-preserving diffeomorphism f by another area-preserving diffeomorphism g such that g is equal to the first order linear approximation of f in a small neighborhood $\mathcal{U}$ of a given point, and equal to f outside a set containing $\mathcal{U}$. Let us present the formal statement.

Theorem 4.3. *(Arbieto-Matheus* [2, Theorem 3.6]*)*
If $f \in \mathit{Diff}^{\,2}_\omega(M)$ and $x \in M$, then for any $0 < \alpha < 1$ and $\varepsilon > 0$, there exists $\tilde{\varepsilon} > 0$ such that any $A_x \in SL(2, \mathbb{R})$ which is $\tilde{\varepsilon}$-close to Df_x satisfies the following; there exists $g \in \mathcal{N}^\omega_\varepsilon(f)$ of class $C^{1+\alpha}$ such that for small neighborhoods $U \supset V$ of x we have, in local charts, that:

- *$g|_V = A_x$ and*
- *$g = f$ outside the set U.*

Actually, Arbieto and Matheus proved that $g|_V = Df_x$ by constructing a perturbation $h(y) = \rho(y)(f(x) + Df_x(y - x)) + (1 - \rho(y))f(y)$ where ρ is a *bell-function* over the annulus $B(x, r) \setminus B(x, r/2)$. Then they make use of a cleaver application of a theorem of Dacorogna and Moser ([12]) to obtain a new $\tilde{h}$ which will be area-preserving. Theorem 4.3 is obtained in the same way by switching Df_x by A_x.

Remark 4.4. We can take, in Theorem 4.3, $A_x = Df_x \cdot S_x$, where S_x is $\frac{\tilde{\varepsilon}}{C}$-close to the identity, where $C := \max_{x \in M} \|Df_x\|$.

Remark 4.5. A similar version of Lemma 4.1 can be obtained directly from Theorem 4.3 if we take f of class C^2.

Finally, we recall the conservative C^1-closing lemma of Arnaud, which in particular assures (a) and (b) bellow. This result is an upgrade of the C^1-closing lemma ([30]) and states that the orbit of a recurrent point x can be approximated for a very long time $\pi > 0$ by a periodic orbit of an area-preserving diffeomorphism $g \in \mathcal{N}^\omega_\varepsilon(f)$.

Theorem 4.6. *(Arnaud* [4]*) Given a recurrent point x, $\varepsilon > 0$ and a C^1-neighborhood $\mathcal{N}^\omega_\varepsilon(f)$, there exists a periodic orbit p of $g \in \mathcal{N}^\omega_\varepsilon(f)$ with period π such that*

(a) $\operatorname{dist}\left(f^i(x), g^i(p)\right) < \varepsilon$ *for all* $i \in \{0, ..., \pi\}$;
(b) $g = f$ *except on the* ε*-neighborhood of the* g*-orbit of* p.

5. C

5.1. **Mixing the eigendirections-Part I.** We start by proving the following result.

Lemma 5.1. *Given a hyperbolic matrix* $A \in SL(2, \mathbb{R})$*, let* $\theta = \sphericalangle(E^s, E^u)$ *be the angle between the matrix* A *eigendirections. Assume that the rotation* R_θ *of angle* θ *takes the unstable direction onto the stable direction of* A*, i.e.,* $R_\theta(E^u) = E^s$*. Then the matrix* $A \cdot R_\theta$ *is elliptic.*

Proof. Let $B := A \cdot R_\theta$. Consider the action of the matrices A and B on the projective line $\mathbb{P}^1 = \mathbb{R}/\pi\,\mathbb{Z}$, described by the diffeomorphisms $f_A : \mathbb{P}^1 \to \mathbb{P}^1$ and $f_B : \mathbb{P}^1 \to \mathbb{P}^1$. Lift these maps to diffeomorphisms $F_A \colon \mathbb{R} \to \mathbb{R}$ and $F_B \colon \mathbb{R} \to \mathbb{R}$ such that $F_A(x + \pi) = F_A(x) + \pi$ and $F_B(x+\pi) = F_B(x)+\pi$, for all $x \in \mathbb{R}$. As $\det(A) = \det(B) = 1$ we get that F_A and F_B are increasing functions. The definition of θ shows that the lifting F_B can be chosen to satisfy the relation $F_B(x) = F_A(x + \theta)$, for all $x \in \mathbb{R}$. Since A is hyperbolic, f_A has two fixed points: an expanding fixed point x^u, and a contracting fixed point x^s. We can choose the lifting F_A so that it has two families of fixed points, $x^u + k\,\pi$ and $x^s + k\,\pi$, with $k \in \mathbb{Z}$, and we may assume that the fixed points x^s, $x^u \in \mathbb{R}$ satisfy $|x^s - x^u| = \theta$. In order to prove that B is elliptic it is enough to show that f_B has non zero rotation number, which amounts to say that $F_B(x) - x$ keeps a constant sign as x runs through $\mathbb{R}$. Two cases may occur: $x^s < x^u$ and $x^u < x^s$. Assume first that $x^s < x^u$. Then $-\theta < F_A(x) - x < 0$ for all $x \in]x^s, x^u[$, and $F_A(x) - x > 0$ for all $x \in]x^u, x^s + \pi[$. This implies that $F_A(x) - x > -\theta$, for all $x \in \mathbb{R}$. Therefore, $F_B(x) - x = F_A(x + \theta) - (x + \theta) + \theta > -\theta + \theta = 0$, for every $x \in \mathbb{R}$, proving that B is elliptic. Assume now that $x^u < x^s$. In this case $0 < F_A(x) - x < \theta$ for all $x \in]x^u, x^s[$, and $F_A(x) - x < 0$ for every $x \in]x^s, x^u + \pi[$. But this implies that $F_A(x) - x < \theta$, for all $x \in \mathbb{R}$. Accordingly, $F_B(x) - x = F_A(x + \theta) - (x + \theta) + \theta < -\theta + \theta = 0$, for every $x \in \mathbb{R}$, which proves that B is elliptic. □

We easily deduce the following result from Lemma 5.1 and Lemma 4.1.

Proposition 5.2. *Given* $\varepsilon > 0$ *and* $f \in \mathit{Diff}_\omega^{\,1}(M)$*, there exists* $\theta > 0$ *such that given any* $x \in Per_{hyp}(f)$ *with period* $\tau > 1$*, and such that* $\sphericalangle(E^u_y, E^s_y) < \theta$ *for some* y *in the* f*-orbit of* x*, then there is some perturbation* $g \in \mathcal{N}_\varepsilon^{\,\omega}(f)$ *such that* x *is an elliptic periodic point for* g *with period* τ.

As we saw in the preceding proposition mixing eigendirections by rotations reveals to be useful to create elliptic periodic orbits for maps near the original one. However, we are only allowed to perform a small perturbation and this can be difficult, or maybe impossible, if the angle between eigendirections is far from zero. In the next lemma we assume some hypotheses under which it will be possible to achieve that objective and, its proof, although easier, follows closely the one in [8, Lemma 3.8].

Lemma 5.3. *Given $f \in Diff_\omega^r(M)$, $r \geq 1$ and $\varepsilon > 0$ let $\theta(f, \varepsilon) = \theta > 0$ be given by Lemma 4.1 (with $\theta < \beta_0$). There is $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$, if $x \in Per_{hyp}(f)$ has period $\tau > m$ and satisfies*

(1) *$\sphericalangle(E^u_{f^n(x)}, E^s_{f^n(x)}) > \theta$, for all $n \in \{1, ..., \tau\}$ and*
(2) *we have $f^n(x) \in \Delta_m(f)$ for some $n \in \{1, ..., \tau\}$,*

then there exist a C^r conservative map $g \in \mathcal{N}_\varepsilon^\omega(f)$ and $y = f^k(x)$ ($k \in \{1, ..., \tau\}$) such that $Dg^m_x(E^u_y) = E^s_{f^m(y)}$.

