Fixed points of endomorphisms of virtually free groups

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March 12, 2012

2010 Mathematics Subject Classification: 20F67, 20E05, 20E36, 68Q45, 37B25

Keywords: virtually free groups, endomorphisms, fixed points, hyperbolic boundary, classification of fixed points

ABSTRACT

A fixed point theorem is proved for inverse transducers, leading to an automata-theoretic proof of the fixed point subgroup of an endomorphism of a finitely generated virtually free group being finitely generated. If the endomorphism is uniformly continuous for the hyperbolic metric, it is proved that the set of regular fixed points in the hyperbolic boundary has finitely many orbits under the action of the finite fixed points. In the automorphism case, it is shown that these regular fixed points are either exponentially stable attractors or exponentially stable repellers.

1 Introduction

Throughout the paper, the ambient groups are assumed to be finitely generated.

Gersten proved in the eighties that the fixed point subgroup of a free group automorphism φ is finitely generated [8]. Using a different approach, Cooper gave an alternative proof, proving also that the fixed points of the continuous extension of φ to the boundary of the free group is in some sense finitely generated [4]. Bestvina and Handel achieved in 1992 a major breakthrough through their innovative train track techniques, bounding the rank of the fixed point subgroup and the generating set for the infinite fixed points [2]. Their approach was pursued by Maslakova in 2003 to prove that the fixed point subgroup can be effectively computed [14].

Gersten's result was generalized to further classes of groups and endomorphisms in subsequent years. Goldstein and Turner extended it to monomorphisms of free groups [11], and later to arbitrary endomorphisms [12]. Collins and Turner extended it to automorphisms of free products of freely indecomposable groups [3] (see the survey by Ventura [22]). With respect to automorphisms, the widest generalization is to hyperbolic groups and is due to Paulin [15].

In 2002, Sykiotis extended Collins and Turner's result to arbitrary endomorphisms of virtually free groups using symmetric endomorphisms [19] (see also [20] for further results on symmetric endomorphisms). In [18], the author generalized Goldstein and Turner's authoma-theoretic proof to arbitrary endomorphisms of free products of cyclic groups. In the present paper, this result is extended to arbitrary endomorphisms of virtually free groups, providing an automata-theoretic alternative to Sykiotis' result.

This is done by reducing the problem to the rationality of some languages associated to a finite inverse transducer, and subsequent application of Anisimov and Seifert's Theorem.

Infinite fixed points of automorphisms of free groups were also discussed by Bestvina and Handel in [2]. Gaboriau, Jaeger, Levitt and Lustig remarked in [7] that some of the results on infinite fixed points would hold for virtually free groups with some adaptations.

In [17], we discussed infinite fixed points for monomorphisms of free products of cyclic groups, the group case of a more general setting based on the concept of special confluent rewriting system. These results are now extended to endomorphisms with finite kernel of virtually free groups (which are precisely the uniformly continuous endomorphisms for the hyperbolic metric), and we discuss the dynamical nature of the regular fixed points in the automorphism case, generalizing the results of [7] on free groups.

The paper is organized as follows. Section 2 is devoted to preliminaries on groups and automata. We discuss inverse transducers in Section 3, proving a useful fixed point theorem. In Section 4 we prove that the fixed point subgroup is finitely generated for arbitrary endomorphisms of a (finitely generated) virtually free group G.

In Section 5 we get a rewriting system with good properties to represent the elements of G, and use it in Section 6 to construct a simple model for the hyperbolic boundary of G. We study uniformly continuous endomorphisms in Section 7 and prove in Section 8 that the infinite fixed points of such endomorphisms are in some sense finitely generated.

The classification of the infinite fixed points of automorphisms is performed in Section 9, and the final Section 10 includes an example and some open problems.

2 Preliminaries

Throughout the whole paper, we assume alphabets to be *finite*.

We start with some group-theoretic definitions. Given an alphabet A, we denote by A^{-1} a set of *formal inverses* of A, and write $\widetilde{A} = A \cup A^{-1}$. We extend the mapping $a \mapsto a^{-1}$ to an involution of the free monoid $\widetilde{A}^*$ in the obvious way. As usual, the *free group on A* is the quotient of $\widetilde{A}^*$ by the congruence generated by the relation $\{(aa^{-1}, 1) \mid a \in \widetilde{A}\}$. We denote by $\theta : \widetilde{A}^* \to F_A$ the canonical morphism.

Let

$$R_A = \widetilde{A}^* \setminus (\cup_{a \in \widetilde{A}} \widetilde{A}^* aa^{-1} \widetilde{A}^*)$$

be the subset of all *reduced words* in $\widetilde{A}^*$. It is well known that, for every $g \in F_A$, $g\theta^{-1}$ contains a unique reduced word, denoted by $\overline{g}$. We write also $\overline{u} = \overline{u\theta}$ for every $u \in \widetilde{A}^*$. Note that the equivalence $u\theta = v\theta \Leftrightarrow \overline{u} = \overline{v}$ holds for all $u, v \in \widetilde{A}^*$.

A group G is *virtually free* if G has a free subgroup F of finite index. In view of Nielsen's Theorem, it is well known that F can be assumed to be normal, and is finitely generated if G is finitely generated itself. Therefore every finitely generated virtually free group G admits a decomposition as a disjoint union

$$G = F \cup Fb_1 \cup \ldots \cup Fb_m,$$

where $F \trianglelefteq G$ is a free group of finite rank and $b_1, \ldots, b_m \in G$.

We shall need also some basic concepts from automata theory:

Let A be a (finite) alphabet. A subset of A^* is called an A-*language*. We say that $\mathcal{A} = (Q, q_0, T, \delta)$ is a (finite) *deterministic* A-*automaton* if:

- Q is a (finite) set;
- $q_0 \in Q$ and $T \subseteq Q$;
- $\delta : Q \times A \to Q$ is a partial mapping.

We extend δ to a partial mapping $Q \times A^* \to Q$ by induction through

$$(q, 1)\delta = q, \quad (q, ua)\delta = ((q, u)\delta, a)\delta \quad (u \in A^*, a \in A).$$

When the automaton is clear from the context, we write $qu = (q, u)\delta$. We can view $\mathcal{A}$ as a directed graph with edges labelled by letters $a \in A$ by identifying $(p, a)\delta = q$ with the edge $p \xrightarrow{a} q$. The set of all such edges is denoted by $E(\mathcal{A}) \subseteq Q \times A \times Q$.

A *finite nontrivial path* in $\mathcal{A}$ is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} p_n$$

with $(p_{i-1}, a_i, p_i) \in E(\mathcal{A})$ for $i = 1, \ldots, n$. Its *label* is the word $a_1 \ldots a_n \in A^*$. It is said to be a *successful* path if $p_0 = q_0$ and $p_n \in T$. We consider also the *trivial path* $p \xrightarrow{1} p$ for $p \in Q$. It is successful if $p = q_0 \in T$.

The *language* $L(\mathcal{A})$ *recognized by* $\mathcal{A}$ is the set of all labels of successful paths in $\mathcal{A}$. Equivalently, $L(\mathcal{A}) = \{u \in A^* \mid q_0 u \in T\}$. If $(p_{i-1}, a_i, p_i) \in E(\mathcal{A})$ for every $i \in \mathbb{N}$, we may consider also the *infinite path*

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{a_3} \ldots$$

Its label is the (right) infinite word $a_1 a_2 a_3 \ldots$ We denote by A^ω the set of all (right) infinite words on the alphabet A, and write also $A^\infty = A^* \cup A^\omega$. We denote by $L_\omega(\mathcal{A})$ the set of labels of all infinite paths $q_0 \longrightarrow \ldots$ in $\mathcal{A}$.

Given $u \in A^*$ and $\alpha \in A^\infty$, we say that u is a *prefix* of α and write $u \leq \alpha$ if $\alpha = u\beta$ for some $\beta \in A^\infty$. By convention, this includes the case $\alpha \leq \alpha$ for $\alpha \in A^\omega$. For every $n \in \mathbb{N}$, we denote by $\alpha^{[n]}$ the prefix of length n of α, applying the convention that $\alpha^{[n]} = \alpha$ if $n > |\alpha|$.

It is immediate that $(A^\infty, \leq)$ is a complete $\wedge$-semilattice: given $\alpha, \beta \in A^\infty$, $\alpha \wedge \beta$ is the longest common prefix of α and β (or α if $\alpha = \beta \in A^\omega$). The operator $\wedge$ will play a crucial role in later sections of the paper.

The *star* operator on A-languages is defined by

$$L^* = \bigcup_{n \geq 0} L^n,$$

where $L^0 = \{1\}$. An A-language L is said to be *rational* if L can be obtained from finite A-languages using finitely many times the operators union, product and star (this is called a *rational expression*). Alternatively, by Kleene's Theorem [1, Section III], L is rational if and only if it is recognized by a finite deterministic A-automaton $\mathcal{A}$. The definition through rational expressions generalizes to subsets of an arbitrary group in the obvious way. Moreover, if we fix a homomorphism $\pi : A^* \to G$, the rational subsets of G are the images by π of the rational A-languages. For obvious reasons, we shall be dealing mostly with matched homomorphisms. A homomorphism $\pi : \widetilde{A}^* \to G$ is said to be *matched* if $a^{-1}\pi = (a\pi)^{-1}$ for every $a \in A$. For details on rational languages and subsets, the reader is referred to [1, 16].

We shall need also the following classical result of Anisimov and Seifert:

Proposition 2.1 [16, Prop. II.6.2] *Let H be a subgroup of a group G. Then H is a rational subset of G if and only if H is finitely generated.*

We end this section with an elementary observation that will help us to establish that fixed point subgroups are finitely generated.

Proposition 2.2 *Let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism and let $X \subseteq G$. Let $\mathcal{A}$ be a finite $\widetilde{A}$-automaton such that:*

(i) $L(\mathcal{A}) \subseteq X\pi^{-1}$*;*

(ii) $L(\mathcal{A}) \cap x\pi^{-1} \neq \emptyset$ *for every* $x \in X$*.*

Then X is a rational subset of G.

Proof. It follows immediately that $X = (L(\mathcal{A}))\pi$, hence X is a rational subset of G. □

3 Inverse transducers

Given a finite alphabet A, we say that $\mathcal{T} = (Q, q_0, \delta, \lambda)$ is a (finite) *deterministic A-transducer* if:

- Q is a (finite) set;
- $q_0 \in Q$;
- $\delta : Q \times A \to Q$ and $\lambda : Q \times A \to A^*$ are mappings.

As in the automaton case, we may extend δ to a mapping $Q \times A^* \to Q$. Similarly, we extend λ to a mapping $Q \times A^* \to A^*$ through

$$(q, 1)\lambda = 1, \quad (q, ua)\lambda = (q, u)\lambda((q, u)\delta, a)\lambda \quad (u \in A^*, a \in A).$$

When the transducer is clear from the context, we write $qa = (q, a)\delta$. We can view $\mathcal{T}$ as a directed graph with edges labelled by elements of $A \times A^*$ (represented in the form $a|w$) by identifying $(p, a)\delta = q$, $(p, a)\lambda = w$ with the edge $p \xrightarrow{a|w} q$. The set of all such edges is denoted by $E(\mathcal{T}) \subseteq Q \times A \times A^* \times Q$. If $pu = q$ and $(p, u)\lambda = v$, we write also $p \xrightarrow{u|v} q$ and call it a path in $\mathcal{T}$.

It is immediate that, given $u \in A^*$, there exists exactly one path in $\mathcal{T}$ of the form $q_0 \xrightarrow{u|v} q$. We write $u\widehat{\mathcal{T}} = v$, defining thus a mapping $\widehat{\mathcal{T}} : A^* \to A^*$.

Assume now that $\mathcal{T} = (Q, q_0, T, \delta, \lambda)$ is a deterministic $\widetilde{A}$-transducer such that

$$p \xrightarrow{a|u} q \text{ is an edge of } \mathcal{T} \text{ if and only if } q \xrightarrow{a^{-1}|u^{-1}} p \text{ is an edge of } \mathcal{T}.$$

Then $\mathcal{T}$ is said to be *inverse*.

Proposition 3.1 *Let $\mathcal{T} = (Q, q_0, \delta, \lambda)$ be an inverse $\widetilde{A}$-transducer. Then:*

(i) $\delta : Q \times \widetilde{A}^* \to Q$ induces a mapping $\widetilde{\delta} : Q \times F_A \to Q$ by $(q, u\theta)\widetilde{\delta} = (q, u)\delta$;

(ii) $\widehat{\mathcal{T}} : \widetilde{A}^* \to \widetilde{A}^*$ induces a partial mapping $\widetilde{\mathcal{T}} : F_A \to F_A$ by $u\theta\widetilde{\mathcal{T}} = u\widehat{\mathcal{T}}\theta$.

Proof. (i) Since the free group congruence $\sim$ is generated by the pairs $(aa^{-1}, 1)$, it suffices to show that $(q, vaa^{-1}w)\delta = (q, vw)\delta$ for all $q \in Q$; $v, w \in \widetilde{A}^*$ and $a \in \widetilde{A}$.

Since δ is a full mapping, we have a path

$$q \xrightarrow{v|v'} q_1 \xrightarrow{a|u} q_2 \xrightarrow{a^{-1}|u'} q_3 \xrightarrow{w|w'} q_4 \tag{1}$$

in $\mathcal{T}$. Since $\mathcal{T}$ is inverse (in particular deterministic), we must have $u' = u^{-1}$ and $q_3 = q_1$, hence we also have a path

$$q \xrightarrow{v|v'} q_1 \xrightarrow{w|w'} q_4$$

and so $(q, vaa^{-1}w)\delta = q_4 = (q, vw)\delta$ as required.

(ii) Similarly to part (i), it suffices to show that $(vaa^{-1}w)\widehat{\mathcal{T}}\theta = (vw)\widehat{\mathcal{T}}\theta$ for all $v, w \in \widetilde{A}^*$ and $a \in \widetilde{A}$.

We consider the path (1) for $q = q_0$. Since $u' = u^{-1}$ and $q_3 = q_1$, we get

$$(vaa^{-1}w)\widehat{\mathcal{T}}\theta = (v'uu^{-1}w')\theta = (v'w')\theta = (vw)\widehat{\mathcal{T}}\theta$$

as required. $\square$

We prove now one of our main results, generalizing Goldstein and Turner's proof [12] to mappings induced by inverse transducers.

