Uncertainty principles for the Kontorovich-Lebedev transform

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Abstract

By using classical uncertainty principles for the Fourier transform and composition properties of the Kontorovich-Lebedev transform, analogs of the Hardy, Beurling, Cowling-Price, Gelfand-Shilov and Donoho-Stark theorems are obtained.

Keywords: *Kontorovich-Lebedev transform, Fourier transform, Laplace transform, modified Bessel functions, Hardy theorem, Cowling-Price theorem, Beurling theorem, Gelfand-Shilov theorem, Donoho-Stark theorem, uncertainty principle*

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1 Introduction

The Kontorovich-Lebedev transformation is defined as follows (cf. [8], [12], [15], [16], [17])

$$g(x) = \int_0^\infty K_{ix}(y) f(y) dy, \ x > 0. \tag{1.1}$$

Here $K_\mu(z)$ is the modified Bessel function [3], which is an independent solution of the differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2 + \mu^2)u = 0.$$

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When $\mu = ix,\ x \in \mathbb{R}, z = y > 0$ it is real valued and even with respect to x. If $f \in L_2(\mathbb{R}_+; ydy)$ then $g \in L_2(\mathbb{R}_+; x\sinh \pi x dx)$ (see [16], [17]) and the Parseval formula holds

$$\int_0^\infty x \sinh \pi x |g(x)|^2 dx = \frac{\pi^2}{2} \int_0^\infty |f(y)|^2 y dy. \tag{1.2}$$

In this case integral (1.1) converges in the mean square sense and can be written making necessary truncations at zero and infinity. Moreover, the inverse transform has the form

$$yf(y) = \frac{2}{\pi^2} \int_0^\infty x \sinh \pi x K_{ix}(y) g(x) dx, \tag{1.3}$$

where integral (1.3) is in the mean square sense with the necessary truncation at infinity.

On the other hand, if $f \in L_1(\mathbb{R}_+; K_0(y)dy)$, where $K_0(y)$ is the modified Bessel function of the index zero then inversion formula (1.3) can be interpreted at each Lebesgue point of f (see in [15]) as

$$yf(y) = \frac{4}{\pi^2} \lim_{\alpha \to \frac{\pi}{2}-} \int_0^\infty x \sinh \alpha x \cosh \frac{\pi x}{2} K_{ix}(y) g(x) dx. \tag{1.4}$$

If also $g \in L_1(\mathbb{R}_+; x \cosh \frac{\pi x}{2} dx)$ one can pass to the limit in (1.4) under the integral sign and we get (1.3) in Lebesgue integrable sense.

The modified Bessel function has the following asymptotic behaviour

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.5}$$

and near the origin

$$z^{|\mathrm{Re}\mu|} K_\mu(z) = 2^{\mu-1}\Gamma(\mu) + o(1),\ z \to 0,\ \mu \neq 0, \tag{1.6}$$

$$K_0(z) = -\log z + O(1),\ z \to 0. \tag{1.7}$$

Meanwhile, when x is restricted to any compact subset of $\mathbf{R}_+$ and τ tends to infinity we have the following asymptotic [16, p. 20]

$$K_{i\tau}(x) = \left(\frac{2\pi}{\tau}\right)^{1/2} e^{-\pi\tau/2} \sin\left(\frac{\pi}{4} + \tau \log \frac{2\tau}{x} - \tau\right) [1 + O(1/\tau)], \qquad \tau \to \infty. \tag{1.8}$$

The modified Bessel function can be represented by the integrals of the Fourier and Mellin types [3], [10], Vol. I, [12], [15], [16]

$$K_\mu(x) = \int_0^\infty e^{-x\cosh u} \cosh \mu u \; du, \tag{1.9}$$

$$K_\mu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\mu \int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-\mu-1} dt, \tag{1.10}$$

$$\sinh\frac{\pi\tau}{2} K_{i\tau}(x) = \int_0^\infty \sin(x\sinh u)\sin\tau u du, \tag{1.11}$$

$$\cosh\frac{\pi\tau}{2} K_{i\tau}(x) = \int_0^\infty \cos(x\sinh u)\cos\tau u du. \tag{1.12}$$

The main aim of the paper is to establish the so-called uncertainty principles for the operator (1.1), which say that a nonzero original and its image under transformation (1.1) cannot be simultaneously too small in the pointwise or integrable decay. This comes as a generalization of the classical Heisenberg uncertainty principle. It was extended to the Fourier transform in [1], [4], [5], [7]. The corresponding principles have been proved also for the Y-transform [9], the Dunkl transform [11] and recently for the Hankel transform [14].

The structure of the paper is as follows: in Section 2 we will prove Hardy's type theorem for the Kontorovich-Lebedev transformation, which will drives us at the Hardy uncertainty principle. Section 3 of the paper will be devoted to the Beurling, Cowling-Price and Gelfand-Shilov theorems. Finally in Section 4 we will prove the Donoho-Stark theorem.

2 Hardy uncertainty principle

Hardy's classical theorem for the Fourier transform [5], [13] says that if $|f(y)| \le Ce^{-ay^2}$ and $|(F_cf)(x)| \le Ce^{-\frac{x^2}{4a}}$, $a > 0$, then $f(y)$ is a multiple of e^{-ay^2}. Here $C > 0$ is a universal constant, which is different in distinct places and

$$(F_cf)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y)\cos xy dy, \tag{2.1}$$

is the cosine Fourier transform.