Proof. Let $C := \max_{x\in M} \|Df_x\|$ and $c > C^2$ depending on the angle θ and obtained according to Lemma 2.3.

Let $x \in Per_{\mathrm{hyp}}(f)$ with period $\tau > m > m_0$ and satisfying (1) and (2). The number m_0 will be very large and will be defined below. By (2) there exists y in the f-orbit of x such that $y \in \Delta_m(f)$, i.e.,

$$\|Df^m_y|_{E^s_y}\|.\|Df^m_y|_{E^u_y}\|^{-1} \geq 1/2. \tag{5.1}$$

$\boxed{\text{Case I}}$

Suppose that for any $i, j \in \{0, 1, ..., m\}$, where $i < j$, we have

$$\|Df^{j-i}_{f^i(y)}|_{E^s_y}\|.\|Df^{j-i}_{f^i(y)}|_{E^u_y}\|^{-1} \leq c. \tag{5.2}$$

Noting that $E^{(\cdot)}_y$ (for $(\cdot) = u/s$) are one-dimensional and using (5.1) and (5.2) we get

$$\frac{\|Df^{j-i}_{f^i(y)}|_{E^s_y}\|}{\|Df^{j-i}_{f^i(y)}|_{E^u_y}\|} = \frac{\|Df^{m-j}_{f^i(y)}|_{E^u_y}\|.\|Df^m_y|_{E^s_y}\|.\|Df^i_y|_{E^u_y}\|}{\|Df^{m-j}_{f^i(y)}|_{E^s_y}\|.\|Df^m_y|_{E^u_y}\|.\|Df^i_y|_{E^s_y}\|} \geq \frac{1}{2c^2}. \tag{5.3}$$

Using (5.2) again we obtain, for $H := 2c^2$, that

$$\frac{1}{H} \leq \frac{\|Df^{j-i}_{f^i(y)}|_{E^s_y}\|}{\|Df^{j-i}_{f^i(y)}|_{E^u_y}\|} \leq H. \tag{5.4}$$

Using (1) we can make a conservative change of coordinates as it was explained in Section 2.2.3 keeping the control on the estimated (depending on $\sin^2\theta$). Hence, by conservativeness, for any

$j \in \{0, 1, ..., m\}$, we have $\|Df_y^j|_{E_y^s}\|.\|Df_y^j|_{E_y^u}\| = \det Df_y^j = 1$. Therefore, using (5.4) we get that $\|Df_y^j|_{E_y^{(\cdot)}}\| \leq 2H = 4c^2$ for $(\cdot) = u/s$ and every j. This implies that for every $j \in \{0, 1, ..., m\}$ we have $\|Df_y^j\| \leq 2H$.

For some $\gamma > 0$ very small, let $\{\theta_j\}_{j=0}^{m-1}$ be such that $0 < \theta_j \leq \gamma$ (for all j) and $\sphericalangle(E_y^s, E_y^u) = \sum_{j=0}^{m-1} \theta_j$. We define, for every $j = \{0, 1, ..., m-1\}$, linear maps $S_j \colon T_{f^j(y)}M \to T_{f^{j+1}(y)}M$ by $S_j := Df_y^{j+1} \cdot R_{\theta_j} \cdot (Df_y^j)^{-1}$. It is straightforward to see that

$$S_{m-1} \cdot S_{m-2} \cdot ... \cdot S_1 \cdot S_0(E_y^u) = Df_y^m \cdot R_{\sphericalangle(E_y^s, E_y^u)}(E_y^u) = E_{f^m(y)}^s.$$

Using Theorem 4.3 we realize[2] these perturbation by m conservative maps g_j in m small self-disjoint balls $B_j := B(f^j(y), r_i)$, $r_i > 0$. Then we define a conservative map g by being equal to g_i in B_i and equal to f outside the union of these balls.

Observe that, since H is fixed, $\|S_i - Id\|$ is small as long as θ_i is close to zero which is equivalent to take γ very small.

We leave it to the reader to verify that, since we have a control on the norm of Df_y^j, g can be chosen ε-close to f and we just have to take m_0 be any positive integer such that $m_0 \geq \frac{2\pi}{\gamma}$.

$\boxed{\text{Case II}}$

We now turn to the case where (5.2) is false, i.e., there exists $i, j \in \{0, 1, ..., m\}$, where $i < j$, such that

$$\|Df_{f^i(y)}^{j-i}|_{E_y^s}\|.\|Df_{f^i(y)}^{j-i}|_{E_y^u}\|^{-1} > c. \tag{5.5}$$

It is understood that $j - i > 1$ because $c > C^2$. Take unit vectors $s \in E_{f^i(y)}^s$ and $u \in E_{f^i(y)}^u$. By (5.5) we are in the hypotheses of Lemma 2.3 for the linear map $Df_{f^i(y)}^{j-i}$, therefore we can find a nonzero vector $v \in T_{f^i(y)}M$ such that $\sphericalangle(v, u) < \theta$ and $\sphericalangle(Df_{f^i(y)}^{j-i} \cdot v, E_{f^j(y)}^s) < \theta$. By making two perturbations at $f^i(y)$ (using Lemma 4.1) and at $f^{j-1}(y)$ (using Remark 4.2) we can obtain $g \in \mathcal{N}_\varepsilon^\omega(f)$, such that:

$$Dg_{f^i(y)} := Df_{f^i(y)} \cdot R_{\sphericalangle(v, E_{f^i(y)}^u)} \text{ and } Dg_{f^{j-1}(y)} := R_{\sphericalangle(Df_{f^i(y)}^{j-i} \cdot v, E_{f^j(y)}^s)} \cdot Df_{f^{j-1}(y)}.$$

[2]In order to use Theorem 4.3 f must be of class C^2. The important point to note here is that we can perturb slightly, using [37], and obtain a C^2 conservative map having the same properties (1) and (2) of Lemma 5.3 for the analytic continuation of the hyperbolic point x.

Moreover, $g = f$ outside two small balls around $f^i(y)$ and $f^j(y)$. Is is easy to verify that by concatenating the tangent maps of g along $\{f^n(y)\}_{n=i}^{j}$ we complete the proof.

□

5.2. **Mixing the eigendirections-Part II.** Our purpose now is to prove the next proposition and its proof will be divided into two main steps; Lemma 5.3 above and Lemma 5.5 below.

Proposition 5.4. *Given $f \in \text{Diff}^1_\omega(M)$, $\varepsilon > 0$ and $\theta > 0$, there exist $m \in \mathbb{N}$ and $T \in \mathbb{N}$ ($T > m$) such that given a periodic hyperbolic point $x \in M$ with period $\tau > T$, satisfying the conditions (1) and (2) of Lemma 5.3, then there is some perturbation $g \in \mathcal{N}^{\omega}_{\varepsilon}(f)$ such that x is an elliptic periodic point for g with period τ.*

The following result allows us, once in the hypotheses of Proposition 5.4, to obtain some control on the growth of the norm of Dg^τ for a large τ, where $g \in \mathcal{N}^{\omega}_{\varepsilon}(f)$.