Theorem 3.2 *Let $\mathcal{T}$ be a finite inverse $\widetilde{A}$-transducer and let $z \in F_A$. Then*

$$L = \{g \in F_A \mid g\widetilde{\mathcal{T}} = gz\}$$

is rational.

Proof. Write $\mathcal{T} = (Q, q_0, \delta, \lambda)$. For every $g \in F_A$, let $P_1(g) = g^{-1}(g\widetilde{\mathcal{T}}) \in F_A$ and write $q_0 g = (q_0, g)\widetilde{\delta}$, $P(g) = (P_1(g), q_0 g)$. Note that $g \in L$ if and only if $P_1(g) = z$. We define a deterministic $\widetilde{A}$-automaton $\mathcal{A}_\varphi = (P, (1, q_0), S, E)$ by

$$P = \{P(g) \mid g \in F_A\};$$

$$S = P \cap (\{z\} \times Q);$$

$$E = \{(P(g), a, P(ga)) \mid g \in F_A,\ a \in \widetilde{A}\}.$$

Clearly, $\mathcal{A}_\varphi$ is a possibly infinite automaton. Note that, since $\mathcal{T}$ is inverse, we have $qaa^{-1} = q$ for all $q \in Q$ and $a \in \widetilde{A}$. It follows that, whenever $(p, a, p') \in E$, then also $(p', a^{-1}, p) \in E$. We say that such edges are the *inverse* of each other.

Since every $w \in \widetilde{A}^*$ labels a unique path $P(1) \xrightarrow{w} P(w\theta)$, it follows that

$$L(\mathcal{A}_\varphi) = L\theta^{-1}.$$

In view of Proposition 2.2, to prove that L is rational it suffices to construct a finite subautomaton $\mathcal{B}_\varphi$ of $\mathcal{A}_\varphi$ such that $\overline{L} \subseteq L(\mathcal{B}_\varphi)$.

We fix now

$$M = \max\{|(q,a)\lambda| : q \in Q,\ a \in \widetilde{A}\}, \quad N = \max\{2M+1, |z|\}$$

and

$$P' = \{P(g) \in P : |P_1(g)| \leq N\}.$$

Since A is finite, so is P'. Given $g \in F_A$, write $g\iota = \overline{g}^{[1]}$. Given $p = (g,q) \in P$, we write also $p\iota = g\iota$. We say that an edge $(p_1, a, p_2) \in E$ is:

- *central* if $p_1, p_2 \in P'$;
- *compatible* if it is not central and $p_1\iota = a$.

We collect in the following lemma some elementary properties involving these concepts:

Lemma 3.3 *(i) There are only finitely many central edges in $\mathcal{A}_\varphi$.*

(ii) If $(p_1, a, p_2) \in E$ is not central, then either (p_1, a, p_2) or (p_2, a^{-1}, p_1) is compatible.

(iii) For every $p \in P$, there is at most one compatible edge leaving p.

Proof. (i) Since A and P' are both finite.

(ii) Assume that (p_1, a, p_2) is neither central nor compatible. Write $p_1 = (g_1, q_1)$ and $p_2 = (g_2, q_2)$. Suppose that $g_1 = 1$. Then $g_2 = P_1(a) = a^{-1}(a\widetilde{\mathcal{T}})$ and so $|g_2| \leq 1 + M \leq N$, in contradiction with (p_1, a, p_2) being non central.

Thus $\overline{g_1} = bu$ for some $b \in \widetilde{A} \setminus \{a\}$ and $u \in R_A$. On the other hand, $g_2 = a^{-1}g_1(q_1, a)\lambda$ and so $\overline{g_2} = \overline{a^{-1}bu(q_1,a)\lambda}$. If $|u| < M$, then $|g_1|, |g_2| \leq 2M+1 \leq N$ and (p_1, a, p_2) would be central, a contradiction. Thus $|u| \geq M \geq |(q_1, a)\lambda|$ and so $g_2\iota = a^{-1}$. Thus (p_2, a^{-1}, p_1) is compatible.

(iii) Since any compatible edge leaving p must be labelled by $p\iota$, and $\mathcal{A}_\varphi$ is deterministic. $\square$

A (possibly infinite) path $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \ldots$ in $\mathcal{A}_\varphi$ is:

- *central* if all the vertices in it are in P';
- *compatible* if all the edges in it are compatible and no intermediate vertex is in P'.

Lemma 3.4 *Let $u \in \overline{L}$. Then there exists a path*

$$(1, q_0) = p'_0 \xrightarrow{u_0} p''_0 \xrightarrow{v_1} p_1 \xrightarrow{w_1^{-1}} p'_1 \xrightarrow{u_1} \ldots \xrightarrow{v_n} p_n \xrightarrow{w_n^{-1}} p'_n \xrightarrow{u_n} p''_n \in S$$

in $\mathcal{A}_\varphi$ such that:

(i) $u = u_0 v_1 w_1^{-1} u_1 \ldots v_n w_n^{-1} u_n$;

(ii) the paths $p'_j \xrightarrow{u_j} p''_j$ are central;

(iii) the paths $p''_{j-1} \xrightarrow{v_j} p_j$ and $p'_j \xrightarrow{w_j} p_j$ are compatible;

(iv) $p_j \notin P'$ *if both* v_j *and* w_j *are nonempty.*

Proof. Since $S \subseteq P'$ by definition of N, there exists a path

$$(1, q_0) = p'_0 \xrightarrow{u_0} p''_0 \xrightarrow{x_1} p'_1 \xrightarrow{u_1} \ldots \xrightarrow{x_n} p'_n \xrightarrow{u_n} p''_n \in S \qquad (2)$$

in $\mathcal{A}_\varphi$ such that $u = u_0 x_1 u_1 \ldots x_n u_n$ and the paths $p'_j \xrightarrow{u_j} p''_j$ (which may be trivial) collect all the occurrences of vertices in P' (and are therefore central).

By Lemma 3.3(ii), if (p, a, r) occurs in a path $p''_{j-1} \xrightarrow{x_j} p'_j$, then either (p, a, r) or (r, a^{-1}, p) is compatible. On the other hand, since x_j is reduced, it follows from Lemma 3.3(iii) that $p''_{j-1} \xrightarrow{x_j} p'_j$ can be factored as

$$p''_{j-1} \xrightarrow{v_j} p_j \xrightarrow{w_j^{-1}} p'_j$$

with $p''_{j-1} \xrightarrow{v_j} p_j$ and $p'_j \xrightarrow{w_j} p_j$ compatible. Clearly, (iv) holds since no intermediate vertex of $p''_{j-1} \xrightarrow{x_j} p'_j$ belongs to P' by construction. □

We say that a compatible path is *maximal* if it is infinite or cannot be extended (to the right) to produce another compatible path.

Lemma 3.5 *For every* $p \in P'$*, there exists in* $\mathcal{A}_\varphi$ *a unique maximal compatible path* M_p *starting at* p*.*

Proof. Clearly, every compatible path can be extended to a maximal compatible path. Uniqueness follows from Lemma 3.3(iii). □

We define now

$$P'_1 = \{p \in P' \mid M_p \text{ has finitely many distinct edges }\}$$

and $P'_2 = P' \setminus P'_1$. Hence M_p contains no cycles if $p \in P'_2$. By Lemma 3.5, if M_p and $M_{p'}$ intersect at vertex $r_{pp'}$, then they coincide from $r_{pp'}$ onwards. In particular, if M_p and $M_{p'}$ intersect, then $p \in P'_1$ if and only if $p' \in P'_1$. Let

$$Y = \{(p, p') \in P'_2 \times P'_2 \mid \; M_p \text{ intersects } M_{p'}\}.$$

For every $(p, p') \in Y$, let $M_p \setminus M_{p'}$ denote the (finite) subpath $p \longrightarrow r_{pp'}$ of M_p. In particular, if $p' = p$, $M_p \setminus M_{p'}$ is the trivial path at p.

Let $\mathcal{B}_\varphi$ be the subautomaton of $\mathcal{A}_\varphi$ containing:

- all vertices in P' and all central edges;
- all edges in the paths M_p ($p \in P'_1$) and their inverses;
- all edges in the paths $M_p \setminus M_{p'}$ ($(p, p') \in Y$) and their inverses.

It follows easily from Lemma 3.3(i) and the definitions of P'_1 and $M_p \setminus M_{p'}$ that $\mathcal{B}_\varphi$ is a finite subautomaton of $\mathcal{A}_\varphi$. As remarked before, it suffices to show that $\overline{L} \subseteq L(\mathcal{B}_\varphi)$.

Let $u \in \overline{L}$. Since $\mathcal{B}_\varphi$ contains all the central edges of $\mathcal{A}_\varphi$, it suffices to show that all subpaths

$$p''_{j-1} \xrightarrow{v_j} p_j \xrightarrow{w_j^{-1}} p'_j$$

appearing in the factorization provided by Lemma 3.4 are paths in $\mathcal{B}_\varphi$.

Without loss of generality, we may assume that $v_j \neq 1$. If $w_j = 1$, then $p''_{j-1} \in P'_1$ and we are done, hence we may assume that also $w_j \neq 1$. Now, if one of the vertices p''_{j-1}, p'_j is in P'_1, so is the other and we are done since $\mathcal{B}_\varphi$ contains all the edges in the paths M_p ($p \in P'_1$) and their inverses. Hence we may assume that $p''_{j-1}, p'_j \in P'_2$. It follows that $p_j = r_{p''_{j-1},p'_j}$ (since $v_j w_j^{-1} \in R_A$, the paths $M_{p''_{j-1}}$ and $M_{p'_j}$ cannot meet before p_j). Thus $p''_{j-1} \xrightarrow{v_j} p_j$ is $M_{p''_{j-1}} \setminus M_{p'_j}$ and $p'_j \xrightarrow{w_j} p_j$ is $M_{p'_j} \setminus M_{p''_{j-1}}$, and so these are also paths in $\mathcal{B}_\varphi$ as required. $\square$

4 The fixed point subgroup

We can now produce an automata-theoretic proof to Sykiotis' theorem:

Theorem 4.1 [19, Proposition 3.4] *Let φ be an endomorphism of a finitely generated virtually free group. Then* $\mathrm{Fix}\,\varphi$ *is finitely generated.*

Proof. We consider a decomposition of G as a disjoint union

$$G = Fb_0 \cup Fb_1 \cup \ldots \cup Fb_m, \tag{3}$$

where $F = F_A \trianglelefteq G$ is a free group with A finite and $b_0, \ldots, b_m \in G$ with $b_0 = 1$.

Let $\varphi_0 : F_A \to F_A$ and $\eta : F_A \to \{0, \ldots, m\}$ be defined by

$$g\varphi = (g\varphi_0)b_{g\eta} \quad (g \in F_A).$$

Since the decomposition (3) is disjoint, $g\varphi_0$ and $g\eta$ are both uniquely determined by $g\varphi$, and so both mappings are well defined.

Write $Q = \{0, \ldots, m\}$. For all $i \in Q$ and $a \in \widetilde{A}$, we have $b_i(a\varphi) = h_{i,a}b_{(i,a)\delta}$ for some (unique) $h_{i,a} \in F_A$ and $(i,a)\delta \in Q$. It follows that, for every $j \in Q$, $\mathcal{A}_j = (Q, 0, j, \delta)$ is a well-defined finite deterministic $\widetilde{A}$-automaton. We define also a finite deterministic $\widetilde{A}$-transducer $\mathcal{T} = (Q, 0, \delta, \lambda)$ by taking $(i,a)\lambda = \overline{h_{i,a}}$ for all $i \in Q$ and $a \in \widetilde{A}$.

Assume that

$$i \xrightarrow{a|\overline{h_{i,a}}} (i,a)\delta = j$$

is an edge of $\mathcal{T}$. Then $b_i(a\varphi) = h_{i,a}b_j$ and so also

$$b_i = b_i(a\varphi)(a^{-1}\varphi) = h_{i,a}b_j(a^{-1}\varphi) = h_{i,a}h_{j,a^{-1}}b_{(j,a^{-1})\delta}.$$

This yields $h_{i,a}h_{j,a^{-1}} = 1$ and $(j,a^{-1})\delta = i$, thus there is an edge $j \xrightarrow{a^{-1}|\overline{h_{i,a}}^{-1}} (j,a^{-1})\delta = i$ in $\mathcal{T}$ and so $\mathcal{T}$ is an inverse transducer. We claim that $\widetilde{\mathcal{T}} = \varphi_0$. Indeed, let $g = a_1 \ldots a_n$ ($a_i \in \widetilde{A_i}$). Then there exists a (unique) path in $\mathcal{T}$ of the form

$$0 = i_0 \xrightarrow{a_1|\overline{h_{i_0,a_1}}} i_1 \xrightarrow{a_2|\overline{h_{i_1,a_2}}} \ldots \xrightarrow{a_n|\overline{h_{i_{n-1},a_n}}} i_n.$$

Moreover, $i_j = (i_{j-1}, a_j)\delta$ for $j = 1, \ldots, n$. It follows that

$$\begin{aligned} g\varphi &= b_{i_0}(a_1\varphi) \ldots (a_n\varphi) = h_{i_0,a_1}b_{i_1}(a_2\varphi) \ldots (a_n\varphi) = h_{i_0,a_1}h_{i_1,a_2}b_{i_2}(a_3\varphi) \ldots (a_n\varphi) \\ &= \ldots = h_{i_0,a_1} \ldots h_{i_{n-1},a_n}b_{i_n} \end{aligned}$$

and so

$$g\varphi_0 = h_{i_0,a_1} \dots h_{i_{n-1},a_n} = (\overline{h_{i_0,a_1}} \dots \overline{h_{i_{n-1},a_n}})\theta = g\widetilde{\mathcal{T}}.$$

Thus $\widetilde{\mathcal{T}} = \varphi_0$.