Let us suppose that transformation (1.1) admits the following series expansion with respect to an index of the modified Bessel functions

$$g(x) = \frac{C}{\cosh(\pi x/2)} \sum_{n=0}^\infty \alpha_n \left[K_{i(\frac{x}{2}+n)}\left(\frac{a}{2}\right) + K_{i(\frac{x}{2}-n)}\left(\frac{a}{2}\right)\right], \ a > 0, \tag{2.2}$$

where $\sum_{n=0}^\infty |\alpha_n| < \infty$. We have

Theorem 1. *Let $g(x)$ satisfy (2.2) and $|f(y)| \le Ce^{-\frac{y^2}{4a}}$. Then $f(y)$ is a multiple of $e^{-\frac{y^2}{4a}}$.*

Proof. Taking (1.9) we find

$$K_{i(\frac{x}{2}+n)}\left(\frac{a}{2}\right) = \int_0^\infty e^{-\frac{a}{2}\cosh u} \cos\left(\frac{x}{2}+n\right) u \ du. \tag{2.3}$$

Hence $\left|K_{i(\frac{x}{2}+n)}\left(\frac{a}{2}\right)\right| \le K_0\left(\frac{a}{2}\right)$ and clearly series (2.2) is uniformly convergent on $\mathbb{R}_+$. Moreover, one can calculate the cosine Fourier transform of the function $\cosh(\pi x/2)g(x)$ by changing the order of integration and summation. Indeed, invoking (2.3) we obtain

$$F_c(\cosh(\pi t/2)g(t))(x) = C\sum_{n=0}^\infty \alpha_n \int_0^\infty \left[K_{i(\frac{x}{2}+n)}\left(\frac{a}{2}\right) + K_{i(\frac{x}{2}-n)}\left(\frac{a}{2}\right)\right] \cos(xt)dt$$

$$= C\sum_{n=0}^\infty \alpha_n \int_{-\infty}^\infty K_{i(\frac{t}{2}-n)}\left(\frac{a}{2}\right) e^{ixt}dt = Ce^{-\frac{a}{2}\cosh 2x}\sum_{n=0}^\infty \alpha_n e^{2ixn}. \tag{2.4}$$

Therefore $|F_c(\cosh(\pi t/2)g(t))(x)| \le Ce^{-\frac{a}{2}\cosh 2x} = O\left(e^{-a\sinh^2 x}\right)$. Further, it is easily seen under conditions of the theorem and asymptotic behaviour of the modified Bessel function (1.5), (1.6), (1.7) that $f \in L_1(\mathbb{R}_+; K_0(y)dy)$. Moreover, by virtue of the asymptotic formula with respect to an index (1.8) we verify that $g \in L_1(\mathbb{R}_+; x\cosh\frac{\pi x}{2}dx)$. Consequently, calling (1.3), (1.4) we arrive at the representation

$$yf(y) = \frac{4}{\pi^2}\int_0^\infty x\sinh\frac{\pi x}{2}\cosh\frac{\pi x}{2} K_{ix}(y)g(x)dx. \tag{2.5}$$

However, since $\sinh\frac{\pi x}{2}K_{ix}(y)$ is bounded for any $y > 0$ (see (1.8)) we take the representation (1.11) and substitute it in (2.5). As a result we obtain

$$yf(y) = \frac{4}{\pi^2}\lim_{N\to\infty}\int_0^N x\cosh\frac{\pi x}{2}g(x)\int_0^\infty \sin(y\sinh u)\sin xududx$$

$$= \frac{4}{\pi^2}\lim_{N\to\infty}\int_0^N x\cosh\frac{\pi x}{2}g(x)\int_0^\infty \sin(yv)\sin(x\log(v+\sqrt{v^2+1}))\frac{dvdx}{\sqrt{v^2+1}}.$$

Via Abel's test we observe that the latter integral is uniformly convergent with respect to $x \in [0, N]$. Thus inverting the order of integration we come out with

$$yf(y) = \frac{4}{\pi^2}\lim_{N\to\infty}\int_0^\infty \frac{\sin(yv)}{\sqrt{v^2+1}}\int_0^N x\cosh\frac{\pi x}{2}g(x)\sin(x\log(v+\sqrt{v^2+1}))dxdv. \tag{2.6}$$

Moreover, the integrability condition $g \in L_1(\mathbb{R}_+; x\cosh\frac{\pi x}{2}dx)$ and the Abel test allow us to pass to the limit under the integra sign in (2.6). Hence returning to the old variables we get

$$yf(y) = \frac{4}{\pi^2}\int_0^\infty \sin(y\sinh u)\int_0^\infty x\cosh\frac{\pi x}{2}g(x)\sin(ux)dxdu$$

$$= -\frac{4}{\pi^2}\int_0^\infty \sin(y\sinh u)\frac{d}{du}\int_0^\infty \cosh\frac{\pi x}{2} g(x)\cos(ux)dxdu. \quad (2.7)$$

