On the Watson L_2-theory for index transforms

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We establish Watson's and Plancherel's theorems for general index transforms in L_2. It involves the familiar Kontorovich-Lebedev, Mehler-Fock, Olevskii transforms and other index transforms with hypergeometric functions as kernels. Symmetric and unsymmetric kernels are considered. As applications general index integrals involving products of Meijer's G-functions are calculated.

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1 Introduction and auxiliary results

In 1933 Watson proved [5, 6, 7] that integral transformations

$$g_k(x) = \frac{d}{dx}\int_0^\infty \frac{k(xu)}{u} f(u)du, \tag{1}$$

$$g_h(x) = \frac{d}{dx}\int_0^\infty \frac{h(xu)}{u} f(u)du, \tag{2}$$

are automorphisms in $L_2(\mathbb{R}_+; dx)$ and have reciprocal inversion formulas for almost all $x \in \mathbb{R}_+$ in the form

$$f(x) = \frac{d}{dx}\int_0^\infty \frac{h(xu)}{u} g_k(u)du, \tag{3}$$

$$f(x) = \frac{d}{dx}\int_0^\infty \frac{k(xu)}{u} g_h(u)du, \tag{4}$$

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if and only if a continual analog of the biorthogonality for sequences

$$\int_0^\infty k(\xi x)h(\eta x)\frac{dx}{x^2} = \min(\xi,\eta), \tag{5}$$

holds. Moreover, in this case the Parseval type equality takes place

$$\int_0^\infty g_k(x)g_h(x)dx = \int_0^\infty [f(x)]^2\, dx. \tag{6}$$

Definition 1. Let

$$\frac{k(x)}{x} \in L_2(\mathbb{R}_+;dx) \quad \text{and} \quad \frac{k^*(s)}{1-s} \in L_2\left(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty\right),$$

$$\frac{h(x)}{x} \in L_2(\mathbb{R}_+;dx) \quad \text{and} \quad \frac{h^*(s)}{1-s} \in L_2\left(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty\right),$$

be reciprocal Mellin pairs in L_2 (see [5]), where

$$\sup_{-\infty<t<\infty}\left|k^*\left(\frac{1}{2}+it\right)\right|^2 \le C_k < \infty,$$

$$\sup_{-\infty<t<\infty}\left|h^*\left(\frac{1}{2}+it\right)\right|^2 \le C_h < \infty.$$

Then conditions (5) and

$$k^*(s)h^*(1-s) = 1, \quad \text{Re } s = \frac{1}{2} \tag{7}$$

are equivalent and k,h are called the conjugate Watson kernels. In the case $h = \overline{k}$ condition (7) becomes $\left|k^*\left(\frac{1}{2}+it\right)\right| = 1$ and k is called the Watson kernel.

Transformations (1), (2) are the Watson transforms and this class contains classical sine and cosine Fourier transforms, the Hankel transform, the Hilbert transform, etc. (cf. [5, 7]). In this paper we will construct an analog of the Watson theory for the index transforms [7, 8] basing on the L_2-properties of the Kontorovich-Lebedev transform. Concerning mapping properties of the index transforms, their composition structure and a relationship with the Mellin convolution type transforms see, for instance, in [9, 10, 11].

In 1949 Lebedev proved [2] the Plancherel theorem for the Kontorovich-Lebedev transform, which says that integral transformation

$$g(\tau) = \frac{d}{d\tau}\int_0^\infty \mathcal{K}(x,\tau)f(x)dx, \tag{8}$$

where

$$\mathcal{K}(x,\tau) = \sqrt{\frac{2}{\pi x}} \int_0^\tau \frac{K_{iy}(x)}{|\Gamma(iy)|} dy, \quad (x,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{9}$$

is an isometric isomorphism in $L_2(\mathbb{R}_+; dx) \equiv L_2(\mathbb{R}_+)$ and for almost all $x \in \mathbb{R}_+$ the inversion formula

$$f(x) = \frac{d}{dx} \int_0^\infty \hat{\mathcal{K}}(x,\tau) g(\tau) d\tau, \tag{10}$$

where

$$\hat{\mathcal{K}}(x,\tau) = \sqrt{\frac{2}{\pi}} \left[|\Gamma(i\tau)|\right]^{-1} \int_0^x \frac{K_{i\tau}(y)}{\sqrt{y}} dy, \quad (x,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{11}$$

and the Parseval equality

$$\int_0^\infty |g(\tau)|^2 d\tau = \int_0^\infty |f(x)|^2 dx, \tag{12}$$

are fulfilled. From the definitions of the kernels (9), (11) we see that

$$\int_0^x \mathcal{K}(x,\tau) dx = \int_0^\tau \hat{\mathcal{K}}(x,\tau) d\tau = \sqrt{\frac{2}{\pi}} \int_0^x \int_0^\tau \frac{K_{iy}(t)}{|\Gamma(iy)|\sqrt{t}} dy dt.$$

Here $\Gamma(i\tau)$ is Euler's gamma-function of the pure imaginary number $i\tau$ and $K_\mu(z)$ is the modified Bessel function [1, Vol. II], which satisfies the differential equation

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \mu^2) u = 0,$$

and has the asymptotic behaviour

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} [1 + O(1/z)], \qquad z \to \infty, \tag{13}$$

and near the origin

$$z^{|\mathrm{Re}\mu|} K_\mu(z) = 2^{\mu-1} \Gamma(\mu) + o(1), \ z \to 0, \ \mu \neq 0, \tag{14}$$

$$K_0(z) = -\log z + O(1), \ z \to 0. \tag{15}$$

By using the same technique as for the Watson transforms (see [5] and Section 2.3 in [7]), it is not difficult to establish an equivalence between the Kontorovich-Lebedev transforms (8), (10) as unitary operators in $L_2(\mathbb{R}_+)$ and the values of the following integrals for their kernels (Watson's equalities)

$$\int_0^\infty \hat{\mathcal{K}}(\xi,\tau) \hat{\mathcal{K}}(\eta,\tau) d\tau = \min(\xi,\eta), \tag{16}$$

$$\int_0^\infty \mathcal{K}(x,\xi)\mathcal{K}(x,\eta)dx = \min(\xi,\eta). \tag{17}$$

These key formulas will be extended in the sequel to characterize L_2-properties of general index transforms, which were introduced in [7, Chapter 7]. In particular, we will get Watson's theorems for the Mehler-Fock, Olevskii transforms, index transforms involving Whittaker's functions and Meijer's G-functions. General index integrals with the product of two G-functions will be calculated.