Note that we have also shown that $g\eta = i_n = (0, a_1 \dots a_n)\delta$, hence

$$L(\mathcal{A}_j) = \{u \in \widetilde{A}^* \mid u\theta\eta = j\}. \tag{4}$$

Next let

$$Y = \{(i,j) \in Q \times Q \mid b_j(b_i\varphi) \in F_A b_i\}.$$

For every $(i,j) \in Y$, let $z_{i,j} \in F_A$ be such that $b_j(b_i\varphi) = z_{i,j}b_i$ and define

$$X_{i,j} = \{g \in F_A \mid gb_i \in \operatorname{Fix}\varphi \text{ and } g\eta = j\}.$$

We claim that $X_{i,j}$ is a rational subset of F_A for every $(i,j) \in Y$. Indeed, $(gb_i)\varphi = (g\varphi)(b_i\varphi) = (g\varphi_0)b_{g\eta}(b_i\varphi)$. Hence

$$\begin{aligned} X_{i,j} &= \{g \in F_A \mid (g\varphi_0)b_j(b_i\varphi) = gb_i \text{ and } g\eta = j\} = \{g \in F_A \mid (g\varphi_0)z_{i,j}b_i = gb_i \text{ and } g\eta = j\} \\ &= \{g \in F_A \mid g\varphi_0 = gz_{i,j}^{-1}\} \cap \{g \in F_A \mid g\eta = j\}. \end{aligned}$$

Writing

$$L_{i,j} = \{g \in F_A \mid g\varphi_0 = gz_{i,j}^{-1}\},$$

it follows from (4) that $X_{i,j} = L_{i,j} \cap (L(\mathcal{A}_j))\theta$. Since $\varphi_0 = \widetilde{\mathcal{T}}$, it follows from Theorem 3.2 that $X_{i,j}$ is an intersection of two rational subsets of F_A, hence rational itself (see [1, Corollary III.2.10]).

Now it is easy to check that

$$\operatorname{Fix}\varphi = \cup_{i\in Q} \left(\cup\{X_{i,j} \mid (i,j) \in Y\}\right)b_i. \tag{5}$$

Indeed, for every $(i,j) \in Y$, we have $X_{i,j}b_i \subseteq \operatorname{Fix}\varphi$ by definition of $X_{i,j}$. Conversely, let $gb_i \in \operatorname{Fix}\varphi$ for some $g \in F_A$ and $i \in Q$. Then $gb_i = (gb_i)\varphi = (g\varphi_0)b_{g\eta}(b_i\varphi)$ and so $b_{g\eta}(b_i\varphi) \in F_A b_i$. Hence $(i, g\eta) \in Y$. Since $g \in X_{i,g\eta}$, (5) holds. Since the $X_{i,j}$ are rational subsets of F_A and therefore of G, it follows that $\operatorname{Fix}\varphi$ is a rational subset of G and therefore finitely generated by Proposition 2.1. □

5 A good rewriting system

We recall that a (finite) *rewriting system* on A is a (finite) subset $\mathcal{R}$ of $A^* \times A^*$. Given $u, v \in A^*$, we write $u \longrightarrow_{\mathcal{R}} v$ if there exist $(r,s) \in \mathcal{R}$ and $x, y \in A^*$ such that $u = xry$ and $v = xsy$. The reflexive and transitive closure of $\longrightarrow_{\mathcal{R}}$ is denoted by $\longrightarrow^*_{\mathcal{R}}$.

We say that $\mathcal{R}$ is:

- *length-reducing* if $|r| > |s|$ for every $(r,s) \in \mathcal{R}$;
- *length-nonincreasing* if $|r| \geq |s|$ for every $(r,s) \in \mathcal{R}$;

- *noetherian* if, for every $u \in A^*$, there is a bound on the length of a chain

$$u \longrightarrow_{\mathcal{R}} v_1 \longrightarrow_{\mathcal{R}} \ldots \longrightarrow_{\mathcal{R}} v_n;$$

- *confluent* if, whenever $u \longrightarrow^*_{\mathcal{R}} v$ and $u \longrightarrow^*_{\mathcal{R}} w$, there exists some $z \in A^*$ such that $v \longrightarrow^*_{\mathcal{R}} z$ and $w \longrightarrow^*_{\mathcal{R}} z$.

A word $u \in A^*$ is an *irreducible* if no $v \in A^*$ satisfies $u \longrightarrow_{\mathcal{R}} v$. We denote by $\mathrm{Irr}\,\mathcal{R}$ the set of all irreducible words in A^* with respect to $\mathcal{R}$.

We introduce now some basic concepts and results from the theory of hyperbolic groups. For details on this class of groups, the reader is referred to [9].

Let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism with A finite. The *Cayley graph* $\Gamma_A(G)$ of G with respect to π has vertex set G and edges $(g, a, g(a\pi))$ for all $g \in G$ and $a \in \widetilde{A}$. We say that a path $p \xrightarrow{u} q$ in $\Gamma_A(G)$ is a *geodesic* if it has shortest length among all the paths connecting p to q in $\Gamma_A(G)$. We denote by $\mathrm{Geo}_A(G)$ the set of labels of all geodesics in $\Gamma_A(G)$. Note that, since $\Gamma_A(G)$ is vertex-transitive, it is irrelevant whether or not we fix a basepoint.

The *geodesic distance* d_1 on G is defined by taking $d_1(g, h)$ to be the length of a geodesic from g to h. Given $X \subseteq G$ nonempty and $g \in G$, we define

$$d_1(g, X) = \min\{d_1(g, x) \mid x \in X\}.$$

A *geodesic triangle* in $\Gamma_A(G)$ is a collection of three geodesics

$$P_1 : g_1 \longrightarrow g_2, \quad P_2 : g_2 \longrightarrow g_3, \quad P_3 : g_3 \longrightarrow g_1$$

connecting three vertices $g_1, g_2, g_3 \in G$. Let $V(P_i)$ denote the set of vertices occurring in the path P_i. We say that $\Gamma_A(G)$ is *δ-hyperbolic* for some $\delta \geq 0$ if

$$\forall g \in V(P_1) \qquad d_1(g, V(P_2) \cup V(P_3)) < \delta$$

holds for every geodesic triangle $\{P_1, P_2, P_3\}$ in in $\Gamma_A(G)$. If this happens for some δ, we say that G is *hyperbolic*. It is well known that the concept is independent from both alphabet and matched epimorphism, but the hyperbolicity constant δ may change. Virtually free groups are among the most important examples of hyperbolic groups.

We now use a theorem of Gilman, Hermiller, Holt and Rees [10] to prove the following result:

Lemma 5.1 *Let G be a finitely generated virtually free group. Then there exist a finite alphabet A, a matched epimorphism $\pi : \widetilde{A}^* \to G$ and a positive integer N_0 such that, for all $u \in \mathrm{Geo}_A(G)$ and $v \in \widetilde{A}^*$:*

(i) there exists some $w \in \mathrm{Geo}_A(G)$ such that $w\pi = (uv)\pi$ and $|u \wedge w| \geq |u| - N_0|v|$;

(ii) there exists some $z \in \mathrm{Geo}_A(G)$ such that $z\pi = (vu)\pi$ and $|u^{-1} \wedge z^{-1}| \geq |u| - N_0|v|$.

Proof. (i) By [10, Theorem 1], there exists a finite alphabet A, a matched epimorphism $\pi : \widetilde{A}^* \to G$ and a finite length-reducing rewriting system $\mathcal{R}$ such that $\mathrm{Geo}_A(G) = \mathrm{Irr}\,\mathcal{R}$. The authors also prove that this property characterizes (finitely generated) virtually free groups.

Let $N_0 = \max\{|r| : (r, s) \in \mathcal{R}\}$. Suppose that

$$uv = w_0 \longrightarrow_{\mathcal{R}} w_1 \longrightarrow_{\mathcal{R}} \ldots \longrightarrow_{\mathcal{R}} w_n = w$$

is a sequence of reductions leading to a geodesic w. Then $(wv^{-1})\pi = u\pi$ and since u is a geodesic we get $|w| \geq |u| - |v|$. Since $\mathcal{R}$ is length-reducing, this yields $n \leq |u| - |w| \leq |v|$.

Trivially, $|u \wedge w_0| \geq |u|$. Since $u \wedge w_{i-1} \in \mathrm{Geo}_A(G)$, it is immediate that $|u \wedge w_i| > |u \wedge w_{i-1}| - N_0$ and so

$$|u \wedge w| = |u \wedge w_n| \geq |u| - nN_0 \geq |u| - N_0|v|.$$

(ii) The inverse of a geodesic is still a geodesic. By applying (i) to u^{-1} and v^{-1}, we get $(u^{-1}v^{-1})\pi = x\pi$ for some $x \in \mathrm{Geo}_A(G)$ satisfying $|u^{-1} \wedge x| \geq |u^{-1}| - N_0|v^{-1}|$. Then we take $z = x^{-1}$. □

We assume for the remainder of the paper that G is a finitely generated virtually free group, $\pi : \widetilde{A}^* \to G$ a matched epimorphism and N_0 a positive integer satisfying the conditions of Lemma 5.1. Since G is hyperbolic, it follows from [6, Theorem 3.4.5] that $\mathrm{Geo}_A(G)$ is an automatic structure for G with respect to π (see [6] for definitions), and so the *fellow traveller property* holds for some constant $K_0 > 0$ (which can be taken as $2(\delta + 1)$, if δ is the hyperbolicity constant). This amounts to say that

$$\forall u, v \in \mathrm{Geo}_A(G)\ (d_1(u\pi, v\pi) \leq 1 \Rightarrow \forall n \in \mathbb{N}\ d_1(u^{[n]}\pi, v^{[n]}\pi) \leq K_0).$$

We fix a total ordering of $\widetilde{A}$. The *shortlex ordering* of $\widetilde{A}^*$ is defined by

$$u \leq_{sl} v \text{ if } \begin{cases} |u| < |v| \\ \text{or} \\ |u| = |v| \text{ and } u = wau', v = wbv' \text{ with } a < b \text{ in } \widetilde{A} \end{cases}$$

This is a well-known well-ordering of $\widetilde{A}^*$, compatible with multiplication on the left and on the right. Let

$$L = \{u \in \mathrm{Geo}_A(G) \mid u \leq_{sl} v \text{ for every } v \in u\pi\pi^{-1}\}.$$

By [6, Theorem 2.5.1], L is also an automatic structure for G with respect to π. We note that L is *factorial* (a factor of a word in L is still in L).

Given $g \in G$, let $\overline{g}$ denote the unique word of L representing g. This corresponds precisely to free group reduction if $G = F_A$ and $\pi = \theta$. Since we shall not need free group reduction from now on, we write also $\overline{u} = \overline{u\pi}$ for every $u \in \widetilde{A}^*$ to simplify notation.

Theorem 5.2 *Consider the finite rewriting system $\mathcal{R}'$ on A defined by*

$$\mathcal{R}' = \{(u, \overline{u}) : u \in \widetilde{A}^*,\ |u| \leq K_0N_0 + 1,\ u \neq \overline{u}\}.$$

Then:

(i) $\mathcal{R}'$ is length-nonincreasing, noetherian and confluent;

(ii) $Irr\,\mathcal{R}' = L$.

Proof. (i) $\mathcal{R}'$ is trivially length-nonincreasing, and noetherian folows from

$$(u, \overline{u}) \in \mathcal{R}' \Rightarrow u >_{sl} \overline{u} \tag{6}$$

and $\widetilde{A}^*$ being well-ordered by $\leq_{sl}$.

Next we show that

$$u \longrightarrow^*_{R'} \overline{u} \qquad \text{holds for every } u \in \widetilde{A}^*. \tag{7}$$

We use induction on $|u|$. The case $|u| \leq K_0N_0+1$ follows from the definition of $\mathcal{R}'$, hence assume that $|u| > K_0N_0+1$ and (7) holds for shorter words. Write $u = avb$ with $a, b \in \widetilde{A}$. If $av \notin L$, we have $u \longrightarrow^*_{R'} \overline{av}b$ and $\overline{u} = \overline{\overline{av}b}$, hence $u \longrightarrow^*_{R'} \overline{u}$ follows from $\overline{av}y \longrightarrow^*_{R'} \overline{\overline{av}y}$. Hence we may assume that $av \in L$.

Suppose that $u \notin \mathrm{Geo}_A(G)$. By Lemma 5.1(i), there exists some $w \in \mathrm{Geo}_A(G)$ such that $w\pi = (avb)\pi$ and $|av \wedge w| \geq |av| - N_0 \geq K_0N_0 + 1 - N_0 > 0$. Hence we may write $w = aw'$ and we get $(vb)\pi = (a^{-1}w)\pi = w'\pi$. Since $|w'| < |vb|$ due to $u \notin \mathrm{Geo}_A(G)$, we get $|\overline{vb}| < |vb|$ and so we may apply twice the induction hypothesis to get

$$u = avb \longrightarrow^*_{R'} a\overline{vb} \longrightarrow^*_{R'} \overline{a\overline{vb}} = \overline{u}.$$

Hence we may assume that $u \in \mathrm{Geo}_A(G)$. We claim that $\overline{u}^{[1]} = a$. Let $p = K_0N_0+1$. Since $u, \overline{u} \in \mathrm{Geo}_A(G)$ and $u\pi = \overline{u}\pi$, the fellow traveller property yields $d_1(u^{[p]}\pi, \overline{u}^{[p]}\pi) \leq K_0$ and so $u^{[p]}\pi = (\overline{u}^{[p]}x)\pi$ for some x of length $\leq K_0$. Thus, by Lemma 5.1(i), there exists some $w \in \mathrm{Geo}_A(G)$ such that $w\pi = (\overline{u}^{[p]}x)\pi = u^{[p]}\pi$ and

$$|\overline{u}^{[p]} \wedge w| \geq |\overline{u}^{[p]}| - N_0|x| \geq p - K_0N_0 = 1,$$

hence $\overline{u}^{[1]} = w^{[1]}$. Now $av \in L$ by assumption, hence $u^{[p]} \in L$ and so $u^{[p]} = \overline{u^{[p]}}$. Since $w\pi = u^{[p]}\pi$ and $w \in \mathrm{Geo}_A(G)$, we get $a = u^{[1]} \leq w^{[1]} = \overline{u}^{[1]}$ in $(\widetilde{A}, \leq)$. On the other hand, $\overline{u} \leq_{sl} u$ yields $\overline{u}^{[1]} \leq a$ in $(\widetilde{A}, \leq)$ and so $\overline{u}^{[1]} = a$ as claimed.