We note that the differentiation under the integral sign in (2.7) is motivated by the uniform convergence by $u \in \mathbb{R}_+$ of the latter integral with respect to x. Hence integrating by parts in (2.7) and eliminating the outer terms owing to the Riemann-Lebesgue lemma we take into account (2.1) to derive the representation

$$f(y) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}}\int_0^\infty \cos(y\sinh u)\cosh u F_c(\cosh(\pi t/2)g(t))(u)du. \quad (2.8)$$

Hence invoking the above estimates and the value of an elementary integral we find for any complex variable $z,\ |z| = r$,

$$|f(z)| < C\int_0^\infty \cosh(r\sinh u)\cosh u|F_c(\cosh(\pi t/2)g(t))(u)|du$$

$$\leq C\int_0^\infty \cosh(r\sinh u)\cosh u e^{-a\sinh^2 u}du = C\int_0^\infty \cosh(rt)e^{-at^2}dt = Ce^{\frac{r^2}{4a}}.$$

Thus $f(\sqrt{z})$ is an entire function, which is $O(e^{\frac{|z|}{4a}})$ for all $z \in \mathbb{C}$ and $f(\sqrt{y}) = O(e^{-\frac{y}{4a}}), y \in \mathbb{R}_+$. Therefore according to [13], Theorem 128 $f(y) = Ce^{-\frac{y^2}{4a}}$. Theorem 1 is proved.

Corollary 1. *Under conditions of Theorem 1*

$$g(x) = C\ \mathrm{sech}(\pi x/2)K_{ix/2}\left(\frac{a}{2}\right) = O(e^{-\frac{3\pi}{4}x}), \quad x \to +\infty.$$

Proof. Indeed, substituting the value $f(y) = Ce^{-\frac{y^2}{4a}}$ into (1.1) we just call the relation 2.16.8.3 from [10], Vol. 2 to get the result. The required asymptotic behavior at infinity immediately follows from (1.8). Corollary 1 is proved.

Remark 1. As we see $g(x)$ from the corollary admits the representation (2.2) with $\alpha_0 \neq 0,\ \alpha_n = 0,\ n = 1, 2\dots$.

As a consequence we are ready to state an analog of the Hardy uncertainty principle for the Kontorovich-Lebedev transformation (1.1).

Corollary 2. *Let* $|f(y)| \leq Ce^{-by^2},\ b > \frac{1}{4a}$. *Then* $f(y) = 0$.

This principle can be formulated in terms of the composition $F_c(\cosh(\pi t/2)g(t))$. Precisely, we have

Corollary 3. *One cannot have both* $|f(y)| \leq Ce^{-ay^2},\ a > 0$ *and* $|F_c(\cosh(\pi t/2)g(t))(x)| \leq Ce^{-b\sinh^2 x},\ b > 0$, *where* $ab > \frac{1}{4}$ *and g is the Kontorovich-Lebedev transform* (1.1) *unless* $f(y) = 0$.

As a consequence of Theorem 1 and Corollary 1 we get

Corollary 4. *Let* $|f(y)| \leq Ce^{-ay^2},\ a > 0$ *and* $|F_c(\cosh(\pi t/2)g(t))(x)| \leq Ce^{-b\sinh^2 x},\ b > 0$, *where* $0 < ab \leq \frac{1}{4}$. *If* $|g(x)| \leq Ce^{-cx},\ x > 0, c > \frac{3\pi}{4}$, *then* $f(y) = 0$.

3 Beurling, Cowling-Price and Gelfand-Shilov theorems

The Beurling condition related to the cosine Fourier transform (2.1) says (cf. [7]), that if

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y)(F_c f)(x)| e^{xy} dx dy < \infty, \tag{3.1}$$

then $f = 0$.

Here we will prove an analog of the Beurling theorem for the Kontorovich-Lebedev transformation (1.1).

Theorem 2. *Let*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y)g(x)| K_x(y) dx dy < \infty. \tag{3.2}$$

Then $f = 0$.

Proof. In fact, representation (1.9) for the modified Bessel function yields the inequality $K_x(y) > K_0(y)$. Consequently, condition (3.2) implies

$$\infty > \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y)g(x)| K_x(y) dx dy \geq \int_{\mathbb{R}_+} |f(y)| K_0(y) dy \int_{\mathbb{R}_+} |g(x)| dx.$$

Therefore, $f \in L_1(\mathbb{R}_+; K_0(y)dy),\ g(x) \in L_1(\mathbb{R}_+; dx)$. The latter condition guarantees the existence of the cosine Fourier transform of g. We will show that

$$(F_c g)(\lambda) = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-y\cosh\lambda} f(y) dy. \tag{3.3}$$

Indeed, denoting by $h(\lambda)$ the right-hand side of (3.3) we find

$$\int_{\mathbb{R}_+} |h(\lambda)|\, d\lambda \leq \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} e^{-y\cosh\lambda} |f(y)| dy d\lambda = \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}_+} |f(y)| K_0(y) dy < \infty.$$

So $h \in L_1(\mathbb{R}_+; d\lambda)$ and $(F_c h)(x)$ can be now easily calculated by using (1.9) and Fubini's theorem. Thus we obtain

$$(F_c h)(x) = \int_0^\infty \cos x\lambda \int_0^\infty e^{-y\cosh\lambda} f(y) dy d\lambda = \int_0^\infty K_{ix}(y) f(y) dy = g(x).$$

Since $g \in L_1(\mathbb{R}_+; dx)$ the inversion theorem for the cosine Fourier transform gives $(F_c g)(\lambda) = h(\lambda)$ and we establish equality (3.3).