Finally in this section we give useful integral representations of the real-valued modified Bessel function $K_{i\tau}(x)$ in terms of the Mellin-Barnes integrals, which will be employed below. Indeed, we have (see [7])

$$\frac{K_{i\tau}(x)}{\sqrt{x}} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{s-5/2}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{i\tau}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{i\tau}{2}\right)x^{-s}ds,\ \gamma>\frac{1}{2}, \tag{18}$$

$$\frac{1}{\sqrt{\pi}}\ \frac{e^{-x^2}K_{i\tau/2}(x^2)}{\sqrt{x}} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{-\frac{s}{2}-\frac{3}{4}}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{i\tau}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{i\tau}{2}\right)$$
$$\times\frac{x^{-s}}{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)}ds,\ \gamma>\frac{1}{2}, \tag{19}$$

$$\frac{\sqrt{\pi}}{\cosh(\pi\tau/2)}\frac{e^{x^2}K_{i\tau/2}(x^2)}{\sqrt{x}} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{-\frac{s}{2}-\frac{3}{4}}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{i\tau}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{i\tau}{2}\right)$$
$$\times\Gamma\left(\frac{3}{4}-\frac{s}{2}\right)x^{-s}ds,\ 0<\gamma<\frac{1}{2}. \tag{20}$$

2 Watson's theorems for index transforms

We begin with

Definition 2. Let $\frac{k(x)}{x} \in L_2(\mathbb{R}_+)$ and $\frac{k^*(s)}{1-s} \in L_2\left(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty\right)$ be its Mellin transform, where $k^*(s)$ is bounded on the line Re $s=\frac{1}{2}$. Let the Kontorovich-Lebedev kernels $\mathcal{K}(x,y)$, $\hat{\mathcal{K}}(x,y)$ be respectively defined by (9), (11). The functions

$$W_k(x,y) = \frac{d}{dy}\int_0^\infty \mathcal{K}(t,y)\frac{k(xt)}{t}dt, \tag{21}$$

$$\Omega_k(x,y) = \frac{d}{dx}\int_0^\infty \mathcal{K}(t,y)\frac{k(xt)}{t}dt, \tag{22}$$

$$\hat{W}_k(x,y) = \frac{d}{dy}\int_0^\infty \hat{\mathcal{K}}(x,t)\frac{k(yt)}{t}dt, \tag{23}$$

$$\hat{\Omega}_k(x,y) = \frac{d}{dx}\int_0^\infty \hat{\mathcal{K}}(x,t)\frac{k(yt)}{t}dt, \tag{24}$$

where $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ are called General Kontorovich-Lebedev kernels (GKL-kernels).

We have

Theorem 1. *The following relations take place*

$$\int_0^\infty W_k(\xi,y)W_h(\eta,y)dy = min(\xi,\eta), \tag{25}$$

$$\int_0^\infty \Omega_k(x,\xi)\Omega_h(x,\eta)dx = min(\xi,\eta), \tag{26}$$

$$\int_0^\infty \hat{W}_k(\xi,y)\hat{W}_h(\eta,y)dy = min(\xi,\eta), \tag{27}$$

$$\int_0^\infty \hat{\Omega}_k(x,\xi)\hat{\Omega}_h(x,\eta)dx = min(\xi,\eta), \tag{28}$$

if and only if k, h *are conjugate Watson's kernels.*

Proof. To prove the sufficiency we observe, that integrals (21), (24) can be considered as the Kontorovich-Lebedev transforms (8), (10) of functions $\frac{k(xt)}{t}, \frac{k(yt)}{t} \in L_2(\mathbb{R}_+)$, respectively. Hence via Parseval identity (12) and biorthogonality condition (5) we get the chain of equalities

$$\int_0^\infty W_k(\xi,y)W_h(\eta,y)dy = \int_0^\infty k(\xi t)h(\eta t)\frac{dt}{t^2}$$

$$= \int_0^\infty \hat{\Omega}_k(x,\xi)\hat{\Omega}_h(x,\eta)dx = \min(\xi,\eta),$$

which leads to (25), (28). Analogously, integrals (22), (23) represent Watson's transforms (see (1)) of the kernels (9), (11), which belong to L_2 by the first and second variable, respectively. So using (6) and (16), (17) we derive conditions (26), (27).

For the necessity we assume that (25), (26), (27), (28) hold. Hence from (25), (28) we get equality (5). This yields immediately that k, h are conjugate Watson's kernels. To treat relation (26) we write its left-hand side by using the Mellin -Parseval equality [5]

$$\int_0^\infty \Omega_k(x,\xi)\Omega_h(x,\eta)dx = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}-i\infty} \Omega_k^*(s,\xi)\Omega_h^*(1-s,\eta)ds, \tag{29}$$

where

$$\Omega^*_{\left\{\substack{k\\h}\right\}}\left(s, \left\{\begin{matrix}\xi\\ \eta\end{matrix}\right\}\right) = \left\{\begin{matrix}k^*(s)\\ h^*(s)\end{matrix}\right\}\Phi\left(1-s, \left\{\begin{matrix}\xi\\ \eta\end{matrix}\right\}\right),$$

and $\Phi(s,y)$ is the Mellin transform in $L_2(\mathbb{R}_+)$ of the Kontorovich-Lebedev kernel $\mathcal{K}(x,y)$, namely

$$\Phi(s,y) = \text{l.i.m.}_{N\to\infty} \int_{1/N}^{N} \mathcal{K}(x,y)\; x^{s-1}\; dx,\; \text{Re}\; s = \frac{1}{2}. \tag{30}$$