Now it follows easily that $\overline{u} = a\overline{a^{-1}u} = a\overline{vb}$ and the induction hypothesis yields $vb \longrightarrow^*_{R'} \overline{vb}$ and therefore $u = avb \longrightarrow^*_{R'} a\overline{vb} = \overline{u}$. Therefore (7) holds.

Assume now that $u \longrightarrow^*_{R'} v$ and $u \longrightarrow^*_{R'} w$. By (7), we get $v \longrightarrow^*_{R'} \overline{v} = \overline{u}$ and $w \longrightarrow^*_{R'} \overline{w} = \overline{u}$, hence $\mathcal{R}'$ is confluent.

(ii) It follows from (7) that $\mathrm{Irr}\,\mathcal{R}' \subseteq L$. The converse inclusion follows from the implication

$$u \longrightarrow_{R'} v \Rightarrow u >_{sl} v,$$

which follows in turn from (6). □

We establish now some technical results which will be useful in later sections:

Lemma 5.3 *Let $u, v \in L$ and let $w \in \widetilde{A}^*$ be such that $vw \in \mathrm{Geo}_A(G)$ and $(vw)\pi = u\pi$. Then $|u \wedge v| \geq |v| - K_0N_0$.*

Proof. Let $k = |v|$ and write $u = u^{[k]}u'$. Since $v = (vw)^{[k]}$, it follows from the fellow traveller property that $d_1(v\pi, u^{[k]}\pi) \leq K_0$, hence we may write $v\pi = (u^{[k]}z)\pi$ with $|z| \leq K_0$. Since $u^{[k]}$ is itself a geodesic, it follows from Lemma 5.1(i) that there exists a geodesic $u^{[p]}z'$ satisfying $(u^{[p]}z')\pi = (u^{[k]}z)\pi = v\pi$ and

$$p = |u^{[k]} \wedge u^{[p]}z'| \geq |u^{[k]}| - N_0|z| \geq |v| - K_0N_0.$$

Now $v \in L$ yields $v \leq_{sl} u^{[p]}z'$ and so $v^{[p]} \leq_{sl} u^{[p]}$. On the other hand, $u \in L$ yields $u \leq_{sl} vw$ and so $u^{[p]} \leq_{sl} v^{[p]}$. Thus $u^{[p]} = v^{[p]}$ and so $|u \wedge v| \geq p \geq |v| - K_0N_0$. □

Proposition 5.4 *(i) Let* $uv \in L$ *and let* $w \in \widetilde{A}^*$ *be such that* $|v| \geq K_0N_0 + N_0|w|$*. Then* $\overline{uvw} = u\overline{vw}$*.*

(ii) Let $u \in \widetilde{A}^*$ *and let* $vw, vw' \in L$*. Then* $|\overline{uvw} \wedge \overline{uvw'}| \geq |v| - K_0N_0 - N_0|u|$*.*

Proof. (i) Write $v = v_1v_2$ with $|v_2| = N_0|w|$. By Lemma 5.1(i), there exists some $uv_1z \in \mathrm{Geo}_A(G)$ such that $(uv_1z)\pi = (uvw)\pi$. Let $x = \overline{uvw}$. By Lemma 5.3, we get $|x \wedge uv_1| \geq |uv_1| - K_0N_0$. Since $|v_1| = |v| - |v_2| \geq K_0N_0$, then $u \leq x$ and we may write $x = uy$ for some y. Since L is factorial, we have $y \in L$. In view of $y\pi = (u^{-1}x)\pi = (vw)\pi$, we get $y = \overline{vw}$ and so $\overline{uvw} = u\overline{vw}$.

(ii) We may assume that $|v| > K_0N_0 + N_0|u|$. Write $v = v_1v_2$ with $|v_1| = N_0|u|$. Let $x = \overline{uv_1}$ and write $p = |x| + |v_2|$. By the proof of Lemma 5.1, we have $xv_2w, xv_2w' \in \mathrm{Geo}_A(G)$.

Let $y = \overline{uvw}$. Since $(xv_2w)\pi = y\pi$, it follows from the fellow traveller property that $d_1((xv_2)\pi, y^{[p]}\pi) \leq K_0$, hence we may write $(xv_2)\pi = (y^{[p]}s)\pi$ with $|s| \leq K_0$. Since $y^{[p]}$ is itself a geodesic, it follows from Lemma 5.1(i) that there exists a geodesic $y^{[p-K_0N_0]}s'$ satisfying $(y^{[p-K_0N_0]}s')\pi = (y^{[p]}s)\pi = (xv_2)\pi$. To complete the proof, it suffices to show that

$$|y \wedge \overline{xv_2}| \geq p - K_0N_0. \tag{8}$$

Indeed, together with the corresponding inequality for $y' = \overline{uvw'}$, this implies

$$|\overline{uvw} \wedge \overline{uvw'}| \geq p - K_0N_0 \geq |v_2| - K_0N_0 = |v| - K_0N_0 - N_0|u|$$

and we obtain the desired inequality.

To prove (8), we consider the geodesic $y^{[p-K_0N_0]}s'$. Since $(y^{[p-K_0N_0]}s')\pi = (xv_2)\pi$, we get $\overline{xv_2} \leq_{sl} y^{[p-K_0N_0]}s'$ and so $\overline{xv_2}^{[p-K_0N_0]} \leq_{sl} y^{[p-K_0N_0]}$. On the other hand, xv_2w is also a geodesic, hence $y = \overline{uvw} = \overline{xv_2w} \leq_{sl} \overline{xv_2}w$ yields $y^{[p-K_0N_0]} \leq_{sl} \overline{xv_2}^{[p-K_0N_0]}$. Therefore $y^{[p-K_0N_0]} = \overline{xv_2}^{[p-K_0N_0]}$ and so (8) holds as required. □

6 A new model for the boundary

We can now present a new model for the boundary of a finitely generated virtually free group which will prove itself fit do study infinite fixed points in forthcoming sections. The notion of boundary is indeed one of the important features associated to hyperbolic groups. To present it, we shall define a second distance in G by means of the *Gromov product* (taking 1 as basepoint). We keep all the notation introduced in Section 5. In particular, G is a finitely generated virtually free group and $L = \mathrm{Irr}\,\mathcal{R}'$.

Given $g, h \in G$, we define

$$(g|h) = \frac{1}{2}(d_1(1,g) + d_1(1,h) - d_1(g,h)).$$

Fix $\varepsilon > 0$ such that $\varepsilon\delta \leq \frac{1}{5}$. Write $z = e^{\varepsilon}$ and define

$$\rho(g,h) = \begin{cases} z^{-(g|h)} & \text{if } g \neq h \\ 0 & \text{otherwise} \end{cases}$$

for all $g, h \in G$. In general, ρ is not a distance because it fails the triangular inequality. This problem is overcome by defining

$$d_2(g,h) = \inf\{\rho(g_0,g_1) + \ldots + \rho(g_{n-1},g_n) \mid g_0 = g,\ g_n = h;\ g_1,\ldots,g_{n-1} \in G\}.$$

By [21, Proposition 5.16] (see also [9, Proposition 7.10]), d_2 is a distance on G and the inequalities

$$\frac{1}{2}\rho(g,h) \leq d_2(g,h) \leq \rho(g,h) \tag{9}$$

hold for all $g, h \in G$.

In general, the metric space (G, d_2) is not complete. Its completion $(\widehat{G}, \widehat{d_2})$ is essentially unique, and $\partial G = \widehat{G} \setminus G$ is the *boundary* of G. The elements of the boundary admit several standard descriptions, such as equivalence classes of rays (infinite words whose finite factors are geodesics) when two rays are equivalent if the Hausdorff distance between them is finite [9, Section 7.1]. We won't need precise definitions for these concepts or $\widehat{d_2}$ since, as we shall see next, we can get a simpler description of $\widehat{G}$ for virtually free groups.

Lemma 6.1 *There exists some $M_0 > 0$ such that, for all $g, h \in G$:*

(i) $|\overline{g}| \leq |\overline{g} \wedge \overline{gh}| + K_0N_0 + N_0|\overline{h}|$;

(ii) $d_1(g,h) \geq \frac{|\overline{g}| - |\overline{g} \wedge \overline{h}|}{N_0} - K_0$;

(iii) $|\overline{g} \wedge \overline{h}| \leq (g|h) \leq |\overline{g} \wedge \overline{h}| + M_0$.

Proof. (i) By applying Lemma 5.1 to the product $\overline{g}\overline{h}$, there exists some factorization $\overline{g} = vz$ and some geodesic $vw \in (gh)\pi^{-1}$ such that $|v| \geq |\overline{g}| - N_0|\overline{h}|$. Now we apply Lemma 5.3 to $u = \overline{gh}$ and vw to get $|u \wedge v| \geq |v| - K_0N_0$. Hence

$$|\overline{g} \wedge \overline{gh}| = |u \wedge v| \geq |v| - K_0N_0 \geq |\overline{g}| - N_0|\overline{h}| - K_0N_0$$

and (i) holds.

(ii) Let $u = \overline{g} \wedge \overline{h}$. Applying (i) to g and $g^{-1}h$, and in view of $d_1(g,h) = |\overline{g^{-1}h}|$, we get

$$|\overline{g}| \leq |\overline{g} \wedge \overline{h}| + K_0N_0 + N_0d_1(g,h)$$

and so (ii) holds.

(iii) We define $M_0 = \delta + (2\delta + 1 + K_0)N_0 - \frac{1}{2}$, assuming that $\mathrm{Geo}_A(G)$ is δ-hyperbolic. Let $u = \overline{g} \wedge \overline{h}$, and write $\overline{g} = uv$, $\overline{h} = uw$. It is easy to check that

$$(g|h) = \frac{1}{2}(d_1(1,g) + d_1(1,h) - d_1(g,h)) = \frac{1}{2}(|u| + d_1(u\pi,g) + |u| + d_1(u\pi,h) - d_1(g,h)).$$

Since $d_1(g,h) \leq d_1(g,u\pi) + d_1(u\pi,h)$, we get $|\overline{g} \wedge \overline{h}| = |u| \leq (g|h)$.

Consider now the geodesic triangle determined by the paths

$$P_1 : u\pi \xrightarrow{v} g, \quad P_2 : u\pi \xrightarrow{w} h, \quad P_3 : g \xrightarrow{\overline{g^{-1}h}} h.$$

Since $\mathrm{Geo}_A(G)$ is δ-hyperbolic, then

$$d_1(q, V(P_1) \cup V(P_2)) < \delta \text{ for every } q \in V(P_3). \tag{10}$$

Assume that $P_3 : g = q_0 \xrightarrow{a_1} \ldots \xrightarrow{a_n} q_n = h$ with $a_i \in \widetilde{A}$. Since $d_1(q_0, V(P_1)) = 0 < \delta$ and $d_1(q_n, V(P_2)) = 0 < \delta$, it follows from (10) that there exist some $j \in \{0, \ldots, n-1\}$ and $p_1 \in V(P_1)$, $p_2 \in V(P_2)$ such that $d_1(q_j, p_1), d_1(q_{j+1}, p_2) \leq \delta$. Since P_1 and P_2 are geodesics, we get

$$\begin{aligned}(g|h) &= \tfrac{1}{2}(d_1(1,g) + d_1(1,h) - d_1(g,h)) \\ &= \tfrac{1}{2}(|u| + d_1(u\pi, p_1) + d_1(p_1, g) \\ &\qquad +|u| + d_1(u\pi, p_2) + d_1(p_2, h) - d_1(g, q_j) - 1 - d_1(q_{j+1}, h)) \\ &= |\overline{g} \wedge \overline{h}| + \tfrac{1}{2}(d_1(u\pi, p_1) + d_1(u\pi, p_2)) \\ &\qquad +\tfrac{1}{2}(d_1(p_1, g) - d_1(g, q_j)) + \tfrac{1}{2}(d_1(p_2, h) - d_1(q_{j+1}, h)) - \tfrac{1}{2}.\end{aligned}$$

Since $d_1(p_1, g) \leq d_1(p_1, q_j) + d_1(q_j, g) \leq \delta + d_1(q_j, g)$, we have

$$\frac{1}{2}(d_1(p_1, g) - d_1(g, q_j)) \leq \frac{\delta}{2}.$$

Similarly,

$$\frac{1}{2}(d_1(p_2, h) - d_1(q_{j+1}, h)) \leq \frac{\delta}{2}.$$

Out of symmetry, it suffices to show that $d_1(u\pi, p_1) \leq (2\delta + 1 + K_0)N_0$.

Applying (ii) to p_1 and p_2, we get

$$d_1(p_1, p_2) \geq \frac{|\overline{p_1}| - |\overline{p_1} \wedge \overline{p_2}|}{N_0} - K_0.$$

Since $\overline{p_1}$ (respectively $\overline{p_2}$) is a prefix of $\overline{g}$ (respectively $\overline{h}$), it follows easily that $\overline{p_1} \wedge \overline{p_2} = u$ and $|\overline{p_1}| - |\overline{p_1} \wedge \overline{p_2}| = d_1(u\pi, p_1)$. Hence

$$\begin{aligned}d_1(u\pi, p_1) &\leq (d_1(p_1, p_2) + K_0)N_0 \leq (d_1(p_1, q_j) + d_1(q_j, q_{j+1}) + d_1(q_{j+1}, p_2) + K_0)N_0 \\ &\leq (2\delta + 1 + K_0)N_0\end{aligned}$$

and we are done. $\square$

We recall that an automaton is said to be trim if every vertex occurs in some successful path. Let $\mathcal{A} = (Q, q_0, T, E)$ be a finite trim deterministic $\widetilde{A}$-automaton recognizing L (e.g. the minimal automaton of L, see [1]). Since L is factorial, we must have $T = Q$. Let

$$\partial L = \{\alpha \in \widetilde{A}^\omega \mid \alpha^{[n]} \in L \text{ for every } n \in \mathbb{N}\}.$$

Equivalently, since $\mathcal{A}$ is trim and deterministic, and $T = Q$, we have $\partial L = L_\omega(\mathcal{A})$. Write $\widehat{L} = L \cup \partial L$. We define a mapping $d_3 : \widehat{L} \times \widehat{L} \to \mathbb{R}_0^+$ by

$$d_3(\alpha, \beta) = \begin{cases} 2^{-|\alpha \wedge \beta|} & \text{if } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases}$$

It is immediate that d_3 is a distance in $\widehat{L}$, indeed an ultrametric since

$$|\alpha \wedge \gamma| \geq \min\{|\alpha \wedge \beta|, |\beta \wedge \gamma|\}$$

holds for all $\alpha, \beta, \gamma \in \widehat{L}$. We shall commit a slight abuse of notation by denoting also by d_3 the restriction of d_3 to $L \times L$.