Let us verify the Beurling condition (3.1) for $g, F_c g$. We have

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |g(x)(F_c g)(\lambda)| e^{x\lambda} dx d\lambda < \sqrt{2\pi} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |g(x)| \cosh x\lambda$$

$$\times \int_0^\infty e^{-y\cosh\lambda}|f(y)|dydxd\lambda = \sqrt{2\pi}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}|f(y)g(x)|K_x(y)dxdy < \infty.$$

Thus $g = 0$. Combining with (3.3) the latter condition yields

$$\int_0^\infty e^{-y\cosh\lambda} f(y)dy = 0,\ \lambda \in \mathbb{R}_+ \tag{3.4}$$

for any $f \in L_1(\mathbb{R}_+; K_0(y)dy)$. We will show that in this case $f = 0$. In fact, choosing any $\lambda_0 > 1$ we treat the left-hand side of equality (3.4) as the Laplace integral $(Lf)(\cosh\lambda)$, where

$$(Lf)(z) = \int_0^\infty e^{-yz} f(y)dy,$$

which is zero via (3.4) at least at the countable set of points satisfying the condition $\cosh\lambda_n = \lambda_0 + jn,\ j > 0, n = 0, 1, 2, \ldots$. Moreover, since (see (1.5), (1.7))

$$\int_0^\infty e^{-y\cosh\lambda_n}|f(y)|dy < \infty,\ n = 0, 1, 2, \ldots\ ,$$

then by virtue of [2], Chapter I we get that $f(y) = 0$ almost for all $y \in \mathbb{R}_+$, i.e. $f = 0$ in the Lebesgue sense.

Theorem 2 is proved.

Let us prove an analog of the Gelfand-Shilov uncertainty principle for the transformation (1.1). Indeed, it was shown in [4] that if

$$\int_{\mathbb{R}_+}|f(y)|e^{(ay)^p/p}dy < \infty, \quad \int_{\mathbb{R}_+}|(F_cf)(x)|e^{(bx)^q/q}dx < \infty, \tag{3.5}$$

with $1 < p, q < \infty,\ p^{-1} + q^{-1} = 1$ and $ab > 1/4$, then $f = 0$.

We have accordingly

Theorem 3. *Let $1 < p, q < \infty,\ p^{-1} + q^{-1} = 1,\ [q]$ be an integer part of q and*

$$\int_{\mathbb{R}_+}|f(y^2)|e^{\frac{(2([q]+1))!}{4y^2}}dy < \infty, \quad \int_{\mathbb{R}_+}|g(x)|e^{x^p/p}dx < \infty. \tag{3.6}$$

Then $f = 0$.

Proof. By using the Young inequality $xy \le \frac{x^p}{p} + \frac{y^q}{q}$ and representation (1.9) for the modified Bessel function we derive

$$K_x(y) = \int_0^\infty e^{-y\cosh u}\cosh xu\ du \le \int_0^\infty e^{-y\cosh u + xu}\ du$$

$$\le e^{x^p/p}\int_0^\infty e^{-y\cosh u + \frac{u^q}{q}}\ du = e^{x^p/p}\left(\int_0^1 + \int_1^\infty\right) e^{-y\cosh u + \frac{u^q}{q}}\ du. \tag{3.7}$$

Meanwhile,

$$\int_0^1 e^{-y\cosh u+\frac{u^q}{q}}du < e\int_0^1 e^{-y\cosh u}du < eK_0(y),$$

$$\int_1^\infty e^{-y\cosh u+\frac{u^q}{q}}du < ([q]+1)\int_1^\infty e^{-y\cosh u+u^{[q]+1}}u^{[q]}du.$$

Therefore an elementary inequality $\cosh u > \frac{u^{2([q]+1)}}{(2([q]+1))!}$ gives the following estimation of the latter integral

$$\int_1^\infty e^{-y\cosh u+\frac{u^q}{q}}du < ([q]+1)\int_1^\infty e^{-y\cosh u+u^{[q]+1}}u^{[q]}du$$

$$< ([q]+1)\int_1^\infty e^{-\frac{yu^{2([q]+1)}}{(2([q]+1))!}+u^{[q]+1}}u^{[q]}du = \int_1^\infty e^{-\frac{yv^2}{(2([q]+1))!}+v}dv$$

$$< \frac{C}{\sqrt{y}}e^{(2([q]+1))!/(4y)}.$$

Combining with (3.7) and taking into account the asymptotic formulas (1.5), (1.7) we obtain the estimate

$$e^{-x^p/p}K_x(y) < eK_0(y) + \frac{C}{\sqrt{y}}e^{(2([q]+1))!/(4y)} < \frac{C}{\sqrt{y}}e^{(2([q]+1))!/(4y)}.$$

Consequently, with conditions (3.2), (3.6) it yields

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}|f(y)g(x)|K_x(y)dxdy < C\int_{\mathbb{R}_+}|g(x)|e^{x^p/p}dx\int_{\mathbb{R}_+}|f(y)|e^{(2([q]+1))!/(4y)}\frac{dy}{\sqrt{y}}$$

$$= C\int_{\mathbb{R}_+}|g(x)|e^{x^p/p}dx\int_{\mathbb{R}_+}|f(y^2)|e^{\frac{(2([q]+1))!}{4y^2}}dy < \infty.$$

Applying Theorem 2 we get the result. Theorem 3 is proved.