The function $\Phi(s,y)$ can be expressed directly by using the Mellin transform of the modified Bessel function (see (18)). So we have

$$\Phi(s,y) = \frac{2^{s-2}}{\sqrt{\pi}} \int_0^y \Gamma\left(\frac{s}{2} - \frac{1}{4} + \frac{iu}{2}\right) \Gamma\left(\frac{s}{2} - \frac{1}{4} - \frac{iu}{2}\right) \frac{du}{|\Gamma(iu)|},\; \text{Re}\; s = \frac{1}{2}. \tag{31}$$

Further, from (17), (29) and (30) we find

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Omega_k^*(s,\xi)\Omega_h^*(1-s,\eta)ds - \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}-i\infty} \Phi\left(1-s,\xi\right)\Phi\left(s,\eta\right) ds = 0.$$

Hence,

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} [k^*(s)h^*(1-s) - 1]\Phi\left(1-s,\xi\right)\Phi\left(s,\eta\right) ds = 0. \tag{32}$$

Fixing a positive η and observing that $[k^*(s)h^*(1-s)-1]\Phi\left(s,\eta\right) \in L_2\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$, we return to the originals in (32) and differentiate then with respect to ξ. Thus it gives the equation

$$\frac{d}{d\xi} \int_0^\infty H_\eta(x)\mathcal{K}(x,\xi)dx = 0,$$

with

$$H_\eta(x) = \text{l.i.m.}_{T\to\infty} \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} [k^*(s)h^*(1-s) - 1]\Phi\left(s,\eta\right) x^{-s} ds \in L_2(\mathbb{R}_+),$$

where its left-hand side is the Kontorovich-Lebedev transform (8) of the function $H_\eta(x)$. Therefore $H_\eta(x) = 0$ and reciprocally we get immediately the equality (7), which is equivalent to (5) and proves that k, h are necessarily conjugate Watson kernels.

Analogously, we write the left-hand side of condition (27) in terms of the Mellin transform

$$\int_0^\infty \hat{W}_k(\xi,y)\hat{W}_h(\eta,y)dy = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \hat{W}_k^*(\xi,s)\hat{W}_h^*(\eta,1-s)ds,$$

where

$$\hat{W}^*_{\left\{\substack{k\\h}\right\}}\left(\left\{\begin{matrix}\xi\\ \eta\end{matrix}\right\}, s\right) = \left\{\begin{matrix}k^*(s)\\ h^*(s)\end{matrix}\right\} \Psi\left(\left\{\begin{matrix}\xi\\ \eta\end{matrix}\right\}, 1-s\right),$$

and $\Psi(y,s)$ is the Mellin transform in $L_2(\mathbb{R}_+)$ of the Kontorovich-Lebedev kernel $\hat{\mathcal{K}}(y,x)$ with respect to x, namely

$$\Psi(y,s) = \text{l.i.m.}_{N\to\infty} \int_{1/N}^{N} \hat{\mathcal{K}}(y,x)\, x^{s-1}\, dx,\ \text{Re}\, s = \frac{1}{2}.$$

Consequently, appealing to (16) we obtain

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} [k^*(s)h^*(1-s)-1]\Psi\,(\xi, 1-s)\,\Psi\,(\eta, s)\, ds = 0,$$

which leads to the zero Kontorovich-Lebedev transform (10)

$$\frac{d}{d\xi}\int_0^\infty \hat{H}_\eta(x)\hat{\mathcal{K}}(\xi,x)dx = 0,$$

of the function $\hat{H}_\eta(x)$

$$\hat{H}_\eta(x) = \text{l.i.m.}_{T\to\infty}\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} [k^*(s)h^*(1-s)-1]\Psi\,(\eta, s)\, x^{-s} ds \in L_2(\mathbb{R}_+),$$

which is zero as well by the same discussions as above. Therefore condition (7) takes place and k, h are conjugate Watson's kernels. Theorem 1 is proved.

Let us consider for GKL- kernels (21), (22), (23), (24) the corresponding four pairs of integral transformations in $L_2(\mathbb{R}_+)$ (the index transforms)

$$g_{\{{k \atop h}\}}(x) = \frac{d}{dx}\int_0^\infty W_{\{{k \atop h}\}}(x,y)f(y)dy, \tag{33}$$

$$\hat{g}_{\{{k \atop h}\}}(x) = \frac{d}{dx}\int_0^\infty \hat{W}_{\{{k \atop h}\}}(x,y)f(y)dy, \tag{34}$$

$$\omega_{\{{k \atop h}\}}(y) = \frac{d}{dy}\int_0^\infty \Omega_{\{{k \atop h}\}}(x,y)f(x)dx, \tag{35}$$

$$\hat{\omega}_{\{{k \atop h}\}}(y) = \frac{d}{dy}\int_0^\infty \hat{\Omega}_{\{{k \atop h}\}}(x,y)f(x)dx. \tag{36}$$

The Watson kernel k is called absolutely continuous if

$$k(x) = \int_0^x \hat{k}(x)dx,$$

We are ready to prove Watson's type theorems for these pairs of integral transforms. Taking, for instance, pairs (33), (35) we have the following results.

Theorem 2. *Let k, h be conjugate absolutely continuous Watson's kernels and $\hat{k}, \hat{h}$ be bounded. Then index transforms* (33) *exist for almost all $x > 0$, form automorphisms in $L_2(\mathbb{R}_+)$ with reciprocal inversions* (35) *for almost all $y > 0$*

$$f(y) = \frac{d}{dy}\int_0^\infty \Omega_{\left\{ {h \atop k} \right\}}(x,y) g_{\left\{ {k \atop h} \right\}}(x) dx \tag{37}$$

and the Parseval equality

$$\int_0^\infty g_k(x) g_h(x) dx = \int_0^\infty \left[f(x) \right]^2 dx,$$

if and only if condition (26) *holds.*

Proof. *Necessity.* Taking

$$f_\xi(y) = \begin{cases} 1, & \text{if } y \in [0,\xi], \\ 0, & \text{if } y \in (\xi,\infty), \end{cases}$$

which belongs to $L_2(\mathbb{R}_+)$ we have from (21), (22), (33)

$$g^\xi_{\left\{ {k \atop h} \right\}}(x) = \frac{d}{dx}\int_0^\xi W_{\left\{ {k \atop h} \right\}}(x,y) dy = \frac{d}{dx}\int_0^\infty \mathcal{K}(t,\xi) \frac{\left\{ {k(xt) \atop h(xt)} \right\}}{t} dt = \Omega_{\left\{ {k \atop h} \right\}}(x,\xi).$$

Consequently, by Parseval equality we find

$$\int_0^\infty \Omega_k(x,\xi) \Omega_h(x,\eta) dx = \int_0^\infty g^\xi_k(x) g^\eta_h(x) dx$$

$$= \int_0^\infty f_\xi(y) f_\eta(y) dy \ = \min(\xi,\eta),$$

which proves (26).