Lemma 5.5. *Let $f \in \text{Diff}^1_\omega(M)$, $\varepsilon > 0$ and $\theta > 0$ be given. Let $m = m(\varepsilon, \theta) \in \mathbb{N}$ be given by Lemma 5.3. Then there exists $K = K(\theta, m) \in \mathbb{R}$ such that given any hyperbolic periodic point x with period $\tau > m$ satisfying (1) and (2) of Lemma 5.3, then there exists $g \in \mathcal{N}^{\omega}_{\varepsilon}(f)$ such that x is also a periodic orbit for g with period τ and $\|Dg^\tau_y\| < K$, for some y in the g-orbit of x.*

Proof. For $f \in \text{Diff}^1_\omega(M)$ and $\varepsilon > 0$ given, there exists $C > 1$ such that, if $g \in \mathcal{N}^{\omega}_{\varepsilon}(f)$ then $\|Dg\| \leq C$. We define

$$K(m(\theta)) := 4C^{m+2} \sin^{-2} \theta. \tag{5.6}$$

Take any hyperbolic periodic point x with period $\tau > m$. Let $g \in \text{Diff}^1_\omega(M)$ be the perturbation provided by Lemma 5.3, corresponding to the same ε and θ of this lemma. We assume that the point y given in Lemma 5.3 is $y = x$. According to Section 2.2 we take matrix representations diagonalizing the hyperbolic decomposition and along the orbit.

Given $k \in \{1, ..., \tau - m\}$ let $y = f^{-k}(x)$. Take a finite sequence $\{F(y,i)\}_{i=1}^{k} \subset \mathbb{R}$ such that the matrix Df^i_y written in the diagonal[3] form associated to the eigendirections is,

$$Df^i_y = \begin{pmatrix} F(y,i) & 0 \\ 0 & F(y,i)^{-1} \end{pmatrix},$$

[3]In fact, we are abusing the notation since we should denote this representation by $\widetilde{Df^i_y}$ instead of Df^i_y.

and let $\sigma = F(y, \tau) > 1$. Observe that by Lemma 2.2 (2), for $i \in \{1, ..., \tau\}$, we have

$$\begin{aligned}\max\{|F(y,i)|, |F(y,i)|^{-1}\} &= \lfloor\!\lfloor Df_y^i \rfloor\!\rfloor \le \|Df_y^i\| \sin^{-1}\theta \\ &\le \prod_{j=0}^{i-1} \|Df_{f^j(y)}\| \sin^{-1}\theta \le C^i \sin^{-1}\theta.\end{aligned}$$

We will consider two cases:

$\boxed{\text{Case I}}$ If $\sigma \le C^{m+1}\sin^{-1}\theta$ then observing that $\lfloor\!\lfloor Df_y^\tau \rfloor\!\rfloor = \sigma$ and applying Lemma 2.2 (1), we obtain

$$\|Df_y^\tau\| \le 4\sin^{-1}\theta \lfloor\!\lfloor Df_y^\tau \rfloor\!\rfloor = 4\sigma\sin^{-1}\theta \le 4C^{m+1}\sin^{-2}\theta \le K,$$

and the lemma is proved by just choosing $g = f$.

$\boxed{\text{Case II}}$ On the other hand, if $\sigma > C^{m+1}\sin^{-1}\theta$, we will use the following calculus lemma whose proof we postpone to the end of the proof of Lemma 5.5.

Lemma 5.6. *Given $\tau, m \in \mathbb{N}$, $\tau > m$, $C > 1$ and $\{a_i\}_{i=1}^\tau$ such that $|a_i|^{\pm 1} < C$ we define $\sigma := |\Pi_{i=1}^\tau a_i|$. If $\sigma > C^{m+1}$, then there exists $k \in \{1, ..., \tau - m\}$ such that*

$$\left|\frac{\Pi_{i=k+m}^\tau a_i}{\Pi_{i=1}^k a_i}\right|^{\pm 1} \le C^2.$$

We feed Lemma 5.6 with $a_i = F(y, i)$ and let $k \in \{1, ..., \tau - m\}$ be given by this lemma. Take $y = f^{-k}(x)$. Since

$$Df_y^\tau = Df_{f^m(x)}^{\tau-m-k} \cdot Df_x^m \cdot Df_y^k$$

we may write Df_y^τ as the following diagonal matrix product representation

$$\begin{pmatrix} F(f^m(x), \tau-m-k) & 0 \\ 0 & \frac{1}{F(f^m(x),\tau-m-k)}\end{pmatrix}\begin{pmatrix} F(x,m) & 0 \\ 0 & \frac{1}{F(x,m)}\end{pmatrix}\begin{pmatrix} F(y,k) & 0 \\ 0 & \frac{1}{F(y,k)}\end{pmatrix}. \tag{5.7}$$

Recall that g, given by Lemma 5.3, is a conservative perturbation of f, supported in a small neighborhood of $\{f^i(x) \colon i \in \{0, ..., m\}\}$, and such that $Dg_x^m(E_x^u) = E_{f^m(x)}^s$. Taking in account the notation of section 2.2.2 we get that $\xi \colon E_x^u \to E_{f^m(x)}^u$ must be the null map, where

$$Dg_x^m := \begin{pmatrix} \xi & \alpha \\ \beta & \gamma \end{pmatrix}, \tag{5.8}$$

for some constants ξ, α, β and γ. That is, the unstable component of the image by Dg^m_x of E^u_x must be zero and so $\xi = 0$.

Now, one just replaces the middle matrix in (5.7) and we obtain that

$$Dg^\tau_y = Df^{\tau-s-m}_{f^m(x)} \cdot Dg^m_x \cdot Df^k_y,$$

is given by

$$Dg^\tau_y = \begin{pmatrix} 0 & \alpha \frac{F(f^m(x),\tau-m-k)}{F(y,k)} \\ \beta \frac{F(y,k)}{F(f^m(x),\tau-m-k)} & \gamma \frac{1}{F(y,k)F(f^m(x),\tau-m-k)} \end{pmatrix}.$$

Notice that,

$$\begin{aligned} \frac{1}{F(y,k)F(f^m(x),\tau-m-k)} &= \frac{F(x,m)}{\sigma} \leq \frac{\lfloor\!\lfloor Df^m_x \rfloor\!\rfloor}{\sigma} \leq \frac{\|Df^m_x\|}{\sigma \sin\theta} \\ &\leq \frac{C^m}{\sigma \sin\theta} < \frac{1}{C}. \end{aligned}$$

Moreover, by Lemma 2.2 (2)

$$\max\{|\alpha|, |\beta|, |\gamma|\} = \lfloor\!\lfloor Dg^m_x \rfloor\!\rfloor \leq \sin^{-1}\theta \|Dg^m_x\| \leq C^m \sin^{-1}\theta.$$

Using Lemma 5.6 we get $\lfloor\!\lfloor Dg^\tau_y \rfloor\!\rfloor \leq \max\{|\alpha|, |\beta|, |\gamma|\} C^2 \leq C^{m+2}\sin^{-1}\theta$. Finally, using Lemma 2.2 (1) we get

$$\|Dg^\tau_y\| < 4\sin^{-1}\theta \lfloor\!\lfloor Dg^\tau_y \rfloor\!\rfloor < 4C^{m+2}\sin^{-2}\theta = K,$$

and the lemma is proved.

□

Proof. (of Lemma 5.6)
For $k = 1$ since we have $\sigma > C^{m+1}$ we obtain

$$\left| \frac{\prod_{i=m+1}^{\tau} a_i}{a_1} \right| = \frac{\sigma}{|a_1 \prod_{i=1}^{m} a_i|} \geq \frac{\sigma}{C^{m+1}} > 1.$$

For $k = \tau - m$ we have

$$\left| \frac{a_\tau}{\prod_{i=1}^{\tau-m} a_i} \right| = \frac{|a_\tau \prod_{i=\tau-m+1}^{\tau} a_i|}{\sigma} \leq \frac{C^{m+1}}{\sigma} < 1.$$

Let

$$\Phi(k) = \left| \frac{\prod_{i=k+m}^{\tau} a_i}{\prod_{i=1}^{k} a_i} \right|.$$

We chose $k \in \{1, ..., \tau - m - 1\}$ such that $\Phi(k) > 1$ and $\Phi(k+1) < 1$. Since $\Phi(k)^{-1} < 1 < C^2$ we are left to the task of proving that $\Phi(k) \leq C^2$.

$$\Phi(k) = \left| \frac{\prod_{i=k+m}^{\tau} a_i}{\prod_{i=1}^{k} a_i} \right| = \Phi(k+1)|a_{k+1}||a_{k+m}| \leq C^2.$$

□

Remark 5.7. The important thing to note here is that Lemma 5.5 allows us to fix a uniform bound K such that we can pick a periodic hyperbolic point with very large period and, nevertheless, the tangent map (on the period) is bounded by K for a C^1-arbitrarily close conservative map.