Finally in this section we establish the Cowling -Price theorem for the Kontorovich-Lebedev transform (1.1). This will be an analog of the following result for the Fourier transform (2.1) (cf. [1]): if $1 \le p, q < \infty$ and

$$\left|\left|e^{ax^2}f(x)\right|\right|_{L_p(\mathbb{R}_+)} + \left|\left|e^{b\lambda^2}(F_cf)(\lambda)\right|\right|_{L_q(\mathbb{R}_+)} < \infty$$

with $ab > 1/4$, then $f = 0$.

We have

Theorem 4. *If*

$$\left|\left|e^{ax^2}g(x)\right|\right|_{L_p(\mathbb{R}_+)} < \infty, \quad \left|\left|e^{6b^2/y^2}f(y^2)\right|\right|_{L_1(\mathbb{R}_+)} < \infty, \tag{3.8}$$

where $p \in [1, \infty)$ *and* $ab > 1/4$, *then* $f = 0$.

Proof. Indeed, choosing a_0, b_0 such that $0 < a_0 < a$ $0 < b_0 < b$, $a_0 b_0 > 1/4$ we easily find that $a_0 x^2 + b_0 y^2 \geq 2\sqrt{a_0 b_0} xy \geq xy$. Furthermore, with the Hölder inequality it gives

$$\int_{\mathbb{R}_+} |g(x)| e^{a_0 x^2} dx \leq \left|\left| e^{ax^2} g(x) \right|\right|_{L_p(\mathbb{R}_+)} \left|\left| e^{-(a-a_0)x^2} \right|\right|_{L_{p'}(\mathbb{R}_+)} < \infty,$$

where p' is the conjugate exponent ($p^{-1} + p'^{-1} = 1$). Taking (3.2) we deduce similar to (3.7)

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |f(y) g(x)| K_x(y) dx dy < \int_{\mathbb{R}_+} |g(x)| e^{a_0 x^2} dx \int_{\mathbb{R}_+} |f(y)| \int_0^\infty e^{-y \cosh u + b_0 u^2} du dy$$

$$< C \left|\left| e^{ax^2} g(x) \right|\right|_{L_p(\mathbb{R}_+)} \int_{\mathbb{R}_+} |f(y)| \left(\int_0^1 + \int_1^\infty \right) e^{-y \cosh u + b_0 u^2} du dy.$$

But

$$\left(\int_0^1 + \int_1^\infty \right) e^{-y \cosh u + b_0 u^2} du < C K_0(y) + 2 \int_1^\infty e^{-y \frac{u^4}{4!} + bu^2} u du$$

$$= C K_0(y) + \int_1^\infty e^{-y \frac{v^2}{4!} + bv} dv < C \frac{e^{6b^2/y}}{\sqrt{y}}.$$

Hence

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |f(y) g(x)| K_x(y) dx dy < C \left|\left| e^{ax^2} g(x) \right|\right|_{L_p(\mathbb{R}_+)} \int_{\mathbb{R}_+} |f(y)| \frac{e^{6b^2/y}}{\sqrt{y}} dy$$

$$= C \left|\left| e^{ax^2} g(x) \right|\right|_{L_p(\mathbb{R}_+)} \left|\left| e^{6b^2/y^2} f(y^2) \right|\right|_{L_1(\mathbb{R}_+)} < \infty.$$

Thus Theorem 2 gives the result. Theorem 4 is proved.

4 Donoho-Stark theorem

As it is known in [17], when $f \in L_2(\mathbb{R}_+; y dy)$, then $g \in L_2(\mathbb{R}_+; x \sinh \pi x dx)$ and vice versa. Moreover, by virtue of (1.2) $||g||_{L_2(\mathbb{R}_+; x \sinh \pi x dx)} = \frac{\pi}{\sqrt{2}} ||f||_{L_2(\mathbb{R}_+; y dy)}$ and the Kontorovich-Lebedev integrals (1.1), (1.3) can be interpreted accordingly in the mean convergence sense with respect to the related norm

$$g(x) = \text{l.i.m.}_{N \to \infty} \int_{1/N}^N K_{ix}(y) f(y) dy, \tag{4.1}$$

$$f(y) = \frac{2}{\pi^2}\text{l.i.m.}_{N\to\infty}\int_0^N x\sinh\pi x\frac{K_{ix}(y)}{y}g(x)dx. \tag{4.2}$$