Sufficiency. Let condition (26) be true. Considering $f(y)$ from the dense set in L_2 of smooth functions with compact support belonging to some segment $[0, Y],\ Y > 0$, we substitute the expression of the kernel (21) into (33) and after integration by parts we obtain the representation

$$g_{\left\{ {k \atop h} \right\}}(x) = -\frac{d}{dx}\int_0^\infty \int_0^\infty \mathcal{K}(t,y) \frac{\left\{ {k(xt) \atop h(xt)} \right\}}{t} f'(y)\ dt dy. \tag{38}$$

Our goal now is to put the operator $\frac{d}{dx}$ inside the integral sign in (38). In fact, by using the inequality $|K_{iy}(t)| \le K_0(t),\ y \ge 0$ [1], asymptotic formulas (13), (15) for the modified Bessel functions and the property for $\hat{k}$ to be bounded, we get the estimate

$$\int_0^\infty \left| \mathcal{K}(t,y) \hat{k}(xt) \right| dt < \text{const.} \int_0^\infty \left| \mathcal{K}(t,y) \right| dt$$

$$< \text{const.} \int_0^Y \frac{du}{|\Gamma(iu)|} \int_0^\infty K_0(t) \frac{dt}{t^{1/2}} < \infty.$$

Hence we see that kernels $\Omega_{\{ {k \atop h} \}}(x,y)$ are continuous on $\mathbb{R}_+ \times \text{supp}f$, since via (22)

$$\Omega_k(x,y) = \frac{d}{dx} \int_0^\infty \mathcal{K}(t,y) \frac{k(xt)}{t} dt = \int_0^\infty \mathcal{K}(t,y) \hat{k}(xt) dt$$

and the latter integral is absolutely and uniformly convergent on the set $\mathbb{R}_+ \times \text{supp}f$ (see the estimate above). Therefore the integral in the right-hand side of (38) with the derivative inside converges absolutely and uniformly with respect to $x \in \mathbb{R}_+$ and we have

$$g_{\{ {k \atop h} \}}(x) = - \int_{\text{supp}f} \Omega_{\{ {k \atop h} \}}(x,y) f'(y)\, dy. \tag{39}$$

Thus using (26) we deduce

$$\int_0^\infty g_k(x) g_h(x) dx = \int_0^\infty \int_0^\infty \Omega_k(x,u)\ f'(u)\ du \int_0^\infty \Omega_h(x,v)\ f'(v)\ dv dx$$

$$= \int_0^\infty f'(u)\ du \int_0^\infty \min(u,v)\ f'(v)\ dv = \int_0^\infty f'(u)\ du \int_0^u f'(v)\ v dv$$

$$- \int_0^\infty f'(u)\ f(u) u\ du = -2 \int_0^\infty f'(u)\ f(u) u\ du = \int_0^\infty [f(u)]^2 du$$

and we prove the Parseval type equality. Further, changing the order of integration in (38) by Fubini's theorem and then integrating by parts in the inner integral with respect to y, we take into account formula (9) to obtain

$$g_{\{ {k \atop h} \}}(x) \equiv g_{\{ {k \atop h} \}}[f](x) = \frac{d}{dx} \int_0^\infty \frac{\left\{ {k(xt) \atop h(xt)} \right\}}{t} \sqrt{\frac{2}{\pi t}} \int_0^\infty \frac{K_{iy}(t)}{|\Gamma(iy)|} f(y) dy\ dt. \tag{40}$$

This means that integral transforms (33) are compositions of the Watson transforms (1), (2) and the Kontorovich-Lebedev transform (10). Since the Watson transform is an automorphism in $L_2(\mathbb{R}_+)$, we appeal to the L_2-theory of the Mellin transform [5], Definition 1 and Parseval's identity (12) to establish the following inequalities for the square of L_2-norm of the composition (40)

$$C^{-1}_{\{ {h \atop k} \}} ||f||_2^2 \le \left|\left| g_{\{ {k \atop h} \}}[f] \right|\right|_2^2 \le C_{\{ {k \atop h} \}} ||f||_2^2. \tag{41}$$

Let $f \in L_2(\mathbb{R}_+)$. Then there exists a sequence $\{f_n\}$ of smooth functions with compact support such that $||f - f_n||_2 \to 0,\ n \to \infty$. Hence from estimates (41) we immediately obtain that $\left\{g_{\left\{ k \atop h \right\}}[f_n]\right\}$ are Cauchy sequences, which have corresponding limits in L_2, namely $g_{\left\{ k \atop h \right\}} = \text{l.i.m}_{n\to\infty} \left\{g_{\left\{ k \atop h \right\}}[f_n]\right\}$. Therefore,

$$\lim_{n\to\infty} \int_0^\infty g_k[f_n](x) g_h[f_n](x) dx = \int_0^\infty g_k(x) g_h(x) dx$$

$$= \lim_{n\to\infty} \int_0^\infty [f_n(y)]^2 dy = \int_0^\infty [f(y)]^2 dy.$$

So it is proved that the Parseval equality is true for any $f \in L_2(\mathbb{R}_+)$. Moreover, via (41) we show that (33) are automorphisms in L_2. Now,

$$\lim_{n\to\infty} \int_0^x g_k[f_n](x) dx = \lim_{n\to\infty} \int_0^\infty W_k(x,y) f_n(y) dy$$

$$= \int_0^\infty W_k(x,y) f(y) dy = \int_0^x g_k(x) dx$$

and differentiating with respect to x we obtain formulas (33), which are valid for almost all $x > 0$ and any $f \in L_2(\mathbb{R}_+)$. Further, taking

$$\psi(u) = \begin{cases} 1, & \text{if } u \in [0, y], \\ 0, & \text{if } u \in (y, \infty), \end{cases}$$

we have (see (9), (21), (22))

$$g_{\left\{ h \atop k \right\}}[\psi](x) = \frac{d}{dx} \int_0^y W_{\left\{ h \atop k \right\}}(x,u) du = \frac{d}{dx} \int_0^\infty \mathcal{K}(t,y) \frac{\left\{ h(xt) \atop k(xt) \right\}}{t} dt = \Omega_{\left\{ h \atop k \right\}}(x,y).$$