Proposition 6.2 *(i) The mutually inverse mappings* $(G, d_2) \to (L, d_3) : g \mapsto \overline{g}$ *and* $(L, d_3) \to (G, d_2) : u \mapsto u\pi$ *are uniformly continuous;*

(ii) $(\widehat{L}, d_3)$ *is the completion of* (L, d_3)*;*

(iii) $(\partial L, d_3)$ *is homeomorphic to the boundary of* G.

Proof. (i) In view of (9), it suffices to show that

$$\forall M > 0 \; \exists N > 0 : \; ((g|h) > N \Rightarrow |\overline{g} \wedge \overline{h}| > M),$$

$$\forall M > 0 \; \exists N > 0 : \; (|\overline{g} \wedge \overline{h}| > N \Rightarrow (g|h) > M).$$

Now we apply Lemma 6.1(iii).

(ii) Let $(\alpha_n)_n$ be a Cauchy sequence in $(\widehat{L}, d_3)$. For every $k \in \mathbb{N}$, the sequence $(\alpha_n^{[k]})_n$ stabilizes when $n \to +\infty$. Moreover, $\lim_{n\to+\infty} \alpha_n^{[k]}$ is a prefix of $\lim_{n\to+\infty} \alpha_n^{[k+1]}$. Let $\beta \in A^\infty$ be the unique word satisfying $\beta^{[k]} = \lim_{n\to+\infty} \alpha_n^{[k]}$ for every $k \in \mathbb{N}$. It is immediate that $\beta \in \widehat{L}$ and $\beta = \lim_{n\to+\infty} \alpha_n$, hence $(\widehat{L}, d_3)$ is complete. Since $\alpha = \lim_{n\to+\infty} \alpha^{[n]}$ for every $\alpha \in \partial L$, $(\widehat{L}, d_3)$ is the completion of (L, d_3).

(iii) By (i) and (ii), the uniformly continuous mappings $(G, d_2) \to (L, d_3) : g \mapsto \overline{g}$ and $(L, d_3) \to (G, d_2) : u \mapsto u\pi$ admit (unique) continuous extensions to their completions (see [5, Section XIV.6]), say

$$\Phi : \widehat{G} \to \widehat{L}, \quad \Psi : \widehat{L} \to \widehat{G}.$$

Hence $\Phi\Psi$ is a continuous extension of the identity on G to its completion $\widehat{G}$. Since such an extension is unique, $\Phi\Psi$ must be the identity mapping on $\widehat{G}$. Similarly, $\Psi\Phi$ must be the identity mapping on $\widehat{L}$ and so Φ and Ψ are mutually inverse homeomorphisms. Therefore the restriction $\Phi|_{\partial G} : \partial G \to \partial L$ must be also a homeomorphism. □

We have just proved that our construction of $\widehat{L}$ constitutes a model for the hyperbolic completion of G. But we must import also to $\widehat{L}$ the algebraic operations of $\widehat{G}$ since we shall be considering homomorphisms soon. Clearly, the binary operation on L is defined as

$$L \times L \to L : (u, v) \mapsto \overline{uv}$$

so that $(G, d_2) \to (L, d_3) : g \mapsto \overline{g}$ is also a group isomorphism. But there is another important algebraic operation involved. Indeed, for every $g \in G$, the left translation $\tau_g : G \to G : x \mapsto gx$ is uniformly continuous for d_2 and so admits a continuous extension $\widehat{\tau}_g : \widehat{G} \to \widehat{G}$. It follows that the left action of G in its boundary, $G \times \partial G \to \partial G : (g, \alpha) \mapsto \alpha\widehat{\tau}_g$, is continuous. We can also replicate this operation in $\widehat{L}$ as follows:

Proposition 6.3 *Let* $u \in L$. *Then* $\tau_u : L \to L : v \mapsto \overline{uv}$ *is uniformly continuous.*

Proof. It suffices to show that

$$\forall M > 0 \; \exists N > 0 : \; (|v \wedge w| > N \Rightarrow |\overline{uv} \wedge \overline{uv}| > M).$$

By Proposition 5.4(ii), we can take $N = M + K_0N_0 + N_0|u|$. □

Therefore τ_u admits a continuous extension $\widehat{\tau}_u : \widehat{L} \to \widehat{L}$ and the left action $L \times \partial L \to \partial L : (u, \alpha) \mapsto \alpha\widehat{\tau}_u$ is continuous. Write $\overline{u\alpha} = \alpha\widehat{\tau}_u$. For every $\alpha \in \partial L$, we have

$$\overline{u\alpha} = \overline{u \lim_{n\to+\infty} \alpha^{[n]}} = \lim_{n\to+\infty} \overline{u\alpha^{[n]}},$$

hence $(\widehat{L}, d_3)$ serves as a model for $(\widehat{G}, \widehat{d_2})$ both topologically and algebraically. From now on, we shall pursue our work within $(\widehat{L}, d_3)$.

7 Uniformly continuous endomorphisms

We keep all the notation introduced in Section 5. In particular, G is a finitely generated virtually free group and $L = \operatorname{Irr}\mathcal{R}'$. Following the program announced above, we work within $(\widehat{L}, d_3)$.

Given an endomorphism φ of G, we denote by $\overline{\varphi}$ the corresponding endomorphism of L for the binary operation induced by the product in G, i.e. $u\overline{\varphi} = \overline{(u\pi)\varphi}$. To simplify notation, we shall often write $u\varphi$ instead of $u\pi\varphi$ for $u \in \widetilde{A}^*$.

We say that φ satisfies the *bounded reduction property* if $\{|u\overline{\varphi}| - |u\overline{\varphi} \wedge (uv)\overline{\varphi}| : uv \in L\}$ is bounded. In that case, we denote its maximum by B_φ. This property was considered originally for free group automorphisms by Cooper [4].

We fix also the notation $D_\varphi = \max\{|\overline{a\varphi}| : a \in \widetilde{A}\}$.

Theorem 7.1 *Let φ be an endomorphism φ of G with finite kernel. Then φ satisfies the bounded reduction property.*

Proof. Suppose that φ does not satisfy the bounded reduction property. Then

$$\forall m \in \mathbb{N}\ \exists u_m v_m \in L : |u_m\overline{\varphi}| - |u_m\overline{\varphi} \wedge (u_m v_m)\overline{\varphi}| > m.$$

Let $X_0 = (K_0 + D_\varphi)N_0$. We claim that

$$\begin{aligned} \forall m \in \mathbb{N}\ \exists u'_m v'_m \in L : (&|u'_m\overline{\varphi}| - |(u'_m v'_m)\overline{\varphi}| > m \\ &\text{and } |(u'_m v'_m)\overline{\varphi}| - |u'_m\overline{\varphi} \wedge (u'_m v'_m)\overline{\varphi}| \leq X_0). \end{aligned} \tag{11}$$

Indeed, let $m \in \mathbb{N}$. Take $n = m + X_0$ and write $v_n = a_1 \ldots a_k$ $(a_i \in \widetilde{A})$. For $i = 0, \ldots, k$, let $w_i = (u_n a_1 \ldots a_i)\overline{\varphi}$. Let j denote the smallest i such that $|u_n\overline{\varphi} \wedge w_i| \leq |u_n\overline{\varphi} \wedge (u_n v_n)\overline{\varphi}|$. Take $u'_m = u_n$ and $v'_m = a_1 \ldots a_{j-1}$ (since $j > 0$). Since L is factorial, we have $u'_m v'_m \in L$.

Now by minimality of j we get

$$|u_n\overline{\varphi} \wedge w_{j-1}| > |u_n\overline{\varphi} \wedge (u_n v_n)\overline{\varphi}|.$$

Since $|u_n\overline{\varphi} \wedge w_j| \leq |u_n\overline{\varphi} \wedge (u_n v_n)\overline{\varphi}|$, it follows that

$$|w_{j-1} \wedge w_j| \leq |u_n\overline{\varphi} \wedge (u_n v_n)\overline{\varphi}|.$$

Applying Lemma 6.1(i) to $w_{j-1}\pi$ and $a_j\varphi$, we get

$$\begin{aligned} |w_{j-1}| &\leq |w_{j-1} \wedge w_j| + K_0N_0 + N_0|\overline{a_j\varphi}| \leq |w_{j-1} \wedge w_j| + X_0 \\ &\leq |u_n\overline{\varphi} \wedge (u_n v_n)\overline{\varphi}| + X_0 < |u_n\overline{\varphi}| - n + X_0 = |u_n\overline{\varphi}| - m \end{aligned}$$

and so $|u'_m\overline{\varphi}| - |(u'_m v'_m)\overline{\varphi}| = |u_n\overline{\varphi}| - |w_{j-1}| > m$.

Suppose that $|w_{j-1}| - |u_n\overline{\varphi} \wedge w_{j-1}| > X_0$. Since we have seen above that $|w_{j-1}| \leq |w_{j-1} \wedge w_j| + X_0$, we get $|u_n\overline{\varphi} \wedge w_{j-1}| < |w_{j-1} \wedge w_j|$, in contradiction with $|w_{j-1} \wedge w_j| \leq |u_n\overline{\varphi} \wedge (u_n v_n)\overline{\varphi}| < |u_n\overline{\varphi} \wedge w_{j-1}|$. Thus

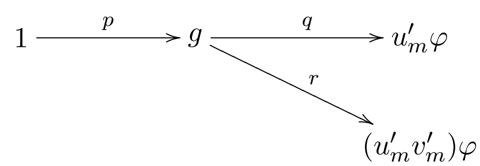
$$|(u'_m v'_m)\overline{\varphi}| - |u'_m\overline{\varphi} \wedge (u'_m v'_m)\overline{\varphi}| = |w_{j-1}| - |u_n\overline{\varphi} \wedge w_{j-1}| \leq X_0$$

and so (11) holds.

We prove that

$$\forall m \in \mathbb{N}\ \exists u''_m v''_m \in L : |u''_m\overline{\varphi}| > m \text{ and } |(u''_m v''_m)\overline{\varphi}| \leq X_0 + N_0 D_\varphi. \tag{12}$$

Indeed, let $m \in \mathbb{N}$. We have in $\Gamma_A(G)$ geodesics



where $pq = u'_m\overline{\varphi}$, $pr = (u'_m v'_m)\overline{\varphi}$ and $p = u'_m\overline{\varphi} \wedge (u'_m v'_m)\overline{\varphi}$. Assume that $u'_m = a_1 \ldots a_k$ $(a_i \in \widetilde{A})$. Let

$$I = \{i \in \{0, \ldots, k\} \mid \text{ there exists a geodesic } (a_1 \ldots a_i)\varphi \longrightarrow g \xrightarrow{q} u'_m\varphi \text{ in } \Gamma_A(G)\}.$$

Clearly, $0 \in I$. We claim that

$$(i - 1 \in I \text{ and } d_1((a_1 \ldots a_{i-1})\varphi, g) > N_0 D_\varphi) \Rightarrow i \in I \tag{13}$$

holds for $i = 1, \ldots, k$. Indeed, assume that $i - 1 \in I$ and $(a_1 \ldots a_{i-1})\varphi \xrightarrow{y} g \xrightarrow{q} u'_m\varphi$ is a geodesic with $y \in L$. Applying Lemma 5.1(ii) to the word $a_i^{-1}\overline{\varphi}$ and the geodesic yq, it follows that there exists some geodesic $(a_1 \ldots a_{i-1})\varphi \xrightarrow{z} u'_m\varphi$ such that z and u share a suffix of length $\geq |yq| - N_0|a_i^{-1}\overline{\varphi}| \geq |yq| - N_0 D_\varphi > |q|$. Since $\Gamma_A(G)$ is deterministic, then our geodesic $(a_1 \ldots a_{i-1})\varphi \xrightarrow{z} u'_m\varphi$ factors through g and so (13) holds.

Since $k \notin I$ due to $|q| > 0$, it follows from (13) that $d_1((a_1 \ldots a_i)\varphi, g) \leq N_0 D_\varphi$ for some $i \in \{1, \ldots, k\}$. let j denote the smallest such i. We define $u''_m = a_{j+1} \ldots a_k$ and $v''_m = v'_m$. Since L is factorial and $u'_m v'_m \in L$, we have also $u''_m v''_m \in L$.

By minimality of j, we have $d_1((a_1 \ldots a_i)\varphi, g) > N_0 D_\varphi$ for $i = 0, \ldots, j - 1$. By (13), we get $1, \ldots, j \in I$ and so there exists a geodesic $(a_1 \ldots a_j)\varphi \longrightarrow g \xrightarrow{q} u'_m\varphi$ in $\Gamma_A(G)$. Hence

$$\begin{aligned} |u''_m\overline{\varphi}| &= d_1(1, u''_m\varphi) = d_1((a_1 \ldots a_j)\varphi, u'_m\varphi) \\ &\geq |q| = |u'_m\overline{\varphi}| - |(u'_m v'_m)\overline{\varphi}| > m. \end{aligned}$$

Finally,

$$\begin{aligned} |(u''_m v''_m)\overline{\varphi}| &= d_1(1, (u''_m v''_m)\varphi) = d_1((a_1 \ldots a_j)\varphi, (u'_m v'_m)\varphi) \\ &\leq d_1((a_1 \ldots a_j)\varphi, g) + d_1(g, (u'_m v'_m)\varphi) \leq N_0 D_\varphi + |r| \\ &= N_0 D_\varphi + |(u'_m v'_m)\overline{\varphi}| - |u'_m\overline{\varphi} \wedge (u'_m v'_m)\overline{\varphi}| \leq N_0 D_\varphi + X_0 \end{aligned}$$

and so (12) holds.