Let $\mathbb{X} = [0, X]$, $\mathbb{Y} = [1/Y, Y]$ the Lebesgue measurable sets and $|\mathbb{X}|$, $|\mathbb{Y}|$ be their Lebesgue measures. Denoting by $P_{\mathbb{X}}$ the operator

$$(P_{\mathbb{X}}g)(x) = \begin{cases} g(x), & \text{if } x \in \mathbb{X}, \\ 0, & \text{if } x \notin \mathbb{X}, \end{cases}$$

we have

$$||g - P_{\mathbb{X}}g||_{L_2(\mathbb{R}_+; x\sinh\pi x dx)} \le \varepsilon_{\mathbb{X}},$$

and this means that g is $\varepsilon_{\mathbb{X}}$-concentrated on the set $\mathbb{X}$. Plainly $||P_{\mathbb{X}}|| = 1$. Another auxiliary operator is given by the formula

$$(Q_{\mathbb{Y}}g)(x) = \int_{\mathbb{Y}} K_{ix}(y)f(y)dy,$$

where f is the reciprocal inverse Kontorovich-Lebedev transform (4.2). If $h = Q_{\mathbb{Y}}g$ the transform (4.2) $\hat{h}(y)$ is equal to

$$\hat{h}(y) = \begin{cases} f(y), & \text{if } y \in \mathbb{Y}, \\ 0, & \text{if } y \notin \mathbb{Y}. \end{cases}$$

Meanwhile by Parseval's equality (1.2) we find

$$\left|\left|f - \hat{h}\right|\right|_{L_2(\mathbb{R}_+; ydy)} = \frac{\sqrt{2}}{\pi}||g - Q_{\mathbb{Y}}g||_{L_2(\mathbb{R}_+; x\sinh\pi x dx)}, \tag{4.3}$$

and f is ε-concentrated on $\mathbb{Y}$ if, and only if, $||g - Q_{\mathbb{Y}}g||_{L_2(\mathbb{R}_+; x\sinh\pi x dx)} \le \varepsilon$. Moreover, one can show that $||Q_{\mathbb{Y}}|| = 1$.

Now we are ready to prove the following analog of the Donoho-Stark uncertainty principle (cf. [6])

Theorem 5. *Let g is $\varepsilon_{\mathbb{X}}$-concentrated on $\mathbb{X} = [0, X]$ and its Kontorovich-Lebedev reciprocity f is $\varepsilon_{\mathbb{Y}}$-concentrated on $\mathbb{Y} = [1/Y, Y]$. Then*

$$|\mathbb{X}|^{3/2}\ |\mathbb{Y}| \ge \frac{\pi^{7/4}\sqrt{24}}{\Gamma^2(1/4)}(1 - (\varepsilon_{\mathbb{X}}^2 + \varepsilon_{\mathbb{Y}}^2)^{1/2})^2. \tag{4.4}$$

Proof. Without loss of generality one can suppose that $Y > 1$. Since g is $\varepsilon_{\mathbb{X}}$-concentrated on $\mathbb{X}$ integral (1.3) exists as a Lebesgue integral and is uniformly convergent with respect to $y \in \mathbb{Y}$. Hence we calculate the following composition of operators $(P_{\mathbb{X}}Q_{\mathbb{Y}}g)(x)$. Indeed, we derive

$$(P_{\mathbb{X}}Q_{\mathbb{Y}}g)(x) = \frac{2}{\pi^2}P_{\mathbb{X}}\int_{\mathbb{Y}}\frac{K_{ix}(y)}{y}\int_0^\infty t\sinh\pi t K_{it}(y)g(t)dtdy$$

$$= \frac{2}{\pi^2} P_{\mathbb{X}} \int_0^\infty t \sinh \pi t g(t) \int_{\mathbb{Y}} K_{ix}(y) K_{it}(y) \frac{dy}{y} dt$$
$$= \int_0^\infty \mathcal{K}(x,t) g(t) dt,$$

where

$$\mathcal{K}(x,t) = \begin{cases} \frac{2}{\pi^2} t \sinh \pi t \int_{\mathbb{Y}} K_{ix}(y) K_{it}(y) \frac{dy}{y}, & \text{if } x < X, \\ 0, & \text{if } x \geq X. \end{cases}$$

Further,

$$||P_{\mathbb{X}} Q_{\mathbb{Y}} g||_{L_2(\mathbb{R}_+; x \sinh \pi x dx)} \leq ||P_{\mathbb{X}} Q_{\mathbb{Y}}|| \ ||g||_{L_2(\mathbb{R}_+; x \sinh \pi x dx)}$$

and the norm of composition $P_{\mathbb{X}} Q_{\mathbb{Y}}$ does not exceed its Hilbert-Schmidt norm, which is equal to

$$\left(\int_0^\infty \int_0^\infty |\mathcal{K}(x,t)|^2 \frac{x \sinh \pi x}{t \sinh \pi t} dt dx \right)^{1/2}.$$

Therefore,

$$||P_{\mathbb{X}} Q_{\mathbb{Y}}||^2_{L_2(\mathbb{R}_+; x \sinh \pi x dx)} \leq \int_0^\infty \int_0^\infty |\mathcal{K}(x,t)|^2 \frac{x \sinh \pi x}{t \sinh \pi t} dt dx$$
$$= \int_0^X \int_0^\infty |\mathcal{K}(x,t)|^2 \frac{x \sinh \pi x}{t \sinh \pi t} dt dx. \tag{4.5}$$