Hence,

$$\int_0^y f(y) dy = \int_0^\infty g_{\left\{ k \atop h \right\}}(x) \Omega_{\left\{ h \atop k \right\}}(x,y) dx$$

and by differentiation with respect to y we prove the reciprocal inversion formulas (35) for almost all $y > 0$. Theorem 2 is proved.

Theorem 3. *Under same conditions as in Theorem 2 index transforms* (35) *exist for almost all $x > 0$, form automorphisms in $L_2(\mathbb{R}_+)$ with reciprocal inversions* (33) *for almost all $y > 0$*

$$f(y) = \frac{d}{dy} \int_0^\infty W_{\left\{ k \atop h \right\}}(x,y) \omega_{\left\{ k \atop h \right\}}(y) dy$$

and the Parseval equality

$$\int_0^\infty \omega_k(y) \omega_h(y) dy = \int_0^\infty [f(x)]^2 \, dx,$$

if and only if condition (25) *holds.*

Proof. *Necessity.* Similarly, taking

$$f_\xi(x) = \begin{cases} 1, & \text{if } x \in [0, \xi], \\ 0, & \text{if } x \in (\xi, \infty), \end{cases}$$

which belongs to $L_2(\mathbb{R}_+)$ we have from (21), (22), (35)

$$\omega^\xi_{\left\{ {k \atop h} \right\}}(y) = \frac{d}{dy} \int_0^\xi \Omega_{\left\{ {k \atop h} \right\}}(x, y) dx = \frac{d}{dy} \int_0^\infty \mathcal{K}(t, y) \frac{\left\{ {k(\xi t) \atop h(\xi t)} \right\}}{t} dt = W_{\left\{ {k \atop h} \right\}}(\xi, y).$$

Consequently, by Parseval equality we find

$$\int_0^\infty W_k(\xi, y) W_h(\eta, y) dy = \int_0^\infty \omega^\xi_k(y) \omega^\eta_h(y) dy$$

$$= \int_0^\infty f_\xi(x) f_\eta(x) dx \ = \min(\xi, \eta),$$

which proves (25).

Sufficiency. Assuming that (25) is valid be true, we take again $f(x)$ from the set of smooth functions with compact support and after integration by parts we write (35) in the form

$$\omega_{\left\{ {k \atop h} \right\}}(y) = -\frac{d}{dy} \int_0^\infty \int_0^\infty \mathcal{K}(t, y) \frac{\left\{ {k(xt) \atop h(xt)} \right\}}{t} f'(x) \ dt dx.$$

In this case, differentiating under the integral sign with respect to y, the kernel (21) can be written as (see (9))

$$W_k(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{K_{iy}(t)}{|\Gamma(iy)|} \frac{k(xt)}{t^{3/2}} dt,$$

where the latter integral is convergent absolutely and uniformly on the domain $[0, X] \times [0, Y], \ X, Y > 0$ by virtue of the estimate

$$\int_0^\infty \frac{|K_{iy}(t)|}{|\Gamma(iy)|} \frac{|k(xt)|}{t^{3/2}} dt \le \text{const.} \frac{X}{|\Gamma(iY)|} \int_0^\infty \frac{K_0(t)}{t^{1/2}} dt < \infty.$$

Hence in the same manner as in Theorem 2 we derive the representation

$$\omega_{\left\{ {k \atop h} \right\}}(y) = -\int_{\text{supp} f} W_{\left\{ {k \atop h} \right\}}(x, y) f'(x) \ dx$$

and the composition

$$\omega_{\{{k \atop h}\}}(y) = \frac{d}{dy}\int_0^\infty \mathcal{K}(t,y)\int_0^\infty \left\{ \begin{matrix} \hat{k}(xt) \\ \hat{h}(xt) \end{matrix} \right\} f(x)dx\ dt,$$

which allow to finish the proof of the theorem by the same discussions as above. Theorem 3 is proved.

Corollary 1. *Let* $k(x) = \overline{h(x)}$. *Then the Parseval identity like* (12) *holds for transformations* (33) *and their reciprocities* (35)

$$\int_0^\infty |g_k(x)|^2 dx = \int_0^\infty |f(x)|^2 dx,$$

$$\int_0^\infty |\omega_k(x)|^2 dx = \int_0^\infty |f(x)|^2 dx,$$

which are unitary operators.

Proof. Indeed, since the Kontorovich-Lebedev kernel $\mathcal{K}(x,y)$ is real-valued the result follows just putting $h(x) = \overline{k(x)}$ in the generalized Parseval equality like (6). Corollary 1 is proved.

Now we are going to prove the Watson theorem for the pairs (34), (36) related to GKL-kernels (23), (24). The proof will be slightly different because we will use definitions of these kernels by integrals, which are convergent in the mean square sense. In fact, from (23), (24) we find (see (11))

$$\hat{W}_{\{{k \atop h}\}}(x,y) = \text{l.i.m.}_{N\to\infty} \hat{W}^N_{\{{k \atop h}\}}(x,y),$$

where

$$\hat{W}^N_{\{{k \atop h}\}}(x,y) = \int_{1/N}^N \hat{\mathcal{K}}(x,t) \left\{ \begin{matrix} \hat{k}(yt) \\ \hat{h}(yt) \end{matrix} \right\} dt, \tag{42}$$

and the convergence is with respect to the norm in $L_2(\mathbb{R}_+; dy)$,

$$\hat{\Omega}_{\{{k \atop h}\}}(x,y) = \text{l.i.m.}_{N\to\infty} \hat{\Omega}^N_{\{{k \atop h}\}}(x,y),$$

where

$$\hat{\Omega}^N_{\{{k \atop h}\}}(x,y) = \sqrt{\frac{2}{\pi x}} \int_{1/N}^N \frac{K_{it}(x)}{|\Gamma(it)|} \left\{ \begin{matrix} k(yt) \\ h(yt) \end{matrix} \right\} \frac{dt}{t} \tag{43}$$

and the convergence is with respect to the norm in $L_2(\mathbb{R}_+; dx)$. The problem is that according to Stirling's asymptotic formula for gamma-functions [1, Vol. I] and asymptotic behavior of the modified Bessel functions with respect to a pure imaginary index under restricted domain of the argument [8], we obtain

$$\frac{K_{it}(x)}{\Gamma(it)} = O(1), \quad t \to \infty.$$

Therefore integrands (42), (43) are, generally, not summable functions in the Lebesgue sense. Nevertheless, we have