Now, since $|(u''_m v''_m)\overline{\varphi}|$ is bounded, $u''_m v''_m \in L$ and $\mathrm{Ker}\varphi$ is finite, then $|u''_m v''_m|$ must be bounded and so must be $|u''_m|$. This implies that $|u''_m\overline{\varphi}|$ must be bounded, contradicting $|u''_m\overline{\varphi}| > m$. Therefore φ satisfies the bounded reduction property. $\square$

Proposition 7.2 *The following conditions are equivalent for a nontrivial endomorphism φ of G:*

(i) φ is uniformly continuous for d_2;

(ii) Kerφ *is finite;*

Proof. (i) $\Rightarrow$ (ii). Suppose that Kerφ is infinite. In view of (9), it suffices to show that there exists some $\eta > 0$ such that

$$\forall \xi > 0\ \exists g, h \in G\ (\rho(g, h) < \xi \text{ and } \rho(g\varphi, h\varphi) \geq \eta).$$

By (9), we only need to show that there exists some $M \in \mathbb{N}$ such that

$$\forall N \in \mathbb{N}\ \exists g, h \in G\ ((g|h) > N \text{ and } g\varphi \neq h\varphi \text{ and } ((g\varphi)|(h\varphi)) \leq M).$$

Take $M = (g_0\varphi|1) = 0$ and fix $g_0 \in G \setminus \text{Ker}\varphi$. We prove the claim by showing that

$$\forall N \in \mathbb{N}\ \exists h \in \text{Ker}\varphi :\ ((hg_0)|h) > N. \tag{14}$$

Let $N \in \mathbb{N}$. By Lemma 6.1(iii), we have $|\overline{hg_0} \wedge \overline{h}| \leq ((hg_0)|h)$ for every $h \in G$, hence we only need to find out $h \in \text{Ker}\varphi$ satisfying $|\overline{hg_0} \wedge \overline{h}| > N$. By Lemma 6.1(i), we have $|\overline{hg_0} \wedge \overline{h}| \geq |\overline{h}| - K_0N_0 - N_0|\overline{g_0}|$, hence it suffices that $|\overline{h}| > N + K_0N_0 + N_0|\overline{g_0}|$ for some $h \in \text{Ker}\varphi$, and that is ensured by Kerφ being infinite. Thus (14) holds as required.

(ii) $\Rightarrow$ (i). Suppose that φ is not uniformly continuous for d_2. In view of (9), there exists some $\eta > 0$ such that

$$\forall \xi > 0\ \exists g, h \in G\ (\rho(g, h) < \xi \text{ and } \rho(g\varphi, h\varphi) \geq \eta).$$

Hence, by (9), there exists some $M \in \mathbb{N}$ such that

$$\forall N \in \mathbb{N}\ \exists g, h \in G\ ((g|h) > N \text{ and } g\varphi \neq h\varphi \text{ and } ((g\varphi)|(h\varphi)) \leq M).$$

In view of Lemma 6.1(iii), we have that

$$\forall n \in \mathbb{N}\ \exists u_n, v_n \in L\ (|u_n \wedge v_n| > n \text{ and } u_n\overline{\varphi} \neq v_n\overline{\varphi} \text{ and } |u_n\overline{\varphi} \wedge v_n\overline{\varphi}| \leq M).$$

Let $w_n = u_n \wedge v_n \in L$. Then either $w_n\overline{\varphi} \neq u_n\overline{\varphi}$ or $w_n\overline{\varphi} \neq v_n\overline{\varphi}$. Without loss of generality, we may assume that $w_n\overline{\varphi} \neq u_n\overline{\varphi}$. Suppose that $|w_n\overline{\varphi}| > M + B_\varphi$. By definition of B_φ, we get $|w_n\overline{\varphi}| - |w_n\overline{\varphi} \wedge u_n\overline{\varphi}| \leq B_\varphi$ and so $|w_n\overline{\varphi} \wedge u_n\overline{\varphi}| > M$. Similarly, $|w_n\overline{\varphi} \wedge v_n\overline{\varphi}| > M$ and so $|u_n\overline{\varphi} \wedge v_n\overline{\varphi}| > M$, a contradiction. Therefore $|w_n\overline{\varphi}| \leq M + B_\varphi$ for every n. Since $|w_n| > n$ and L is a cross-section for π, it follows that Kerφ is infinite. □

Given a uniformly continuous endomorphism φ of (G, d_2), then $\overline{\varphi} : L \to L$ is uniformly continuous for d_3. Since $\widehat{L}$ is the completion of (L, d_3), then $\overline{\varphi}$ admits a unique continuous extension $\Phi : \widehat{L} \to \widehat{L}$. By continuity, we have

$$\alpha\Phi = (\lim_{n\to+\infty} \alpha^{[n]})\Phi = \lim_{n\to+\infty} \alpha^{[n]}\overline{\varphi}. \tag{15}$$

Corollary 7.3 *Let φ be a uniformly continuous endomorphism of G and let $u\alpha \in \partial L$. Then $|u\overline{\varphi}| - |u\overline{\varphi} \wedge (u\alpha)\Phi| \leq B_\varphi$.*

Proof. We have $(u\alpha)\Phi = \lim_{n\to+\infty}(u\alpha^{[n]})\overline{\varphi}$ by (15). In view of Proposition 7.2, we have $\lim_{n\to+\infty} |(u\alpha^{[n]})\overline{\varphi}| = +\infty$, hence $|u\overline{\varphi} \wedge (u\alpha)\Phi| = |u\overline{\varphi} \wedge (u\alpha^{[m]})\overline{\varphi}|$ for sufficiently large m. Since $u\alpha^{[m]} \in L$, the claim follows from the definition of $\mathcal{B}_\varphi$. □

8 Infinite fixed points

Keeping all the notation and assumptions introduced in the preceding sections, we fix now a uniformly continuous endomorphism φ of the finitely generated virtually free group G. We adapt notation introduced in [13] for free groups, and the proofs are also adaptations of proofs in [17].

Given $u \in L$, let $u\sigma = u \wedge u\overline{\varphi}$ and write

$$u = (u\sigma)(u\tau), \quad u\overline{\varphi} = (u\sigma)(u\rho).$$

Define also

$$u\sigma' = \wedge\{(uv)\sigma \mid uv \in L\}$$

and write $u\sigma = (u\sigma')(u\sigma'')$.

Lemma 8.1 *Let $uv \in L$. Then:*

(i) $|u\sigma''| \leq B_{\varphi}$;

(ii) $|u\sigma| - |u\sigma \wedge (uv)\overline{\varphi}| \leq |u\sigma''|$;

(iii) $(uv)\overline{\varphi} = (u\sigma')\overline{(u\sigma'')(u\rho)(v\overline{\varphi})}$;

(iv) $(uv)\sigma' = (u\sigma')(\bigwedge_{uvz \in L} (\, \overline{(u\sigma'')(u\rho)((vz)\overline{\varphi})} \wedge (u\sigma'')(u\tau)vz\,)$.

Proof. (i) We may assume that $|u\sigma| > B_{\varphi}$. Let v denote the suffix of length B_{φ} of $u\sigma$ and write $u\sigma = u'v$. Suppose that $uw \in L$. It suffices to show that u' is a prefix of $(uw)\overline{\varphi}$, and this follows from

$$|u'v(u\rho)| - |u'v(u\tau) \wedge (uw)\overline{\varphi}| = |u\overline{\varphi}| - |u\overline{\varphi} \wedge (uw)\overline{\varphi}| \leq B_{\varphi}$$

and $|v| = B_{\varphi}$.

(ii) Since $u\sigma'$ is a prefix of $u\sigma \wedge (uv)\overline{\varphi}$.

(iii) Since $u\sigma'$ is a prefix of $(uv)\overline{\varphi}$ and both sides of the equality are equivalent in G.

(iv) Since $u\sigma'$ is a prefix of $(uv)\sigma'$ by (iii). $\square$

For every $u \in L$, we define

$$u\xi = (u\sigma'', u\tau, u\rho, q_0u).$$

Note that there exists precisely one path of the form $q_0 \xrightarrow{u} q_0u$ in $\mathcal{A}$.

Lemma 8.2 *Let $u, v \in L$ be such that $u\xi = v\xi$ and let $a \in \widetilde{A}$, $\alpha \in \widetilde{A}^{\infty}$. Then:*

(i) *$ua \in L$ if and only if $va \in L$;*

(ii) *if $ua \in L$, then $(ua)\xi = (va)\xi$*

(iii) $\overline{uv^{-1}} \in \mathit{Fix}\overline{\varphi}$;

(iv) *$u\alpha \in \widehat{L}$ if and only if $v\alpha \in \widehat{L}$;*

(v) $u\alpha \in \mathit{Fix}\,\Phi$ *if and only if* $v\alpha \in \mathit{Fix}\,\Phi$*;*

(vi) if $\alpha \in \widehat{L}$*, then* $\alpha = \lim_{n\to+\infty} \overline{\alpha^{[n]}u}$*.*

Proof. (i) Since $u\xi = v\xi$ implies $q_0u = q_0v$.

(ii) Clearly, $q_0u = q_0v$ yields $q_0ua = q_0va$. Considering $v = a$ in Lemma 8.1(iii), we may write $(ua)\sigma = (u\sigma')u'$ and deduce that u', $(ua)\tau$ and $(ua)\rho$ are all determined by $u\xi$. Hence $(ua)\tau = (va)\tau$, $(ua)\rho = (va)\rho$ and $u' = v'$.

Finally, since $q_0u = q_0v$, we have $uaz \in L$ if and only if $vaz \in L$. It follows from Lemma 8.1(iv) that there exists a word $x \in L$ which satisfies both $(ua)\sigma' = (u\sigma')x$ and $(va)\sigma' = (v\sigma')x$. Now $(u\sigma')u' = (ua)\sigma = ((ua)\sigma')((ua)\sigma'') = (u\sigma')x((ua)\sigma'')$, hence $u' = x((ua)\sigma'')$. Similarly, $v' = x((va)\sigma'')$. Since $u' = v'$, we get $(ua)\sigma'' = (va)\sigma''$ and so $(ua)\xi = (va)\xi$.

(iii) Since

$$\begin{aligned}\overline{(uv^{-1})\varphi} &= \overline{(u\varphi)(v\varphi)^{-1}} = \overline{(u\sigma)(u\rho)(v\rho)^{-1}(v\sigma)^{-1}} = \overline{(u\sigma)(v\sigma)^{-1}} \\ &= \overline{(u\sigma)(u\tau)(v\tau)^{-1}(v\sigma)^{-1}} = \overline{uv^{-1}}.\end{aligned}$$

(iv) We have $u\alpha \in \widehat{L}$ if and only if $u\alpha^{[n]} \in L$ for every $n \in \mathbb{N}$. Now we use (i) and induction on n.

(v) We have $u\alpha = (u\sigma')(u\sigma'')(u\tau)\alpha$ and in view of Corollary 7.3 and (15) also

$$(u\alpha)\Phi = (u\sigma') \lim_{n\to+\infty} \overline{(u\sigma'')(u\rho)(\alpha^{[n]}\overline{\varphi})}.$$

Hence $u\alpha \in \mathrm{Fix}\,\Phi$ depends just on $u\xi$ and α and we are done.

(vi) Let $m = K_0N_0 + N_0|u|$. By Lemma 6.1(i), we have $|\alpha^{[n]} \wedge \overline{\alpha^{[n]}u}| \geq n - m$ for every n, hence $\alpha = \lim_{n\to+\infty} \alpha^{[n-m]} = \lim_{n\to+\infty} \overline{\alpha^{[n]}u}$. $\square$

Given $X \subseteq A^\infty$, write

$$\mathrm{Pref}\, X = \{u \in A^* \mid u\alpha \in X \text{ for some } \alpha \in A^\infty\}.$$

Recall the finite trim deterministic $\widetilde{A}$-automaton $\mathcal{A} = (Q, q_0, Q, E)$ recognizing L. We build a (possibly infinite) $\widetilde{A}$-automaton $\mathcal{A}'_\varphi = (Q', q'_0, T', E')$ by taking

- $Q' = \{u\xi \mid u \in \mathrm{Pref}\,\mathrm{Fix}\,\Phi\}$;
- $q'_0 = 1\xi$;
- $T' = \{u\xi \in Q' \mid u\tau = u\rho = 1\}$;
- $E' = \{(u\xi, a, v\xi) \in Q' \times \widetilde{A} \times Q' \mid v = ua \in \mathrm{Pref}\,\mathrm{Fix}\,\Phi\}$.

Note that $\mathcal{A}'_\varphi$ is deterministic by Lemma 8.2(ii) and is also *accessible*: if $u \in \mathrm{Pref}\,\mathrm{Fix}\,\Phi$, then there exists a path $q'_0 \xrightarrow{u} u\xi$ and so every vertex can be reached from the initial vertex.

Let S denote the set of all vertices $q \in Q'$ such that there exist at least two edges in $\mathcal{B}'_\varphi$ leaving q. Let Q'' denote the set of all vertices $q \in Q'$ such that there exists some path $q \longrightarrow p \in S \cup T'$. We define $\mathcal{A}''_\varphi = (Q'', q''_0, T'', E'')$ by taking $q''_0 = q'_0$, $T'' = T' \cap Q''$ and $E'' = E' \cap (Q'' \times \widetilde{A} \times Q'')$.

Lemma 8.3 *S is finite.*

Proof. In view of Lemma 8.1, the unique components of $u\xi$ that may assume infinitely many values are $u\tau$ and $u\rho$. Moreover, we claim that

$$u\tau \neq 1 \Rightarrow |u\rho| \leq B_\varphi \tag{16}$$

holds for every $u \in \operatorname{Pref}\operatorname{Fix}\Phi$. Indeed, suppose that $u\tau \neq 1$ and $|u\rho| > B_\varphi$. Write $\alpha = u\beta$ for some $\alpha \in \operatorname{Fix}\Phi$. In view of Proposition 7.3, $|u\rho| > B_\varphi$ yields $|(u\beta)\Phi \wedge u\overline{\varphi}| > |u\sigma|$ and now $u\tau \neq 1$ yields $((u\beta)\Phi \wedge u\beta) = (u\overline{\varphi} \wedge u) = u\sigma$. Since $\beta \neq 1$, this contradicts $\alpha \in \operatorname{Fix}\Phi$. Therefore (16) holds.