Proof. (of Proposition 5.4)
We know that for any diffeomorphism f_1 C^1-close to f any hyperbolic periodic point x of f has an analytic continuation y for the diffeomorphism f_1 (see e.g. [34]). Moreover, by [37], $\text{Diff}^2_\omega(M)$ is C^1-dense in $\text{Diff}^1_\omega(M)$. Hence, for a diffeomorphism $f_1 \in \text{Diff}^2_\omega(M)$ arbitrarily C^1-close to f, by Lemma 5.3, we take $m_0(f_1)$ (larger than $m_0(f)$ if necessary) such that, if y is a hyperbolic periodic point of period $\tau > m$ for any $m \geq m_0(f_1)$ satisfying

(1) $\measuredangle(E^u_{f_1^n(y)}, E^s_{f_1^n(y)}) \geq \theta$ for all $n \in \{1, ..., \tau\}$ and
(2) $f_1^n(y) \in \Delta_m(f_1)$ for some $n \in \{1, ..., \tau\}$,

then there exist $f_2 \in \text{Diff}^2_\omega(M) \cap \mathcal{N}_\varepsilon^\omega(f)$ and $z = f_1^k(y)$, for $k \in \{1, ..., \tau\}$, such that $(Df_2{}^m)_y(E^u_z) = E^s_{f_1^m(z)}$.

Fix $f_2 \in \text{Diff}^2_\omega(M)$ and any $x \in M$. By Theorem 4.3 followed by Remark 4.4, for $\varepsilon > 0$, there exists $\zeta_0 > 0$ such that any $S_x \in SL(2, \mathbb{R})$ which is ζ-close to the identity (with $\zeta < \zeta_0$) satisfies the following; there exists $g \in \mathcal{N}_\varepsilon^\omega(f_2)$ such that for small neighborhoods $U \supset V$ of x we have, in local charts, that:

- $g|_V = (Df_2)_x \cdot S_x$ and
- $g = f_2$ outside the set U.

Take $K := K(m(\theta))$ according to Lemma 5.5 and depending on $f_1 \in \text{Diff}^2_\omega(M)$, on ε, $m_0(f_1)$ and on θ. Now, for ζ_0 and θ fixed above, set $\sigma := (\eta_\theta)^{-1}(\zeta_0)$, where $\eta_\theta(\zeta_0)$ was defined in (2.2). By definition, the number $\sigma > 1$ has the following property: Given any $\varphi \geq \theta$, we can pick hyperbolic matrices $S \in SL(2, \mathbb{R})$ such that:

(i) $\|S - Id\| \leq \zeta_0$;
(ii) σ and σ^{-1} are the eigenvalues of S; and
(iii) S has an angle φ between its eigenspaces.

Finally, we take $T \in \mathbb{N}$ such that $\sigma^T \geq K$. Now, let $\Gamma = \{f_1^n(x) : n \in \{1, ..., \tau\}\}$ be any hyperbolic periodic orbit, with period $\tau > T$, satisfying (1) and (2) of Lemma 5.3. Let $g \in \mathcal{N}_\varepsilon^\omega(f_1)$ be the diffeomorphism provided by Lemma 5.5 satisfying $\|Dg^\tau_y\| < K$ for some point $y \in \Gamma$. We take $i \in \{0, ..., \tau - 1\}$ and we define $x_i := f_1^i(y)$ and

$x_i' := f_1(x_i) = f_1^{i+1}(y)$. Take the linear maps $Dg_{f_1^i(y)} := Dg_i \colon \mathbb{R}^2_{x_i} \to \mathbb{R}^2_{x_i'}$ and let $\theta_i \geq \theta$ be the angle between the eigenspaces $E^u_{x_i}$ and $E^s_{x_i}$ of the map Dg_i^τ, for each $i \in \{0, ..., \tau - 1\}$. Take now $S_i \in SL(2, \mathbb{R})$ such that $\|S_i - I\| \leq \zeta_0$, and S_i has eigenspace $E^u_{x_i}$ with eigenvalue σ^{-1}, and has eigenspace $E^s_{x_i}$ with eigenvalue σ. Observe that these eigenspaces do make an angle equal to θ_i. The product linear map $(Dg_i{\cdot}S_i) \colon \mathbb{R}^2_{x_i} \to \mathbb{R}^2_{x_i'}$ takes the decomposition $\mathbb{R}^2_{x_i} = E^u_{x_i} \oplus E^s_{x_i}$ onto the decomposition $\mathbb{R}^2_{x_i'} = E^u_{x_i'} \oplus E^s_{x_i'}$. Moreover, we have $\|Dg_i \cdot S_i|_{E^u_{x_i}}\| = \|Dg_i|_{E^u_{x_i}}\|\sigma^{-1}$ and $\|Dg_i \cdot S_i|_{E^s_{x_i}}\| = \|Dg_i|_{E^s_{x_i}}\|\,\sigma$.

Consider a family of smooth deformations of the identity into S_i, that is, let $\{S_{i,t}\}_{i=0,t\in[0,1]}^{\tau-1}$ be defined analogously to S_i but with eigenvalues σ^t and σ^{-t}, where for $t = 0$ we get the identity and for $t = 1$ we get S_i.

By a direct application of Theorem 4.3 we can obtain a family of C^1 area-preserving diffeomorphisms $(h_i)_t$ such that $(h_i)_t \in \mathcal{N}_\varepsilon^\omega(g)$, $g = (h_i)_t$ outside a small neighborhood of the point x_i, and $[D(h_i)_t]_{x_i} = Dg_i \cdot S_{i,t}$. But, since we can produce these perturbations with self-disjoint support, we can glue them into a single conservative C^1 perturbation h_t ($t \in [0, 1]$) of g such that $h_t \in \mathcal{N}_\varepsilon^\omega(g)$ and $g = h_t$ outside a small neighborhood of Γ. By way of construction, the area-preserving diffeomorphism h_t has the same invariant decomposition as g. Moreover, using that $\|Dg^\tau_y\| < K$ and also the unidimensionality of E^u, we have

$$\varphi(t) := \|D(h_t)^\tau_y|_{E^u_y}\| = \|Dg^\tau_y|_{E^u_y}\|\,\sigma^{-\tau t} < K\,\sigma^{-\tau t}\ , \tag{5.9}$$

while, on the other hand, $\|D(h_t)^\tau_y)|_{E^s_y}\| > K\,\sigma^{\tau t}$. For $t = 0$ we have $\varphi(0) = \|Dg^\tau_y|_{E^u_y}\| > 1$. But, since $\sigma^\tau \geq K$ (recall that $\tau > T$), for $t = 1$ we get $\varphi(1) < 1$. Therefore, there is some $t_0 \in]0, 1[$ such that $\varphi(t) = 1$. For such t_0 we must have $\|D(h_{t_0})^\tau_y\| = 1$.

Finally, applying[4] Lemma 4.1 to the periodic orbit y of h_{t_0} we get a conservative C^1 perturbation h of h_{t_0} such that $h \in \mathcal{N}_\varepsilon^\omega(h_{t_0})$ and y is an elliptic periodic orbit of h.

Going back and replacing ε by $\varepsilon/5$ along the proof enables us to conclude the proof of the proposition. □

[4]If the point is parabolic we can perform a small rotation in the tangent space in order to make it elliptic.