Theorem 4. *Let k, h be conjugate absolutely continuous Watson's kernels and $\hat{k}, \hat{h}$ be bounded. Then index transforms* (34) *exist for almost all $x > 0$, form automorphisms in $L_2(\mathbb{R}_+)$ with reciprocal inversions* (36) *for almost all $y > 0$*

$$f(y) = \frac{d}{dy}\int_0^\infty \hat{\Omega}_{\left\{ {h \atop k} \right\}}(x,y)\hat{g}_{\left\{ {k \atop h} \right\}}(x)dx \tag{44}$$

and the Parseval equality

$$\int_0^\infty \hat{g}_k(x)\hat{g}_h(x)dx = \int_0^\infty [f(x)]^2\, dx,$$

if and only if condition (28) *holds. Conversely, index transforms* (36) *exist for almost all $x > 0$, form automorphisms in $L_2(\mathbb{R}_+)$ with reciprocal inversions* (34) *for almost all $y > 0$*

$$f(y) = \frac{d}{dy}\int_0^\infty \hat{W}_{\left\{ {h \atop k} \right\}}(x,y)\hat{w}_{\left\{ {k \atop h} \right\}}(x)dx$$

and the Parseval equality

$$\int_0^\infty \hat{w}_k(x)\hat{w}_h(x)dx = \int_0^\infty [f(x)]^2\, dx,$$

if and only if condition (27) *holds.*

Proof. The necessity of conditions (27), (28) can be proved exactly as in Theorems 2,3. Let us prove the sufficiency. Suppose that condition (28) is valid. Considering $f(y)$ from the set of smooth functions with compact support we obtain, as in the proofs of previous theorems, the following representations of operators (34)

$$\hat{g}_{\left\{ {k \atop h} \right\}}(x) = -\frac{d}{dx}\int_0^\infty \int_0^\infty \hat{\mathcal{K}}(x,t)\frac{\left\{ {k(yt) \atop h(yt)} \right\}}{t} f'(y)\; dtdy. \tag{45}$$

Denoting by

$$\hat{g}^N_{\left\{ {k \atop h} \right\}}(x) = -\frac{d}{dx}\int_0^\infty \int_{1/N}^N \hat{\mathcal{K}}(x,t)\frac{\left\{ {k(yt) \atop h(yt)} \right\}}{t} f'(y)\; dtdy, \tag{46}$$

we can put the differentiation inside the latter integral owing to the absolute and uniform convergence. Hence taking into account (43), we write

$$\hat{g}^N_{\left\{ k \atop h \right\}}(x) = -\int_{\text{supp}f} \hat{\Omega}^N_{\left\{ k \atop h \right\}}(x,y) f'(y)\, dy.$$

Meanwhile, with the generalized Minkowski inequality and via the boundedness of the Kontorovich-Lebedev operator (10) we have the estimate of the L_2-norm $||\ ||_2$ $(M > N)$

$$\left|\left|\hat{g}^N_{\left\{ k \atop h \right\}} - \hat{g}^M_{\left\{ k \atop h \right\}}\right|\right|_2 \le \int_{\text{supp}f} \left|\left|\hat{\Omega}^N_{\left\{ k \atop h \right\}}(\cdot,y) - \hat{\Omega}^M_{\left\{ k \atop h \right\}}(\cdot,y)\right|\right|_2 \ |f'(y)|\ dy$$

$$\le \text{const.} \int_{\text{supp}f} \left[\left(\int_{1/M}^{1/N} \left| \frac{\left\{ k(yt) \atop h(yt) \right\}}{t} \right|^2 dt \right)^{1/2} + \left(\int_{N}^{M} \left| \frac{\left\{ k(yt) \atop h(yt) \right\}}{t} \right|^2 dt \right)^{1/2} \right] |f'(y)|\ dy$$

$$\text{const.} \left[\left(\int_{1/M}^{1/N} \left| \frac{\left\{ k(y_0t) \atop h(y_0t) \right\}}{t} \right|^2 dt \right)^{1/2} + \left(\int_{N}^{M} \left| \frac{\left\{ k(y_0t) \atop h(y_0t) \right\}}{t} \right|^2 dt \right)^{1/2} \right] \to 0, M > N \to \infty,$$

where $\left|\left\{ k(y_0t) \atop h(y_0t) \right\}\right| = \sup_{y\in \text{supp}f} \left|\left\{ k(yt) \atop h(yt) \right\}\right|$. Therefore $\{\hat{g}^N_{\left\{ k \atop h \right\}}(x)\}$ is a Cauchy sequence, which converges to some function $\varphi_{\left\{ k \atop h \right\}}(x)$. Moreover, the use of Schwarz's inequality, definition (43) of the GKL-kernels and straightforward computations show

$$\lim_{N\to\infty} \int_0^\infty \hat{g}^N_k(x)\hat{g}^N_h(x)dx = \int_0^\infty \varphi_k(x)\varphi_h(x)dx$$

$$= \lim_{N\to\infty} \int_0^\infty \int_0^\infty \hat{\Omega}^N_k(x,u)\ f'(u)\ du \int_0^\infty \hat{\Omega}^N_h(x,v)\ f'(v)\ dvdx$$

