A Voronoi type summation formula involving $\sigma_{i\tau}(n)$ and index transforms

Semyon YAKUBOVICH

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Abstract

By using the theory of the Riemann zeta-function, Mellin's and index-convolution transforms we prove a summation formula involving sums of the form $\sum \sigma_{i\tau}(n)n^{-i\tau/2}f(n)$, where $\sigma_\mu(n)$ is the sum of the μ-th powers of the divisors of n. Here $i\tau,\ \tau \in \mathbb{R}$ is a pure imaginary complex number. When $\tau = 0$ it drives to the well- known Voronoi's formula. Some examples of summation formulas like Voronoi's, Koshliakov's, Ramanujan's, Guinand's, Nasim's are demonstrated. Finally, a relationship between the Kontorovich-Lebedev and Tichmarsh transforms is obtained.

Keywords: *Voronoi summation formula, Mellin transform, Riemann zeta-function, Kontorovich-Lebedev transform, Titchmarsh transform, Index transforms, Bessel functions, Lommel functions, Arithmetic functions, Whittaker functions, confluent hypergeometric functions, Gauss hypergeometric function*

AMS subject classification: 44A15, 33C05, 33C10, 33C15, 11M06, 11M36, 11N 37

1 Introduction and auxiliary results

Let $i\tau,\ \tau \in \mathbb{R}$ be a pure imaginary number. As usual, by $\sigma_{i\tau}(n)$ we denote the arithmetic function [9], which is the sum of $i\tau$-th powers of the divisors of $n \in \mathbb{N}$. When $\tau = 0$ it becomes the divisor function $d(n)$. Nasim in [4], [5], [6] proved several summation formulas of the Voronoi type involving these arithmetic functions and Fourier-Watson transforms. Some of them were recently generalized by the author (see in [15], [16]) with the use of the Kontorovich-Lebedev transform [8], [13], [14]. In this paper we will prove a summation formula of the Voronoi type involving $\sigma_{i\tau}(n)$, the so-called index-convolution Kontorovich-Lebedev transform [10], [11] and index-convolution transform of Titchmarsh type with a combination of Bessel functions [11], [12]. When $\tau \to 0$ it will drive us to the Voronoi-Nasim formula involving the divisor function $d(n)$. Finally we will give some examples of new summation formulas with arithmetic and special functions of the hypergeometric type like Koshliakov's, Ramanujan's, Guinand's, Nasim's (cf. [1], [2]), appealing to the basic table of the Mellin transform in [7, Vol. 3].

As it is known [8], [13], the Mellin transform in $L_2(\mathbb{R}_+)$ is defined by the integral

$$f^*(s) = \text{l.i.m.}_{N\to\infty} \int_{1/N}^{N} f(x)x^{s-1}dx, \tag{1.1}$$

where s belongs to the critical line $\left\{s \in \mathbb{C} : \text{Re } s = \frac{1}{2}\right\}$ and the convergence of the integral is in the mean square with respect to the norm of the space $L_2(1/2 - i\infty, 1/2 + i\infty)$. Moreover, the inversion formula takes place

$$f(x) = \text{l.i.m.}_{N\to\infty} \frac{1}{2\pi i} \int_{1/2-iN}^{1/2+iN} f^*(s)x^{-s}ds, \tag{1.2}$$