The absence of elliptic periodic orbits for all nearby perturbations implies uniform bounds on hyperbolic orbits with large enough period. This is an easy consequence of the two previous Proposition 5.2 and Proposition 5.4 which we state for future reference.

Corollary 5.8. *Let $f \in Diff^{\,1}_{\omega}(M)$ and $\varepsilon > 0$ be given and set $\theta = \theta(\varepsilon, f)$, $m = m(\varepsilon, \theta)$ and $T = T(m)$ given by Proposition 5.2 and Proposition 5.4.*

Assume that all area-preserving maps g which are ε-C^1-close to f do not admit elliptic periodic orbits. Then for every such g all closed orbits with period larger than T are hyperbolic, m-dominated and with angle between its stable and unstable directions bounded from below by θ.

6. P T B

In this section we present the proof of Theorem B. Let $f \in \mathrm{Diff}^{\,1}_{\omega}(M)$ be a non Anosov diffeomorphism $\varepsilon > 0$ and U any open subset of M, we will prove that there exists an area-preserving map $g \in \mathcal{N}_{\varepsilon}^{\omega}(f)$ and which exhibits an elliptic orbit passing through U.

Let $\mathcal{P}$ be the residual set given by the general density theorem (see [30]), that is $\mathcal{P}$ is the set of all area-preserving maps f such that $\Omega(f)$ is the closure of the set of periodic orbits, all of them hyperbolic or elliptic, and $\Omega(f) = M$ by the Poincaré recurrence theorem.

We take any $f \in \mathrm{Diff}^{\,1}_{\omega}(M)$ which is not approximated by an Anosov area-preserving map. Then by a small C^1 perturbation we can and will assume that f belongs to $\mathcal{P}$ and that f is still *not* approximated by an Anosov conservative map. We fix some open set $U \subset M$ and $\varepsilon > 0$.

If some elliptic periodic orbit of f intersects U there is nothing to prove, just choose $f = g$. Otherwise we must consider three cases:

$\boxed{\text{Case I}}$ All periodic orbits of f which intersect U are hyperbolic, and some of them has a small angle, less than $\theta = \theta(\varepsilon, f)$ provided by Proposition 5.2, between the stable and unstable eigendirections at one point of the orbit.

$\boxed{\text{Case II}}$ All periodic orbits of f which intersect U are hyperbolic, with angle between the stable and the unstable directions bounded from bellow by θ, but some of them, with period larger than T, do not admits any m-dominated splitting, where $m = m(\varepsilon, \theta)$ and $T = T(m)$ are given by Proposition 5.4, and $\theta = \theta(\varepsilon, f)$ was given as before by Proposition 5.2.

$\boxed{\text{Case III}}$ All periodic orbits of f which intersect U and have period larger than T are hyperbolic, with m-dominated splitting, and with the angle between the stable and unstable directions bounded from bellow by θ, where $m = m(\varepsilon, \theta)$ and $T = T(m)$ are given by Proposition 5.4, and $\theta = \theta(\varepsilon, f)$ was given as before by Proposition 5.2.

Using Proposition 5.2 the Case I implies the desired conclusion for some area-preserving diffeomorphism $g \in \mathcal{N}_\varepsilon^\omega(f)$. Analogously for Case II by the choice of the bounds m, T and by Proposition 5.4.

Finally, we use Theorem F to show that if f is in Case III and we assume that every C^1-nearby area-preserving map g does not admit elliptic periodic orbits through U, then we get a contradiction. This establishes the statement of Theorem B.

If f is in Case III, then from Corollary 5.8 we know that every periodic orbit intersecting U, for area-preserving diffeomorphism $g \in \mathcal{N}_\varepsilon^\omega(f)$, with period larger than T, is hyperbolic with uniform bounds on m and θ.

From Theorem F, since f is not approximated by an Anosov area-preserving map, there exists an area-preserving map g, which is $\frac{\varepsilon}{2}$-C^1-close to f, admitting a full Lebesgue measure subset $\mathcal{Z}$ where all the Lyapunov exponents for g are zero. Moreover, we can assume that g is aperiodic, that is the set of all periodic orbits has zero Lebesgue measure[5].

Let $\hat{U} \subset U$ be a measurable set with positive Lebesgue measure. Let $R \subset \hat{U}$ be the set given by Poincaré Recurrence Theorem with respect to g. Then every $x \in R$ returns to $\hat{U}$ infinitely many times under g and is not a periodic point. Denote by $\mathcal{T}$ the set of positive return times to $\hat{U}$ under g.

Given $x \in \mathcal{Z} \cap R$ and $0 < \delta < \log 2/2m$, from the Oseledets' theorem there exists $n_x \in \mathbb{R}$ such that the upper Lyapunov exponent is near zero, formally, for every $n \geq n_x$ we have

$$e^{-\delta n} < \|Dg_x^n\| < e^{\delta n}.$$

Let us choose $\tau \in \mathcal{T}$ such that $\tau > \max\{n_x, T\}$.

Now, by Arnaud's closing lemma ([4]), given a g-recurrent point x, $\varepsilon > 0$ and a neighborhood $\mathcal{N}_{\varepsilon/2}^\omega(g)$, there exists a periodic orbit p of $h \in \mathcal{N}_{\varepsilon/2}^\omega(g)$ with period π such that

a) $\operatorname{dist}\left(g^i(x), h^i(p)\right) < \varepsilon$ for all $i \in \{0, ..., \pi\}$;

[5]Actually, by the conservative version of the Kupka-Smale theorem (see [31]) we obtain a residual where the periodic points are countable, hence of zero Lebesgue measure.

b) $h = g$ except on the ε-neighborhood of the h-orbit of p.

Letting $\varepsilon > 0$ be small enough we obtain also that

$$e^{-\delta\pi} < \|Dh_p^\pi\| < e^{\delta\pi} \quad \text{with} \quad \pi > T. \tag{6.1}$$

Now it is easy to see that $h \in \mathcal{N}_\varepsilon^\omega(f)$, so that the orbit of p under h satisfies the conclusion of Corollary 5.8. In particular we have that

$$\frac{\left\|Dh_x^m \mid E_x^s\right\|}{\left\|Dh_x^m \mid E_x^u\right\|} \leq \frac{1}{2} \quad \text{for all } x \text{ in the } h\text{-orbit of } p,$$

for otherwise we would use Proposition 5.4 and produce an elliptic periodic orbit for an area-preserving map in $\mathcal{N}_\varepsilon^\omega(f)$. Since the subbundles E^s and E^u are one-dimensional we write $p_i := h^{im}(p)$ for $i = 0, \ldots, \lfloor \pi/m \rfloor = \ell$ with $\lfloor z \rfloor$ denoting the largest integer less or equal than z and

$$\frac{\left\|Dh_p^\pi \mid E_p^s\right\|}{\left\|Dh_p^\pi \mid E_p^u\right\|} = \frac{\left\|Dh^{\pi-m\ell} \mid E_{p_\ell}^s\right\|}{\left\|Dh^{\pi-m\ell} \mid E_{p_\ell}^u\right\|} \cdot \prod_{i=0}^{\ell-1} \frac{\left\|Dh^m \mid E_{p_i}^s\right\|}{\left\|Dh^m \mid E_{p_i}^u\right\|} \leq L(p,h) \cdot \left(\frac{1}{2}\right)^\ell, \tag{6.2}$$

where

$$L(p,h) = \sup_{i \in \{0,\ldots,m\}} \left(\frac{\|Dh^i \mid E_p^s\|}{\|Dh^i \mid E_p^u\|} \right)$$

depends continuously on h in the C^1 topology. Therefore, there exists a uniform bound on $L(p,h)$ for all maps $h \in \mathcal{N}_\varepsilon^\omega(f)$.