It is also easy to see that

$$|u\rho| > B_\varphi \Rightarrow u\xi \notin S \tag{17}$$

for every $u \in \operatorname{Pref}\operatorname{Fix}\Phi$. Indeed, if $|u\rho| > B_\varphi$ and a is the first letter of $u\rho$, then, by definition of B_φ, $(u\sigma)a$ is a prefix of $(u\alpha)\Phi$ whenever $u\alpha \in \operatorname{Fix}\Phi$. Therefore any edge leaving $u\xi$ in $\mathcal{A}'_\varphi$ must have label a and so (17) holds.

In view of Proposition 7.2, we can define

$$W_0 = \max\{|u| : u \in L,\ |u\overline{\varphi}| \leq 2(B_\varphi + D_\varphi - 1)\}.$$

Let $Z_0 = B_\varphi + N_0(K_0 + W_0)D_\varphi$. To complete the proof of the lemma, it suffices to prove that

$$|u\tau| > Z_0 \Rightarrow u\xi \notin S \tag{18}$$

for every $u \in \operatorname{Pref}\operatorname{Fix}\Phi$.

Suppose that $|u\tau| > Z_0$ and $(u\xi, a, (ua)\xi), (u\xi, b, (ub)\xi) \in E'$ for some $u \in \operatorname{Pref}\operatorname{Fix}\Phi$, where $a, b \in \widetilde{A}$ are distinct. We have $(ua)\xi = v\xi$ for some $v \in \operatorname{Pref}\operatorname{Fix}\Phi$. By Lemma 8.2(v), we get $ua\alpha \in \operatorname{Fix}\Phi$ for some $\alpha \in \widehat{L}$. By (15), we get $ua\alpha = \lim_{n\to+\infty}(ua\alpha^{[n]})\overline{\varphi}$ and so $|(ua\alpha^{[n]})\overline{\varphi}| \geq |u|$ for sufficiently large n. Let

$$p = \min\{n \in \mathbb{N} : |(ua\alpha^{[n]})\overline{\varphi}| \geq |u|\}.$$

Note that $p > 0$ since $|u\tau| > Z_0$ and by (16). Since $|(ua\alpha^{[p-1]})\overline{\varphi}| < |u|$ by minimality of p, we get

$$|(ua\alpha^{[p]})\overline{\varphi}| \leq |(ua\alpha^{[p-1]})\overline{\varphi}| + D_\varphi < |u| + D_\varphi. \tag{19}$$

On the other hand,

$$|u| - |(ua\alpha^{[p]})\overline{\varphi} \wedge u| \leq B_\varphi, \tag{20}$$

otherwise, by definition of B_φ, $ua\alpha$ and $(ua\alpha)\Phi$ would differ at position $|(ua\alpha^{[p]})\overline{\varphi} \wedge u| + 1$.

Similarly, $ub\beta \in \operatorname{Fix}\Phi$ for some $\beta \in \widehat{L}$. Defining

$$q = \min\{n \in \mathbb{N} : |(ub\beta^{[n]})\overline{\varphi}| \geq |u|\},$$

we get

$$|(ub\beta^{[q]})\overline{\varphi}| < |u| + D_\varphi \tag{21}$$

and

$$|u| - |(ub\beta^{[q]})\overline{\varphi} \wedge u| \leq B_\varphi. \tag{22}$$

Write $u = u_1u_2$ with $|u_2| = B_\varphi$. Then by (19) and (20) we may write $(ua\alpha^{[p]})\overline{\varphi} = u_1x$ for some x such that $|x| < B_\varphi + D_\varphi$. Similarly, (21) and (22) yield $(ub\beta^{[q]})\overline{\varphi} = u_1y$ for some x

such that $|x| < B_\varphi + D_\varphi$. Writing $w = \overline{(\beta^{[q]})^{-1}b^{-1}a\alpha^{[p]}}$, it follows that $w\varphi = (y^{-1}x)\pi$ and so $|w\overline{\varphi}| \leq 2(B_\varphi + D_\varphi - 1)$. Hence $|w| \leq W_0$. Applying Lemma 6.1(i) to $g = (ub\beta^{[q]})\pi$ and $h = w\pi$, we get

$$|ub\beta^{[q]}| \leq |ub\beta^{[q]} \wedge ua\alpha^{[p]}| + N_0(K_0 + |w|) \leq |u| + N_0(K_0 + W_0)$$

and so $q < N_0(K_0 + W_0)$. Hence, in view of (16), we get

$$\begin{aligned} |u\tau| = |u| - |u\sigma| &\leq |(ub\beta^{[q]})\overline{\varphi}| - |u\sigma| \leq |u\overline{\varphi}| + |(b\beta^{[q]})\overline{\varphi}| - |u\sigma| \\ &\leq |u\rho| + N_0(K_0 + W_0)D_\varphi \leq B_\varphi + N_0(K_0 + W_0)D_\varphi, \end{aligned}$$

contradicting $|u\tau| > Z_0$. Thus (18) holds and the lemma is proved. □

We say that an infinite fixed point $\alpha \in \operatorname{Fix}\Phi \cap \partial L$ is *singular* if α belongs to the topological closure $(\operatorname{Fix}\varphi)^c$ of $\operatorname{Fix}\varphi$. Otherwise, α is said to be *regular*. We denote by $\operatorname{Sing}\Phi$ (respectively $\operatorname{Reg}\Phi$) the set of all singular (respectively regular) infinite fixed points of Φ.

Theorem 8.4 *Let φ be a uniformly continuous endomorphism of a finitely generated virtually free group G. Then:*

(i) the automaton $\mathcal{A}''_\varphi$ is finite;

(ii) $L(\mathcal{A}''_\varphi) = \operatorname{Fix}\overline{\varphi}$;

(iii) $L_\omega(\mathcal{A}''_\varphi) = \operatorname{Sing}\Phi$.

Proof. (i) The set T' is finite and S is finite by Lemma 8.3. On the other hand, by definition of S, there are only finitely many paths in $\mathcal{A}'_\varphi$ of the form $p \longrightarrow q$ with $p, q \in S \cup T' \cup \{q'_0\}$ and no intermediate vertex in $S \cup T' \cup \{q'_0\}$. Therefore Q'' is finite and so is $\mathcal{A}''_\varphi$.

(ii) Every $u \in L$ labels at most a unique path $q'_0 = 1\xi \xrightarrow{u} u\xi$ out of the initial vertex in $\mathcal{A}'_\varphi$. On the other hand, if $q'_0 = 1\xi \xrightarrow{u} q'$ is a path in $\mathcal{A}'_\varphi$, then the fourth component of ξ yields a path $q_0 \xrightarrow{u} q$ in $\mathcal{A}$ and so $u \in L$. Hence

$$L(\mathcal{A}'_\varphi) = \{u \in L \mid u\xi \in T'\} = \{u \in L \mid u\tau = u\rho = 1\} = \operatorname{Fix}\overline{\varphi}.$$

Since $L(\mathcal{A}''_\varphi) = L(\mathcal{A}'_\varphi)$, (ii) holds.

(iii) Let $\alpha \in L_\omega(\mathcal{A}''_\varphi)$. Then there exists some $q'' \in Q''$ and some infinite sequence $(i_n)_n$ such that $q''_0 \xrightarrow{\alpha^{[i_n]}} q''$ is a path in $\mathcal{A}''_\varphi$ for every n. Write $u = \alpha^{[i_1]}$ and let $v_n = \overline{\alpha^{[i_n]}u^{-1}}$. By Lemma 8.2(iii), we have $v_n \in \operatorname{Fix}\overline{\varphi}$ for every n. It follows from Lemma 8.2(vi) that $\alpha = \lim_{n\to+\infty} v_n$, thus $\alpha \in \operatorname{Sing}\Phi$.

Conversely, let $\alpha \in \operatorname{Sing}\Phi$. Then we may write $\alpha = \lim_{n\to+\infty} v_n$ for some sequence $(v_n)_n$ in $\operatorname{Fix}\overline{\varphi}$. Let $k \in \mathbb{N}$. For large enough n, we have $\alpha^{[k]} = v_n^{[k]}$ and so there is some path

$$q''_0 \xrightarrow{\alpha^{[k]}} q''_k \xrightarrow{w} t''_k \in T'',$$

where $\alpha^{[k]}w = v_n$. Thus $\alpha \in L_\omega(\mathcal{A}''_\varphi)$ as required. □

Recall now the continuous extensions $\widehat{\tau}_u : \widehat{L} \to \widehat{L}$ of the uniformly continuous mappings $\tau_u : L \to L : v \mapsto \overline{uv}$ defined for each $u \in L$ (see Proposition 6.3). As remarked before, this is equivalent to say that the left action $L \times \partial L \to \partial L : (u, \alpha) \mapsto \overline{u\alpha}$ is continuous. Identifying L with G and ∂L with ∂G, we have a continuous action (on the left) of G on ∂G. Clearly, this action restricts to a left action of $\mathrm{Fix}\,\varphi$ on $\mathrm{Fix}\,\Phi \cap \partial G$: if $g \in \mathrm{Fix}\,\varphi$ and $\alpha \in \mathrm{Fix}\,\Phi \cap \partial G$, with $\alpha = \lim_{n\to+\infty} g_n$ ($g_n \in G$), then

$$\begin{aligned}(g\alpha)\Phi &= (g \lim_{n\to+\infty} g_n)\Phi = (\lim_{n\to+\infty} gg_n)\Phi = \lim_{n\to+\infty}(gg_n)\varphi \\ &= \lim_{n\to+\infty}(g\varphi)(g_n\varphi) = (g\varphi)\lim_{n\to+\infty} g_n\varphi = g(\lim_{n\to+\infty} g_n)\Phi \\ &= g(\alpha\Phi) = g\alpha.\end{aligned}$$

Moreover, the $(\mathrm{Fix}\,\varphi)$-orbits of $\mathrm{Sing}\,\Phi$ and $\mathrm{Reg}\,\Phi$ are disjoint: if $\alpha \in \mathrm{Sing}\,\Phi$, we can write $\alpha = \lim_{n\to+\infty} g_n$ with the $g_n \in \mathrm{Fix}\,\varphi$ and get $g\alpha = \lim_{n\to+\infty} gg_n$ with $gg_n \in \mathrm{Fix}\,\varphi$ for every n; hence $\alpha \in \mathrm{Sing}\,\Phi \Rightarrow g\alpha \in \mathrm{Sing}\,\Phi$ and the action of g^{-1} yields the converse implication.

We can now prove the main result of this section.

Theorem 8.5 *Let φ be a uniformly continuous endomorphism of a finitely generated virtually free group G. Then $\mathrm{Reg}\,\Phi$ has finitely many $(\mathrm{Fix}\,\varphi)$-orbits.*

Proof. Let P be the set of all infinite paths $s'_0 \xrightarrow{a_1} s'_1 \xrightarrow{a_2} \dots$ in $\mathcal{A}'_\varphi$ such that:

- $s'_0 \in S \cup \{q_0\}$;
- $s'_n \notin S \cup \{q_0\}$ for every $n > 0$;
- $s'_n \neq s'_m$ whenever $n \neq m$.

By Lemma 8.3, there are only finitely many choices for s'_0. Since A is finite and $\mathcal{A}'_\varphi$ is deterministic, there are only finitely many choices for s'_1, and from that vertex onwards, the path is univocally determined due to $s'_n \notin S$ ($n \geq 1$). Hence P is finite, and we may assume that it consists of paths $p'_i \xrightarrow{\alpha_i} \dots$ for $i = 1, \dots, m$. Fix a path $q'_0 \xrightarrow{u_i} p_i$ for each i and let $X = \{u_1\alpha_1, \dots, u_m\alpha_m\} \subseteq \partial L$. We claim that $X \subseteq \mathrm{Reg}\,\Phi$.

Let $i \in \{1, \dots, m\}$ and write $\beta = u_i\alpha_i$. To show that $\beta \in \mathrm{Fix}\,\Phi$, it suffices to show that $\lim_{n\to+\infty} \beta^{[n]}\overline{\varphi} = \beta$. Let $k \in \mathbb{N}$. We must show that there exists some $r \in \mathbb{N}$ such that

$$n \geq r \Rightarrow |\beta^{[n]}\overline{\varphi} \wedge \beta| > k. \tag{23}$$

In view of Proposition 7.2, there exists some $r > k$ such that

$$n \geq r \Rightarrow |\beta^{[n]}\overline{\varphi}| > k + B_\varphi.$$

Suppose that $|\beta^{[n]}\overline{\varphi} \wedge \beta| \leq k$ for some $n \geq r$. Then $|\beta^{[n]}\sigma| \leq k$. Since $k < r \leq n$, it follows that $\beta^{[n]}\tau \neq 1$. On the other hand, since $|\beta^{[n]}\overline{\varphi}| > k + B_\varphi$, we get $|\beta^{[n]}\rho| > B_\varphi$. In view of (16), this contradicts $\beta^{[n]}\xi \in Q'$. Therefore (23) holds for our choice of r and so $X \subseteq \mathrm{Fix}\,\Phi$. Since the path $q'_0 \xrightarrow{\beta} \dots$ can visit only finitely often a given vertex, then $\beta \notin L_\omega(\mathcal{A}''_\varphi)$ and so $X \subseteq \mathrm{Reg}\,\Phi$ by Theorem 8.4(iii).

By the previous comments on $(\mathrm{Fix}\,\varphi)$-orbits, the $(\mathrm{Fix}\,\varphi)$-orbits of the elements of X must be contained in $\mathrm{Reg}\,\Phi$. We complete the proof of the theorem by proving the opposite inclusion.