But the inner integral with respect to t in (4.5) can be calculated by the Parseval equality (1.2) regarding $\frac{\mathcal{K}(x,t)}{t \sinh \pi t}$ as the Kontorovich-Lebedev transform (1.1) of

$$\varphi(y) = \begin{cases} \frac{2}{\pi^2} \frac{K_{ix}(y)}{y}, & \text{if } y \in \mathbb{Y}, \\ 0, & \text{if } y \notin \mathbb{Y}. \end{cases}$$

Consequently,

$$\int_0^\infty |\mathcal{K}(x,t)|^2 \frac{dt}{t \sinh \pi t} = \frac{2}{\pi^2} \int_{\mathbb{Y}} K^2_{ix}(y) \frac{dy}{y}$$

and we come out with

$$||P_{\mathbb{X}} Q_{\mathbb{Y}}||^2_{L_2(\mathbb{R}_+; x \sinh \pi x dx)} \leq \frac{2}{\pi^2} \int_{\mathbb{X}} \int_{\mathbb{Y}} x \sinh \pi x K^2_{ix}(y) \frac{dy}{y} dx. \tag{4.6}$$

Let us estimate the right-hand side of (4.6). Applying twice the Schwarz inequality we obtain

$$\frac{2}{\pi^2} \int_{\mathbb{X}} \int_{\mathbb{Y}} x \sinh \pi x K^2_{ix}(y) \frac{dy}{y} dx \leq \frac{2}{\pi^2} \left(Y - \frac{1}{Y} \right)^{1/2} \int_{\mathbb{X}} x \sinh \pi x \left(\int_{\mathbb{Y}} K^4_{ix}(y) dy \right)^{1/2} dx$$

$$\leq \frac{2}{\pi^2\sqrt{3}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}\left(\int_{\mathbb{X}}\int_{\mathbb{Y}}\sinh^2\pi x K_{ix}^4(y)dydx\right)^{1/2}.$$

On the other hand by relation 2.16.52.17 from [10], Vol. 2 and the Parseval equality for the sine Fourier transform we find

$$\int_0^\infty \sinh^2\pi x K_{ix}^4(y)dx = \frac{\pi^3}{8}\int_0^\infty J_0^2\left(2y\sinh(u/2)\right)du,$$

where $J_0(z)$ is the Bessel function of the first kind. Consequently, invoking relation 2.12.31.2 from [10], Vol. 2 and the Hölder inequality we get

$$\frac{2}{\pi^2\sqrt{3}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}\left(\int_{\mathbb{X}}\int_{\mathbb{Y}}\sinh^2\pi x K_{ix}^4(y)dydx\right)^{1/2}$$

$$\leq \frac{1}{\sqrt{6\pi}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}\left(\int_{\mathbb{Y}}\int_0^\infty J_0^2\left(2y\sinh(u/2)\right)dudy\right)^{1/2} = \frac{1}{\sqrt{3\pi}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}$$

$$\times\left(\int_{\mathbb{Y}}\int_0^\infty J_0^2(v)\frac{dvdy}{\sqrt{v^2+4y^2}}\right)^{1/2} \leq \frac{|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}}{\sqrt{6\pi}}\left(\int_{\mathbb{Y}}\frac{dy}{\sqrt{y}}\int_0^\infty J_0^2(v)\frac{dv}{\sqrt{v}}\right)^{1/2}$$

$$= \frac{\Gamma^2(1/4)}{2\pi^{7/4}\sqrt{6}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}\left(\int_{\mathbb{Y}}\frac{dy}{\sqrt{y}}\right)^{1/2} \leq \frac{\Gamma^2(1/4)}{2\pi^{7/4}\sqrt{6}}|\mathbb{X}|^{3/2}|\mathbb{Y}|^{7/8}\left(\int_{\mathbb{Y}}\frac{dy}{y^2}\right)^{1/8}$$

$$= \frac{\Gamma^2(1/4)}{2\pi^{7/4}\sqrt{6}}|\mathbb{X}|^{3/2}|\mathbb{Y}|.$$

Thus combining with (4.5) we derive finally the inequality

$$||P_{\mathbb{X}}Q_{\mathbb{Y}}||_{L_2(\mathbb{R}_+;x\sinh\pi x dx)} \leq \frac{\Gamma(1/4)}{\sqrt{2\sqrt{6}}\pi^{7/8}}|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}.$$

Assuming that $|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2} < \frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\Gamma(1/4)}$ we have $||P_{\mathbb{X}}Q_{\mathbb{Y}}||_{L_2(\mathbb{R}_+;x\sinh\pi x dx)} < 1$, and therefore, $I - P_{\mathbb{X}}Q_{\mathbb{Y}}$ is invertible in $L_2(\mathbb{R}_+;x\sinh\pi x dx)$. Moreover,

$$||(I-P_{\mathbb{X}}Q_{\mathbb{Y}})^{-1}|| \leq \sum_{n=0}^\infty ||P_{\mathbb{X}}Q_{\mathbb{Y}}||^n \leq \sum_{n=0}^\infty\left[\frac{\Gamma(1/4)}{\sqrt{2\sqrt{6}}\pi^{7/8}}|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}\right]^n$$

$$= \frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\sqrt{2\sqrt{6}}\pi^{7/8} - \Gamma(1/4)|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}}.$$