$$= \int_0^\infty \int_0^\infty \hat{\Omega}_k(x,u)\ f'(u)\ du \int_0^\infty \hat{\Omega}_h(x,v)\ f'(v)\ dvdx$$

$$= \int_0^\infty f'(u)\ du \int_0^\infty \min(u,v)\ f'(v)\ dv = \int_0^\infty f'(u)\ du \int_0^u f'(v)\ vdv$$

$$- \int_0^\infty f'(u)\ f(u)u\ du = -2\int_0^\infty f'(u)\ f(u)u\ du = \int_0^\infty [f(u)]^2 du$$

and we prove the Parseval type equality. On the other hand, appealing to equalities (45), (46) we get

$$\lim_{N\to\infty} \int_0^x \hat{g}^N_{\left\{ k \atop h \right\}}(x)dx = \int_0^x \varphi_{\left\{ k \atop h \right\}}(x)dx = -\int_0^\infty \int_0^\infty \hat{\mathcal{K}}(x,t) \frac{\left\{ k(yt) \atop h(yt) \right\}}{t} f'(y)\ dtdy$$

$$= \int_0^x \hat{g}_{\left\{ k \atop h \right\}}(x)dx, \tag{47}$$

where the passage to the limit under the integral sign in (46) is possible via the following estimate

$$\int_0^\infty \int_{1/N}^N \left| \hat{\mathcal{K}}(x,t) \frac{\left\{ k(yt) \atop h(yt) \right\}}{t} f'(y) \right| dt dy \le \int_0^\infty |f'(y)| \left(\int_0^\infty \left| \hat{\mathcal{K}}(x,t) \right|^2 dt \right)^{1/2}$$

$$\times \left(\int_0^\infty \frac{|k(yt)|^2}{t^2} dt \right)^{1/2} dy = x^{1/2} \left(\int_0^\infty \frac{|k(t)|^2}{t^2} dt \right)^{1/2} \int_0^\infty y^{1/2} |f'(y)| dy < \infty.$$

Hence differentiating with respect to x in (47) we immediately obtain that $\varphi_{\left\{ k \atop h \right\}}(x) = \hat{g}_{\left\{ k \atop h \right\}}(x)$ for almost all $x > 0$. Further, changing the order of integration in (45) by Fubini's theorem and then integrating by parts in the inner integral with respect to y, we find

$$\hat{g}_{\left\{ k \atop h \right\}}(x) = \frac{d}{dx} \int_0^\infty \hat{\mathcal{K}}(x,t) \int_0^\infty \left\{ \begin{matrix} \hat{k}(yt) \\ \hat{h}(yt) \end{matrix} \right\} f(y)\ dy dt.$$

Consequently, we complete the proof of Theorem 4 by using boundedness properties of the Kontorovich-Lebedev and Watson transforms similarly as we did above. Analogously we prove the converse part of the theorem. Theorem 4 is proved.

Remark 1. Under condition $k(x) = \overline{h(x)}$ Corollary 1 is true for transformations (34) and their reciprocities (36).

3 Index transforms with non Watson kernels

In this section we will consider transformations (33), (35) with another interpretation of the corresponding GKL-kernels (21), (22), having functions k, h as non Watson kernels in the sense of Definition 1. Namely, despite $k^*(s), h^*(s)$ still satisfy equation (7), one of them is unbounded on the line $\text{Re}s = \frac{1}{2}$.

Definition 3. Let functions

$$\frac{\rho_k(x)}{x} \in L_2(\mathbb{R}_+; dx) \quad \text{and} \quad \frac{\rho_k^*(s)}{1-s} \in L_2\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right),$$

where

$$\rho_k^*(s) = \Gamma\left(\frac{3}{4} - \frac{s}{2}\right) k^*(s),$$

$$\frac{\theta_h(x)}{x} \in L_2(\mathbb{R}_+; dx) \quad \text{and} \quad \frac{\theta_h^*(s)}{1-s} \in L_2\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right),$$

where

$$\theta_h^*(s) = \frac{h^*(s)}{\Gamma\left(\frac{1}{4} + \frac{s}{2}\right)}$$

be reciprocal Mellin pairs in L_2, where

$$\sup_{-\infty<t<\infty} \left|\rho_k^*\left(\frac{1}{2} + it\right)\right| = \sup_{-\infty<t<\infty} \left|\Gamma\left(\frac{1}{2} + \frac{it}{2}\right) k^*\left(\frac{1}{2} - it\right)\right| \leq A_k < \infty,$$

$$\sup_{-\infty<t<\infty} \left|\theta_h^*\left(\frac{1}{2} + it\right)\right| = \sup_{-\infty<t<\infty} \left|\frac{h^*\left(\frac{1}{2} - it\right)}{\Gamma\left(\frac{1}{2} + \frac{it}{2}\right)}\right| \leq A_h < \infty.$$

Let also condition (7) hold. Then we call k, h the extended conjugate Watson kernels.

It follows directly from Definition 3 that $\rho_k,\ \theta_h$ are conjugate Watson kernels. Returning now to (21), (22) and taking into account representation (31) we can identify GKL-kernels $W_k(x, y), \Omega_k(x, y)$ in the Watson case employing again the Mellin-Parseval equality. Precisely, we have (see (30))

$$W_k(x, y) = \frac{d}{dy} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s, y) k^*(1 - s) \frac{x^s}{s} ds$$

$$= \frac{d}{dy} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Phi(s, y)}{\Gamma\left(\frac{1}{4} + \frac{s}{2}\right)} \rho_k^*(1 - s) \frac{x^s}{s} ds, \tag{48}$$

$$\Omega_k(x, y) = \frac{d}{dx} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(1 - s, y) k^*(s) \frac{x^{1-s}}{1 - s} ds$$

$$= \frac{d}{dx} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Phi(1 - s, y)}{\Gamma\left(\frac{3}{4} - \frac{s}{2}\right)} \rho_k^*(s) \frac{x^{1-s}}{1 - s} ds. \tag{49}$$