Let $\beta \in \operatorname{Reg}\Phi$. By Theorem 8.4(iii), we have $\beta \notin L_\omega(\mathcal{A}''_\varphi)$ and so there exists a factorization $\beta = u\alpha$ and a path

$$q'_0 \xrightarrow{u} p' \xrightarrow{\alpha} \ldots$$

in $\mathcal{A}'_\varphi$ such that p' signals the last occurrence of a vertex from $S \cup \{q'_0\}$. We claim that no vertex is repeated after p'. Otherwise, since no vertex of S appears after p', we would get a factorization of $p' \xrightarrow{\alpha} \ldots$ as

$$p' \xrightarrow{v} q' \xrightarrow{w} q' \xrightarrow{w} \ldots$$

and by Lemma 8.2(iii) and (iv) we would get $(uvw^nv^{-1}u^{-1})\pi \in \operatorname{Fix}\varphi$ and

$$\beta = \lim_{n\to+\infty} \overline{uvw^nv^{-1}u^{-1}},$$

contradicting $\beta \in \operatorname{Reg}\Phi$. Thus no vertex is repeated after p' and so we must have $p' = p'_i$ and $\alpha = \alpha_i$ for some $i \in \{1, \ldots, m\}$. It follows that $\beta = u\alpha_i$. By Lemma 8.2(iii), we get $\overline{uu_i^{-1}} \in \operatorname{Fix}\overline{\varphi}$ and we are done. $\square$

Theorem 8.5 is somehow a version for infinite fixed points of Theorem 4.1, which we proved before for finite fixed points. Note however that $\operatorname{Sing}\Phi$ has *not* in general finitely many $(\operatorname{Fix}\varphi)$-orbits since $\operatorname{Sing}\Phi$ may be uncountable (take for instance the identity automorphism on a free group of rank 2).

Since every finite set is closed in a metric space, we obtain the following corollary from Theorem 8.5:

Corollary 8.6 *Let φ be a uniformly continuous endomorphism of a finitely generated virtually free group G with Fixφ finite. Then* $\operatorname{Fix}\Phi$ *is finite.*

9 Classification of the infinite fixed points

We can now investigate the nature of the infinite fixed points of Φ when φ is an automorphism. Since both φ and φ^{-1} are then uniformly continuous by Proposition 7.2, they extend to continuous mappings Φ and Ψ which turn out to be mutually inverse in view of the uniqueness of continuous extensions to the completion. Therefore Φ is a bijection. We say that $\alpha \in \operatorname{Reg}\Phi$ is:

- an *attractor* if

$$\exists \varepsilon > 0\ \forall \beta \in \widehat{L}\ (d_3(\alpha,\beta) < \varepsilon \Rightarrow \lim_{n\to+\infty} \beta\Phi^n = \alpha).$$

- a *repeller* if

$$\exists \varepsilon > 0\ \forall \beta \in \widehat{L}\ (d_3(\alpha,\beta) < \varepsilon \Rightarrow \lim_{n\to+\infty} \beta\Phi^{-n} = \alpha).$$

The latter amounts to say that α is an attractor for Φ^{-1}. There exist other types but they do not occur in our context as we shall see.

We say that an attractor $\alpha \in \operatorname{Reg}\Phi$ is *exponentially stable* if

$$\exists \varepsilon, k, \ell > 0\ \forall \beta \in \widehat{L}\ \forall n \in \mathbb{N}\ (d_3(\alpha,\beta) < \varepsilon \Rightarrow d_3(\alpha,\beta\Phi^n) \le k2^{-\ell n} d_3(\alpha,\beta)).$$

This is equivalent to say that

$$\exists M, N, \ell > 0\ \forall \beta \in \widehat{L}\ \forall n \in \mathbb{N}\ (|\alpha \wedge \beta| > M \Rightarrow |\alpha \wedge \beta\Phi^n| + N > \ell n + |\alpha \wedge \beta|). \tag{24}$$

A repeller $\alpha \in \operatorname{Reg}\Phi$ is exponentially stable if it is an exponentially stable attractor for Φ^{-1}.

Theorem 9.1 *Let φ be an automorphism of a finitely generated virtually free group G. Then* $\operatorname{Reg}\Phi$ *contains only exponentially stable attractors and exponentially stable repellers.*

Proof. Let $\alpha \in \operatorname{Reg}\Phi$ and write $\alpha = a_1a_2\ldots$ with $a_i \in \widetilde{A}$. Then there exists a path

$$1\xi \xrightarrow{a_1} \alpha^{[1]}\xi \xrightarrow{a_2} \alpha^{[2]}\xi \xrightarrow{a_3} \ldots$$

in $\mathcal{A}'_\varphi$. Let $Y_0 = B_\varphi(D_{\varphi^{-1}} + 1) + B_{\varphi^{-1}}(D_\varphi + 1)$ and let

$$V = \{u\xi \in Q' : |u\tau| > Y_0 \text{ or } |u\rho| > Y_0\}.$$

It is easy to see that $Q' \setminus V$ is finite. We saw in the proof of Theorem 8.5 that there are only finitely many repetitions of vertices in a path in $\mathcal{A}'_\varphi$ labelled by a regular fixed point, hence there exists some $n_0 \in \mathbb{N}$ such that

$$\alpha^{[n]}\xi \in V \text{ for every } n \geq n_0. \tag{25}$$

Now we consider two cases:

Case I: $\alpha^{[n_0]}\tau = 1$.

We claim that

$$\alpha^{[n]}\tau = 1 \text{ for every } n \geq n_0. \tag{26}$$

The case $n = n_0$ holds in Case I, so assume that $\alpha^{[n]}\tau = 1$ for some $n \geq n_0$. Then $\alpha^{[n]} \in V$ and so $|\alpha^{[n]}\rho| > Y_0 > 2B_\varphi$. Since $|\alpha^{[n+1]}\overline{\varphi}| \geq |\alpha^{[n]}\overline{\varphi}| - B_\varphi$ by definition of B_φ, then

$$\begin{aligned}|\alpha^{[n+1]}\rho| &\geq |\alpha^{[n+1]}\overline{\varphi}| - |\alpha^{[n+1]}| \geq |\alpha^{[n]}\overline{\varphi}| - B_\varphi - |\alpha^{[n]}| - 1 = |\alpha^{[n]}\rho| - B_\varphi - 1\\ &> Y_0 - B_\varphi - 1 > B_\varphi.\end{aligned}$$

By (16), we get $\alpha^{[n+1]}\tau = 1$ and so (26) holds.

Next we show that

$$((\alpha^{[n]}\gamma)\Phi)^{[n+1]} = \alpha^{[n+1]} \tag{27}$$

if $n \geq n_0$ and $\alpha^{[n]}\gamma \in \widehat{L}$. Indeed, by (26) we have $\alpha^{[n]}\overline{\varphi} = \alpha^{[n]}(\alpha^{[n]}\rho)$ and $|\alpha^{[n]}\rho| > Y_0 > B_\varphi$. By the definition of B_φ and Corollary 7.3, we get $((\alpha^{[n]}\gamma)\Phi)^{[n+1]} = \alpha^{[n]}(\alpha^{[n]}\rho)^{[1]}$. Considering the particular case $\gamma = a_{n+1}$, we also get

$$(\alpha^{[n+1]}\overline{\varphi})^{[n+1]} = \alpha^{[n]}(\alpha^{[n]}\rho)^{[1]} = ((\alpha^{[n]}\gamma)\Phi)^{[n+1]}.$$

Since $\alpha^{[n+1]}\tau = 1$ by (26), we have $(\alpha^{[n+1]}\overline{\varphi})^{[n+1]} = \alpha^{[n+1]}$ and so (27) holds.

Hence we may write $(\alpha^{[n]}\gamma)\Phi = \alpha^{[n+1]}\gamma'$ whenever $\alpha^{[n]}\gamma \in \widehat{L}$. Iterating, it follows that, for all $k \geq n_0$ and $n \in \mathbb{N}$, $\alpha^{[k]}\gamma \in \widehat{L}$ implies $(\alpha^{[k]}\gamma)\Phi^n = \alpha^{[k+n]}\gamma'$ for some γ'. By considering $\beta = \alpha^{[k]}\gamma$ and $\alpha^{[k]} = \alpha \wedge \beta$, we deduce that

$$|\alpha \wedge \beta| \geq n_0 \Rightarrow |\alpha \wedge \beta\Phi^n| \geq n + |\alpha \wedge \beta|$$

holds for all $\beta \in \widehat{L}$ and $n \in \mathbb{N}$. Therefore (24) holds and so α is an exponentially stable attractor in this case.

Now, if $|\alpha^{[t]}\tau| = 1$ for some $t > n_0$, we can always replace n_0 by t and deduce by Case I that α is an exponentially stable attractor. Thus we may assume that:

Case II: $\alpha^{[n]}\tau \neq 1$ for every $n \geq n_0$.

By replacing n_0 by a larger integer if necessary, we may assume that (25) is also satisfied when we consider the equivalents of ξ and V for φ^{-1}.

Since φ is injective, there exists some $n_1 \geq n_0$ such that $|\alpha^{[n_1]}\overline{\varphi}| \geq n_0 + B_\varphi$. Since $\alpha^{[n_1]}\tau \neq 1$, it follows from (16) that $|\alpha^{[n_1]}\rho| \leq B_\varphi$, hence $\alpha^{[n_1]}\sigma = \alpha^{[n_2]}$ for some $n_2 \geq n_0$. Write $x = \alpha^{[n_1]}\rho$. Then $\alpha^{[n_1]}\overline{\varphi} = \alpha^{[n_2]}x$ yields $\alpha^{[n_1]} = \overline{(\alpha^{[n_2]}\overline{\varphi^{-1}})(x\overline{\varphi^{-1}})}$ and so

$$n_1 = |\alpha^{[n_1]}| \leq |\alpha^{[n_2]}\overline{\varphi^{-1}}| + |x\overline{\varphi^{-1}}| \leq |\alpha^{[n_2]}\overline{\varphi^{-1}}| + B_\varphi D_{\varphi^{-1}}.$$

On the other hand, $|\alpha^{[n_1]}\rho| \leq B_\varphi < Y_0$ and $\alpha^{[n_1]} \in V$ together yield $Y_0 < |\alpha^{[n_1]}\tau| = n_1 - n_2$ and so

$$n_2 + B_{\varphi^{-1}} < n_1 - Y_0 + B_{\varphi^{-1}} < n_1 - B_\varphi D_{\varphi^{-1}} \leq |\alpha^{[n_2]}\overline{\varphi^{-1}}|.$$

In view of (16), we can apply Case I to φ^{-1}, hence α is an exponentially stable attractor for φ^{-1} and therefore an exponentially stable repeller for φ. $\square$

10 Example and open problems

We include a simple example which illustrates some of the constructions introduced earlier:

Example. Let $G = \mathbb{Z} \times \mathbb{Z}_2$ and let $A = \{a, b, c\}$. Note that this is not the canonical set of generators, which would not work. Then the matched homomorphism $\pi : \widetilde{A}^* \to G$ defined by

$$a\pi = (1, 0), \quad b\pi = (0, 1), \quad c\pi = (1, 1)$$

yields

$$\mathrm{Geo}_A(G) = (a \cup c)^* \cup (a^{-1} \cup c^{-1})^* \cup \{b, b^{-1}\}$$

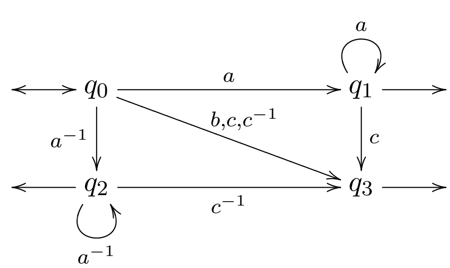
and we can take

$$\begin{aligned} \mathcal{R} = &\{(xx^{-1}, 1) \mid x \in \widetilde{A}\} \cup \{(a^\varepsilon b^\delta, c^\varepsilon), (b^\delta a^\varepsilon, c^\varepsilon), (c^\varepsilon b^\delta, a^\varepsilon), (b^\delta c^\varepsilon, a^\varepsilon) \mid \delta, \varepsilon = \pm 1\} \\ &\cup \{(ac^{-1}, b), (c^{-1}a, b), (a^{-1}c, b), (ca^{-1}, b), (b^2, 1), (b^{-2}, 1)\} \end{aligned}$$

to get $\mathrm{Geo}_A(G) = \mathrm{Irr}\,\mathcal{R}$. Ordering $\widetilde{A}$ by $a < c < a^{-1} < c^{-1} < b < b^{-1}$, we get

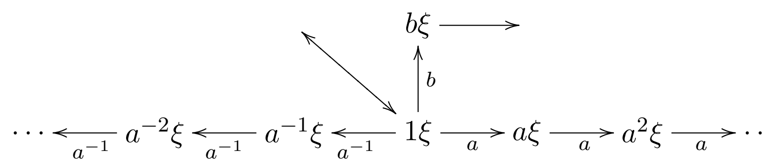
$$L = a^*(1 \cup c) \cup (a^{-1})^*(1 \cup c^{-1}) \cup b,$$

recognized by the automaton $\mathcal{A}$ depicted by



Hence $\partial L = L_\omega(\mathcal{A}) = \{a^\omega, (a^{-1})^\omega\}$.

Let φ be the endomorphism of G defined by $(m,n)\varphi = (2m,n)$. Then φ is injective and therefore uniformly continuous, admiting a continuous extension Φ to $\widehat{L}$. Since $B_\varphi = 0$, it is easy to check that $\mathcal{A}'_\varphi$ is the automaton



and

$$1\xi = (1,1,1,q_0), \quad b\xi = (1,1,1,q_3), \quad a^n\xi = (1,1,a^n,q_1), \quad a^{-n}\xi = (1,1,a^{-n},q_2)$$

for $n \geq 1$. Note that in general we ignore how to compute $\mathcal{A}'_\varphi$, our proofs being far from constructive!

It is immediate that $\mathrm{Fix}\,\Phi = \{1, b, a^\omega, (a^{-1})^\omega\}$. Moreover, the regular infinite fixed points a^ω and $(a^{-1})^\omega$ are both exponentially stable attractors.

Finally, we end the paper with some easily predictable open problems:

Problem 10.1 *Is it possible to generalize Theorems 4.1, 8.5 and 9.1 to arbitrary finitely generated hyperbolic groups?*

Paulin proved that Theorem 4.1 holds for automorphisms of hyperbolic groups [15].

Problem 10.2 *Is* $\mathrm{Fix}\,\varphi$ *effectively computable when* φ *is an endomorphism of a finitely generated virtually free group?*

For the moment, only the case of free group automorphisms is known (Maslakova, [14]).

Acknowledgements

The author acknowledges support from the European Regional Development Fund through the programme COMPETE and from the Portuguese Government through FCT – Fundação para a Ciência e a Tecnologia, under the project PEst-C/MAT/UI0144/2011.

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