We note that we can take $\pi > T$ arbitrarily large by letting $\varepsilon > 0$ be small enough in the above arguments. Therefore (6.2) ensures that

$$\frac{1}{\pi} \log \left\|Dh^\pi \mid E_p^s\right\| \leq \frac{1}{\pi} \log L(p,h) + \frac{\ell}{\pi} \log \frac{1}{2} + \frac{1}{\pi} \log \left\|Dh^\pi \mid E_p^u\right\|.$$

Moreover, since h is area-preserving and recalling (2.8), we have that the sum of the Lyapunov exponents along the h-orbit of p is zero, that is (we recall that π is the period of p)

$$\frac{1}{\pi} \log \|Dh^\pi \mid E_p^s\| = -\frac{1}{\pi} \log \|Dh^\pi \mid E_p^u\|.$$

The constants in (6.2) are independent of π so taking the period very large and noting that $\|Dh_p^\pi\| = \|Dh^\pi \mid E_p^u\|$ we deduce that

$$\frac{1}{\pi} \log \|Dh_p^\pi\| \geq \frac{1}{2m} \log 2 > \delta.$$

This contradicts (6.1) and Theorem B follows.

7. M -

7.1. **Robust transitivity.** Here we present an alternative proof of Theorem C using the next well-know theorem (see for example [32, Theorem 5.2]).

Theorem 7.1. *(KAM) Let $f \in Diff_{\omega}^{\infty}(M)$, p a periodic elliptic orbit with period π and assume that the two eigenvalues of Df_p^{π}, denoted by λ_1 and λ_2, are such that $\lambda_1 = e^{2\pi i\theta}$ and $\lambda_1 = e^{-2\pi i\theta}$ for $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \in Diff_{\omega}^{\infty}(M)$ such that $f_k \underset{k\to+\infty}{\longrightarrow} f$ (in the C^1-topology) such that each f_k has an elliptic periodic orbit p_k admitting a f_k-invariant tori.*

Proof. (of Theorem C) Assume that $f \in \mathrm{Diff}_{\omega}^{1}(M)$ is non Anosov and C^1-robustly transitive. Hence, there exists a C^1-neighborhood of f, $\mathcal{V} \subset \mathrm{Diff}_{\omega}^{1}(M)$, such that every $h \in \mathcal{V}$ is transitive. By Theorem B given a non Anosov diffeomorphism $f \in \mathrm{Diff}_{\omega}^{1}(M)$, $\varepsilon > 0$, $x \in M$ and any open subset U of M, then there exists $g \in \mathcal{N}_{\varepsilon}^{\omega}(f)$ and exhibiting an elliptic orbit passing through U. Choose, ε such that $g \in \mathcal{V}$. Since elliptic orbits are stable, we use Zehnder's Theorem [37] and we take $\tilde{g} \in \mathrm{Diff}_{\omega}^{\infty}(M) \cap \mathcal{V}$ and exhibiting an elliptic orbit passing through U.

If the eigenvalues of this elliptic point are in $\mathbb{Q}$, then by using Lemma 4.1, we can perturb in order to get these eigenvalues in $\mathbb{R} \setminus \mathbb{Q}$.

Therefore, we are in the conditions of Theorem 7.1. So, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \in \mathrm{Diff}_{\omega}^{\infty}(M)$ such that $f_k \underset{k\to+\infty}{\longrightarrow} \tilde{g}$ (in the C^1-topology) such that each f_k has an elliptic periodic orbit p_k admitting a f_k-invariant tori. Of course that, for $k \geq k_0$, we have $f_k \in \mathcal{V}$ and the property of having f_k-invariant tori contradicts the C^1-robust transitivity. ◻

We say that $f \in \mathrm{Diff}_{\omega}^{1}(M)$ is *ergodic* if given any measurable f-invariant set it has full or zero Lebesgue measure. Stable ergodicity means persistence of the ergodicity for perturbations of f. It is easy to see that stable ergodicity implies robust transitivity within the conservative context. However, we note that this implication is false if the (stable) ergodicity is with respect to some atomic invariant measure (c.f. the next example).

Example 7.2. *Consider the gradient flow on $\mathbb{S}^2 \subset \mathbb{R}^3$ generated by the height function $h(x, y, z) = -z$. The points $N = (0, 0, 1)$ and $S = (0, 0, -1)$ are a source and a sink respectively. The Dirac measure δ_N (or δ_S) is ergodic, however the flow is non-transitive.*

Corollary 7.3. *If $f \in Diff_{\omega}^{1}(M)$ is C^1-stably ergodic, then f is Anosov.*

As we said in the introduction the KAM phenomena contrasts with stable ergodicity, since it prevails persistence of invariant tori with positive Lebesgue measure.

We end this section with the following yet unknown problem.

Question: Is ergodicity C^1-generic among conservative surface diffeomorphisms?

7.2. **Area-preserving star diffeomorphisms.** Let $f \in \mathrm{Diff}^1_\omega(M)$ be a conservative star-diffeomorphism, that is, there exists a neighborhood $\mathcal{V}$ of f in $\mathrm{Diff}^1_\omega(M)$ such that any $g \in \mathcal{V}$, has all the periodic orbits hyperbolic. We denote this set by $\mathcal{F}^1_\omega(M)$ and, as we said in Section 3, $\mathcal{A}^2_\omega$ denotes the set of conservative Anosov diffeomorphisms on the surface M.

It is clear that $\mathcal{F}^1(M) \cap \mathrm{Diff}^1_\omega(M) \subset \mathcal{F}^1_\omega(M)$; Theorem D implies that

$$\mathcal{F}^1(M) \cap \mathrm{Diff}^1_\omega(M) = \mathcal{F}^1_\omega(M) = \mathcal{A}^2_\omega.$$

As a consequence of Theorem D we also obtain the following result.

Corollary 7.4. *The boundary of $\mathcal{A}^2_\omega$ has no isolated points.*

A diffeomorphism $f \in \mathrm{Diff}^1_\omega(M)$ is said to be C^1-structurally stable in the conservative setting if there exists a C^1 neighborhood, $\mathcal{V}$, of f in $\mathrm{Diff}^1_\omega(M)$ such that every $g \in \mathcal{V}$ is topological equivalent to f (see [32]).

Combining Theorem D with Theorem A we are able to obtain the next result.

Theorem G. *If f is a C^1-structurally stable surface area-preserving diffeomorphism, then f is Anosov.*

We assume Theorem D for a moment and we conclude the proof of Theorem G but before that we present an abstract result about finite product of $SL(2,\mathbb{R})$ matrices that will be used in the proof of Theorem G.

Lemma 7.5. *(*[10, Lemme 6.6]*) For all $\varepsilon > 0$, there exists $N \geq 1$ such that, for all $n \geq N$ and every family $\{A_i\}_{i=1}^n \subset SL(2,\mathbb{R})$, there exists $\{\alpha_i\}_{i=1}^n$ (where each $\alpha_i \in]-\varepsilon, \varepsilon[$) satisfying the following property: For all $i \in \{1, \dots, n\}$ we denote $B_i = R_{\alpha_i} \cdot A_i$ and we have that*

$$B_n \cdot B_{n-1} \cdot \ldots \cdot B_1$$

has real eigenvalues.

Proof. (of Theorem G) Let us fix a C^1-structurally stable area-preserving diffeomorphism in $\text{Diff}^1_\omega(M)$ and choose a neighborhood $\mathcal{V}$ of f whose elements are topologically equivalent to f. If $f \notin \mathcal{A}^2_\omega = \mathcal{F}^1_\omega(M)$, then it follows that $\mathcal{V} \cap \mathcal{A}^2_\omega = \emptyset$. Using Theorem A one gets that there exists a residual subset $\mathcal{R} \subset \mathcal{V}$ such that for every $f_0 \in \mathcal{R}$ the set of elliptic periodic orbits is dense in M. Let us fix $f_0 \in \mathcal{R}$ and choose a small neighborhood of f_0, $\mathcal{W} \subset \mathcal{V}$.