However,

$$I = P_{\mathbb{X}} + P_{\mathbb{R}_+\setminus\mathbb{X}} = P_{\mathbb{X}}Q_{\mathbb{Y}} + P_{\mathbb{X}}Q_{\mathbb{R}_+\setminus\mathbb{Y}} + P_{\mathbb{R}_+\setminus\mathbb{X}}$$

and the orthogonality $P_{\mathbb{X}}$ and $P_{\mathbb{R}_+\backslash\mathbb{X}}$ gives

$$||P_{\mathbb{X}}Q_{\mathbb{R}_+\backslash\mathbb{Y}}g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)} + ||P_{\mathbb{R}_+\backslash\mathbb{X}}g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)}$$

$$= ||P_{\mathbb{X}}Q_{\mathbb{R}_+\backslash\mathbb{Y}}g + P_{\mathbb{R}_+\backslash\mathbb{X}}g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)}.$$

Taking into account that $||P_{\mathbb{X}}|| = 1$ we find

$$||g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)} \le ||(I - P_{\mathbb{X}}Q_{\mathbb{Y}})^{-1}||^2||(I - P_{\mathbb{X}}Q_{\mathbb{Y}})g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)}$$

$$\le \left(\frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\sqrt{2\sqrt{6}}\pi^{7/8} - \Gamma(1/4)|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}}\right)^2 \left[||P_{\mathbb{X}}Q_{\mathbb{R}_+\backslash\mathbb{Y}}g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)}\right.$$

$$\left. + ||P_{\mathbb{R}_+\backslash\mathbb{X}}g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)}\right] \le \left(\frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\sqrt{2\sqrt{6}}\pi^{7/8} - \Gamma(1/4)|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}}\right)^2$$

$$\times \left[||Q_{\mathbb{R}_+\backslash\mathbb{Y}}g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)} + ||P_{\mathbb{R}_+\backslash\mathbb{X}}g||^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)}\right].$$

Now since g is $\varepsilon_{\mathbb{X}}$-concentrated then $||P_{\mathbb{R}_+\backslash\mathbb{X}}g||_{L_2(\mathbb{R}_+;x\sinh\pi x dx)} \le \varepsilon_{\mathbb{X}}$. Furthermore, since f is $\varepsilon_{\mathbb{Y}}$-concentrated then owing to (4.3) $||Q_{\mathbb{R}_+\backslash\mathbb{Y}}g||_{L_2(\mathbb{R}_+;x\sinh\pi x dx)} \le \varepsilon_{\mathbb{Y}}$. Therefore considering g of unit norm we arrive at the inequality

$$1 \le \left(\frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\sqrt{2\sqrt{6}}\pi^{7/8} - \Gamma(1/4)|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}}\right)^2 (\varepsilon^2_{\mathbb{X}} + \varepsilon^2_{\mathbb{Y}}),$$

which is equivalent to (4.4). Theorem 5 is proved.

References

1. M.G.Cowling and J.F.Price, Generalization of Heisenberg's inequality, In: Harmonic Analysis, LNM, No 992 (1983), Springer, Berlin, 443-449.

2. V.A. Ditkin and A.P. Prudnikov, *Operational Calculus* , Vysh. Shkola, Moscow (1975) (in Russian).

3. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, London and Toronto (1953).

4. I.M. Gelfand and G.E. Shilov, Fourier transforms of rapidly increasing functions and questions of uniqueness of the solution of Cauchy's problem, *Uspekhi Mat. Nauk*, **8** (1953), 3-54.

5. G. Hardy, A theorem concerning Fourier transform, *J. London Math. Soc.*, **8** (1933), 227-231.

6. J. A. Hogan and J.D. Lakey, *Time- Frequency and Time-Scale Methods. Adaptive Decompositions, Uncertainty Principles, and Sampling.* Birkhäuser, Boston, Basel, Berlin (2005).

7. L. Hörmander, A uniqueness theorem of Beurling for Fourier transform pairs, *Ark. Mat.*, **29** (1991), 237-240.

8. N.N. Lebedev, Sur une formule d'inversion, *C.R.Acad. Sci. URSS* **52**(1946), 655-658.

9. F. Al-Musallam, and Vu Kim Tuan, An uncertainty principle for a modified Y-transform, *Arch. of Ineq. and Appl.*, **1** (2003), 441-452.

10. A.P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: Vol. I: Elementary Functions; Vol. 2: Special Functions*, Gordon and Breach, New York (1986).

11. M. Rösler, An uncertainty principle for the Dunkl transform, *Bull. Austral. Math. Soc.* , **99** (1999), 353-360.

12. I.N. Sneddon, *The Use of Integral Transforms*, McGray Hill, New York (1972).

13. E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* , Clarendon Press, Oxford (1937).

14. Vu Kim Tuan, Uncertainty principles for the Hankel transform, *Integral Transforms and Special Functions* , (to appear).

15. S.B. Yakubovich and Yu.F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*, (Kluwers Ser. Math. and Appl.: Vol. 287), Dordrecht, Boston, London (1994).

16. S.B. Yakubovich, *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong (1996).

17. S.B. Yakubovich, On the least values of Lp-norms for the Kontorovich-Lebedev transform and its convolution. *Journal of Approximation Theory*, **131** (2004), 231-242.

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