Let x be an elliptic periodic point of large period, say π (given by Lemma 7.5) depending on ε (depending on $\mathcal{V}$) and on $A_i := Df_{f^i(x)}$ for $i = 1, ..., \pi$. Define, for $t \in [0, 1]$, $B_{i,t} := R_{t\alpha_i} \cdot A_i$. By Lemma 7.5 we obtain that

$$B_1^\pi := B_{\pi,1} \cdot B_{\pi-1,1} \cdot ... \cdot B_{1,1}$$

has real eigenvalues. Since $B_0^\pi = A^\pi = Df_x^\pi$ has complex eigenvalues, there must be $t_0 \in]0, 1[$ such that $B_{t_0}^\pi$ has a parabolic behavior. Finally, we apply Lemma 4.1 several times, in order to realize an area-preserving map $f_1 \in \mathcal{V}$ exhibiting a parabolic periodic orbit. Since the existence of a parabolic point prevents structural stability and $f_1 \in \mathcal{W}$ we get a contradiction. Therefore $f \in \mathcal{A}^2_\omega$, which ends the proof.

□

Proof. (of Theorem D)

We observe that $\mathcal{F}^1_\omega(M)$ is C^1 open in $\text{Diff}^1_\omega(M)$. Let $f \in \mathcal{F}^1_\omega(M) \setminus \mathcal{A}^2_\omega$.

We recall Corollary 5.8 and we consider a C^1-neighborhood $\mathcal{V}$ of f in $\mathcal{F}^1_\omega(M)$ where any $g \in \mathcal{V}$ do not admit elliptic closed orbits. Then, from Corollary 5.8 there exist constants $\theta = \theta(\varepsilon, g)$, $m = m(\varepsilon, \theta)$ and $T = T(m)$ such that, for each periodic orbit with period greater than T, one has:

- m-dominated splitting and
- angle between its stable and unstable directions bounded from below by θ.

Observe that, since $g \in \mathcal{F}^1_\omega(M)$, these periodic orbits are hyperbolic.

We will get a contradiction with the fact that there exists a positive measure set without domination. For that we consider the following claim.

Claim 7.6. *For all $m \in \mathbb{N}$, there exists an f-invariant and positive Lebesgue measure set $\Gamma_m \subset M$ without m-dominated splitting.*

If the claim was false, then there would exist $m \in \mathbb{N}$ and $\Lambda_m \subset M$ such that $\text{Leb}(M \setminus \Lambda_m) = 0$ and Λ_m has an m-dominated splitting. Since the m-dominated splitting extends to the closure and we are considering the Lebesgue measure it follows that M has an m-dominated splitting. But the existence of an m-dominated splitting implies, by Lemma 2.5, that f is Anosov which contradicts our assumption.

Now, we recall the core of the dynamical principle involved in the proof of Theorem F; given any $\varepsilon > 0$, there exists (a sufficiently large) $m \in \mathbb{N}$ such that for any $\eta > 0$ arbitrarily close to 0, for a.e. $x \in \Gamma_m$ there exists g, ε-C^1-close to f, such that $e^{-n\eta} < \|Dg_x^n\| < e^{n\eta}$, for every arbitrarily large $n \in \mathbb{N}$.

Repeating the arguments in the proof of Theorem B we get a periodic point with period π for an area-preserving map $h \in \mathcal{V}$ and such that:

$$e^{-\delta\pi} < \|Dh_p^\pi\| < e^{\delta\pi}, \tag{7.1}$$

and in the same way we obtain a contradiction. Therefore, f has a dominated splitting over M and, by Lemma 2.5, we conclude that f is Anosov.

□

Proof. (of Corollary 7.4) Take an isolated point f in the interior of the boundary of $\mathcal{A}^2_\omega$ and a small neighborhood $\mathcal{V}$ of f such that any $g \in \mathcal{V}$ is Anosov. The diffeomorphism f must satisfy Claim 7.6 otherwise f is Anosov. We follow the proof of Theorem D and we conclude that under a small C^1-perturbation we find $g \in \mathcal{V}$ exhibiting an elliptic periodic orbit which is a contradiction. □

7.3. **Homoclinic tangencies.** For surface area-preserving diffeomorphisms the existence of smooth invariant curves is associated to the existence of elliptic points. Actually, Mora and Romero ([22]) developed a mechanism to create open sets containing a dense set of maps exhibiting homoclinic tangencies once one has a smooth invariant curve. A key step to prove this result is [22, Proposition 7]. To state this proposition let us define

$$\mathbb{A} = \{(\theta, r) \colon \theta \in \mathbb{S}^1,\ r \in \mathbb{R}\} \text{ and } \mathbb{A}_\delta = \{(\theta, r) \colon \theta \in \mathbb{S}^1,\ r \in]-\delta, \delta[\}.$$

Theorem 7.7. *Let $f \colon \mathbb{A}_\delta \to \mathbb{A}$ be a C^∞ area-preserving map of the annulus leaving invariant some C^∞ curve*

$$\Lambda = \{(\theta, \Phi(\theta)), \theta \in \mathbb{S}^1\},$$

where $\Phi\colon \mathbb{S}^1 \to \mathbb{R}$*, and such that* $f|_\Lambda$ *has an irrational rotation number. Then, for* $s \geq 1$ *and* $\varepsilon > 0$*,* f *can be* ε*-*C^s*-approximated by an area-preserving* g *exhibiting homoclinic tangencies such that for some* $\delta' < \delta$ *we have*

$$g|_{\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'}} = f|_{\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'}}.$$

Let $f_0 \in \mathrm{Diff}^1_\omega(M)$ be such that it cannot be C^1-approximated by a diffeomorphism in $\mathcal{A}^2_\omega$. Using Theorem A, we approximate, in the C^1-topology, f_0 by $f_1 \in \mathrm{Diff}^1_\omega(M)$ such that the elliptic points of f_1 are dense on the surface. Now, using Zehnder Theorem ([37]) and the stability of elliptic orbits, we approximate, in the C^1-topology, f_1 by $f_2 \in \mathrm{Diff}^\infty_\omega(M)$ having an elliptic point p of period π.

Now we consider the linear action $Df_2^\pi\colon T_pM \to T_pM$ defined by the rotation R_θ, in a small neighborhood of the orbit, and a direct application of Theorem 4.3 allows us to C^1-approximate f_2 by $f_3 \in \mathrm{Diff}^\infty_\omega(M)$ such that p is still an elliptic point of period π and there exists an f_3-invariant neighborhood $\mathfrak{T}$ where the first return map at p (not the tangent map) is a rotation of angle θ. We can assume that θ is irrational, otherwise, we could perturb f_3, by using Lemma 4.1, in order to get $f_4 \in \mathrm{Diff}^\infty_\omega(M)$, C^1-close to f_3, with the same properties but with irrational rotation angle. This area-preserving diffeomorphism is in the hypotheses of Theorem 7.7 and Theorem E is proved.

7.4. **Lots of Chaos or lack of it?** We recall one of the most common definitions of chaos due to Devaney (see [13, Definition 8.5]): $f\colon M \to M$ is *chaotic* if:

(a) f is transitive;
(b) the periodic points are dense in M and
(c) f is *sensitive to the initial conditions*, i.e., there exists $\delta > 0$ such that for all $x \in M$ and all neighborhood of x, V_x, there exists $y \in V_x$ and an integer n where $\mathrm{dist}(f^n(y), f^n(x)) > \delta$.

In this case we also say that f is *chaotic in the topological sense*.

It was proved in [6] that (a) and (b) implies (c), and so in order to be chaotic in the sense of Devaney the system only has to satisfy the transitivity property and the density of periodic points.

The other definition of chaotic map that we are going to use is the one that says that there are no zero Lyapunov exponents for Lebesgue almost every point. When, in our conservative surface setting, we have two non-zero (thus symmetric) Lyapunov exponents we say that f is *chaotic in the measurable sense*.

Theorem H. *Let M is any closed surface aside from the two-torus. There exists a C^1-residual $\mathcal{R} \subset Diff_{\omega}^{1}(M)$ such that, if $f \in \mathcal{R}$, then f is chaotic in the topological sense and nonchaotic in the measurable sense.*

Proof. As an outcome of [10] we obtain that there exists a residual subset $\mathcal{R}_1$ of $\mathrm{Diff}_{\omega}^{1}(M)$ such that if $f \in \mathcal{R}_1$, then f is transitive. Furthermore, by the general density theorem [30] we get there exists a residual subset $\mathcal{R}_2$ of $\mathrm{Diff}_{\omega}^{1}(M)$ such that if $f \in \mathcal{R}_2$, then the periodic points of f are dense in M. Therefore, defining $\mathcal{R}_3 = \mathcal{R}_1 \cap \mathcal{R}_2$ and recalling [6] we conclude that there exists a residual subset $\mathcal{R}_3$ of $\mathrm{Diff}_{\omega}^{1}(M)$ such that if $f \in \mathcal{R}_3$, then f is chaotic in the topological sense.

By Franks' classical result about the rigidity of Anosov diffeomorphisms (see [14]) we know that the only surfaces that support Anosov diffeomorphisms are the tori. Therefore, if M is any closed surface except the two-torus, then by Theorem F, there exists a C^1-residual subset $\mathcal{R}_4$ of $\mathrm{Diff}_{\omega}^{1}(M)$ such that, if $f \in \mathcal{R}_4$, then f has zero Lyapunov exponents for almost every points, thus is nonchaotic in the measurable sense.

Finally, $\mathcal{R} := \mathcal{R}_3 \cap \mathcal{R}_4$ is the residual set required by the statement of the theorem.

□

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[1] V. Araújo and M. Bessa, *Dominated splitting, singularities and zero volume for incompressible 3-flows*, Nonlinearity, **21**, (2008), 1637–1653.

[2] A. Arbieto and C. Matheus, A pasting lemma and some applications for conservative systems, *Ergodic Theory Dynam. Systems*, **27**, 5, (2007), 1399–1417.

[3] M.-C., Arnaud, The generic symplectic C^1-diffeomorphisms of four-dimensional symplectic manifolds are hyperbolic, partially hyperbolic or have a completely elliptic periodic point, *Ergodic Theory Dynam. Systems*, **22**, 6, (2002), 1621–1639.

[4] M.-C., Arnaud, Le "closing lemma" en topologie C^1, *Mm. Soc. Mat. Fr.* (N.S.) **74** (1998), vi+120pp.

[5] V. I. Arnold, " Mathematical methods of classical mechanics", Springer Verlag, 1978.

[6] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacy, On Devaney's definition of chaos. *Amer. Math. Montly*, **99**, (1992), 332–334.

[7] M. Bessa and P. Duarte, Abundance of elliptic dynamics on conservative three-flows, *Dynamical Systems - An international Journal*, **23**, 4, (2008), 409–424.

[8] J. Bochi, Genericity of zero Lyapunov exponents, *Ergodic Theory Dynam. Systems*, **22**,6, (2002), 1667–1696.

[9] J. Bochi and M. Viana, The Lyapunov exponents of generic volume-preserving and symplectic maps, *Ann. of Math. (2)*, **161**, 3, (2005), 1423–1485.

[10] C. Bonatti and S. Crovisier, Récurrence et généricité, *Invent. Math.*, **158**, 1, (2004), 33–104.

[11] C. Bonatti, L.J. Díaz J. and M. Viana, "Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective", Encycl. of Math. Sc. **102**. Math. Phys. 3. Springer-Verlag, 2005.

[12] B. Dacorogna and J. Moser, On a partial differential equation involving the Jacobian determinant, *Ann. Inst. Henri Poincaré*, **7**, 1, (1990), 1–26.

[13] R. Devaney, "An introduction to Chaotic Dynamical Systems". Addison-Wesley, 1989.

[14] J. Franks, Anosov diffeomorphisms, *Global Analysis* (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) 61–93, Amer. Math. Soc., Providence, R.I. 1070.

[15] K. Kuratowski, "Topology, vol. **1**," Academic Press, 1966.

[16] S. D. Liao, On the stability conjecture, *Chinese Ann. Math.*, **1**, (1980), 9–30.

[17] R. Mañé, Persistent manifolds are normally hyperbolic, *Bull. Amer. Math. Soc.*, **80**, (1974), 90–91.

[18] R. Mañé, An ergodic closing lemma, *Annals of Math.*, **116**, (1982), 503–540.

[19] R. Mañé, Oseledec's theorem from the generic viewpoint, In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 1269–1276, Warsaw, 1984. PWN.

[20] R. Mañé, The Lyapunov exponents of generic area preserving diffeomorphisms, In *International Conference on Dynamical Systems (Montevideo, 1995)*, volume 362 of *Pitman Res. Notes Math. Ser.*, pages 110–119. Longman, Harlow, 1996.

[21] R. Mañé, "Ergodic theory and differentiable dynamics", Springer-Verlag, Berlin, 1987.

[22] L. Mora and N. Romero, Persistence of homoclinic tangencies for area-preserving maps, *Ann. Fac. Sci. Toulouse Math.*, **6**, 4, (1997), 711–725.

[23] S. Newhouse, Quasi-elliptic periodic points in conservative dynamical systems, *Am. J. Math.*, **99**, (1977), 1061–1087.

[24] V. I. Oseledets, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, *Trans. Moscow Math. Soc.*, **19**, (1968), 197–231.

[25] J. Palis and F. Takens, "Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors",. Cambridge Studies in Advanced Mathematics, 35. Cambridge University Press, Cambridge, 1993.

[26] J. Palis and S. Smale, Structural stability theorems,. 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) pp. 223–231 Amer. Math. Soc., Providence, R.I.

[27] J. Palis, *Open questions leading to a global perspective in dynamics*, Nonlinearity, **21**, (2008), 37–43.

[28] V. A. Pliss, On a conjecture of Smale, *Differ. Uravnenija*, **8**, (1972), 268–282.

[29] M. Pollicott, “Lectures on Ergodic Theory and Pesin Theory on compact manifolds”, London Mathematical Society Lecture Notes Series, 180, Cambridge 1993.

[30] C. Pugh and C. Robinson, The C^1 closing lemma, including Hamiltonians, *Ergodic Theory Dynam. Systems*, **3**, (1983), 261–313.

[31] C. Robinson, Generic properties of conservative systems, *Am. J. Math.*, **92**, (1970), 562–603.

[32] C. Robinson, Dynamical systems. Stability, symbolic dynamics, and chaos, (Second edition). *Stud. Adv. Math.. CRC Press, Boca Raton, FL*, 1999.

[33] R. Saghin and Z. Xia, Partial Hyperbolicity or dense elliptical periodic points for C^1-generic symplectic diffeomorphisms, *Trans. A.M.S.*, **358**, (2006), 5119–5138.

[34] M. Shub, “Global stability of dynamical systems”, Springer-Verlag, New York, 1987.

[35] F. Takens, Homoclinic points in conservative systems, *Invent. Math.*, **18**, (1972), 267–292.

[36] J. C. Yoccoz, Travaux de Herman sur les Tores invariants, *Astérisque*, 206 Exp: 754, 4 , (1992), 311–344.

[37] E. Zehnder, Note on smoothing symplectic and volume-preserving diffeomorphisms, In *Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976*, volume 597 *Lecture Notes in Math.*, 828–854. Springer, Berlin, 1977.

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