On the other hand, making use integral representation (19), we return to original functions under the inverse Mellin transform and we write GKL-kernels (48), (49) in the form

$$W_k(x, y) = \frac{d}{dy} \int_0^\infty \mathcal{H}(t, y) \rho_k(xt) \frac{dt}{t}, \tag{50}$$

$$\Omega_k(x, y) = \frac{d}{dx} \int_0^\infty \mathcal{H}(t, y) \rho_k(xt) \frac{dt}{t}, \tag{51}$$

where

$$\mathcal{H}(t, y) = \frac{e^{-t^2/8}}{\pi\sqrt{2t}} \int_0^y \frac{K_{iu/2}(t^2/8)}{|\Gamma(iu)|} du \tag{52}$$

is the kernel of the modified Kontorovich-Lebedev transform. Analogously, basing on the formula (20) we derive the following relations

$$W_h(x,y) = \frac{d}{dy}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(1-s,y)h^*(s)\frac{x^{1-s}}{1-s}ds$$

$$= \frac{d}{dy}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(1-s,y)\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)\theta_h^*(s)\frac{x^{1-s}}{1-s}ds, \tag{53}$$

$$\Omega_h(x,y) = \frac{d}{dx}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(1-s,y)h^*(s)\frac{x^{1-s}}{1-s}ds$$

$$= \frac{d}{dx}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(1-s,y)\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)\theta_h^*(s)\frac{x^{1-s}}{1-s}ds, \tag{54}$$

$$W_h(x,y) = \frac{d}{dy}\int_0^\infty \hat{\mathcal{H}}(t,y)\theta_h(xt)\frac{dt}{t}, \tag{55}$$

$$\Omega_h(x,y) = \frac{d}{dx}\int_0^\infty \hat{\mathcal{H}}(t,y)\theta_h(xt)\frac{dt}{t}, \tag{56}$$

where

$$\hat{\mathcal{H}}(t,y) = \frac{e^{t^2/8}}{\sqrt{2t}}\int_0^y \frac{K_{iu/2}(t^2/8)}{\cosh(\pi u/2)|\Gamma(iu)|}du \tag{57}$$

is the conjugate modified Kontorovich-Lebedev kernel. Our goal is to prove analogs of Theorems 1, 2, 3 for the kernels (50), (51), (55), (56).

Consider, for instance, transformations (35) with kernels (51), (56). Hence taking into account (49), (54) by the Mellin-Parseval equality integrals (35) can be written in the form

$$\omega_k(y) \equiv \omega_k(y,f) = \frac{d}{dy}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Phi(s,y)}{\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)}\rho_k^*(1-s)f^*(s)ds, \tag{58}$$

$$\omega_h \equiv \omega_h(y,f) = \frac{d}{dy}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s,y)\Gamma\left(\frac{3}{4}-\frac{s}{2}\right)\theta_h^*(1-s)f^*(s)ds, \tag{59}$$

where $f^*(s) \in L_2\left(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty\right)$ is the Mellin transform in $L_2(\mathbb{R}_+)$. Hence coming back to the original functions and invoking (52), (57) we obtain the representations

$$\omega_k(y,f) = \frac{d}{dy}\int_0^\infty \mathcal{H}(t,y)g_{\rho_k}(t)dt, \tag{60}$$

$$\omega_h(y,f) = \frac{d}{dy}\int_0^\infty \hat{\mathcal{H}}(t,y)g_{\theta_h}(t)dt, \tag{61}$$

where $g_{\rho_k}(t),\ g_{\theta_h}(t)$ are conjugate Watson transforms like (1), (2) given in the form of Mellin´s integrals in L_2

$$g_{\left\{ \rho_k \atop \theta_h \right\}}(t) = \text{l.i.m.}_{N\to\infty}\frac{1}{2\pi i}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \left\{ \begin{matrix} \rho_k \\ \theta_h \end{matrix} \right\}^* (s) f^*(1-s)t^{-s}ds. \tag{62}$$

So we have proved the composition of index transforms (35) in terms of the Watson transforms, the modified Kontorovich-Lebedev transform

$$H(y,f) = \frac{d}{dy}\int_0^\infty \mathcal{H}(t,y)f(t)dt \tag{63}$$

and its conjugate

$$\hat{H}(y,f) = \frac{d}{dy}\int_0^\infty \hat{\mathcal{H}}(t,y)f(t)dt. \tag{64}$$

A relationship of (63), (64) with the Kontorovich-Lebedev transform (8) can be given appealing again to (30), (31) and the Mellin-Parseval equality. Thus after simple manipulations we obtain,

$$H(y,f) = \frac{d}{dy}\int_0^\infty \mathcal{K}(t,y)(L^{-1}f)(t)dt \tag{65}$$

and its conjugate

$$\hat{H}(y,f) = \frac{d}{dy}\int_0^\infty \mathcal{K}(t,y)(Lf)(t)dt, \tag{66}$$

where $(Lf)(x),\ (L^{-1}f)(x)$ can be interpreted as a generalized Laplace transform and its inverse [7], namely

$$(Lf)(x) = \text{l.i.m.}_{N\to\infty}\frac{1}{2\pi i}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \Gamma\left(\frac{1}{4}+\frac{s}{2}\right) f^*(1-s)x^{-s}ds, \tag{67}$$

$$(L^{-1}f)(x) = \text{l.i.m.}_{N\to\infty}\frac{1}{2\pi i}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \frac{f^*(1-s)}{\Gamma\left(\frac{3}{4}-\frac{s}{2}\right)}x^{-s}ds \tag{68}$$

and we assume that

$$||Lf||^2_{L_2(\mathbb{R}_+)} = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\Gamma\left(\frac{1}{2}+\frac{it}{2}\right)\right|^2 \left|f^*\left(\frac{1}{2}+it\right)\right|^2 dt$$

$$= \frac{1}{2}\int_{-\infty}^\infty \left|f^*\left(\frac{1}{2}+it\right)\right|^2 \frac{dt}{\cosh(\pi t/2)} < \infty, \tag{69}$$

$$||L^{-1}f||^2_{L_2(\mathbb{R}_+)} = \frac{1}{2\pi^2}\int_{-\infty}^\infty \left|f^*\left(\frac{1}{2}+it\right)\right|^2 \cosh\left(\frac{\pi t}{2}\right) dt < \infty. \tag{70}$$

In particular, for $f \in L_2(\mathbb{R}_+)$ the right-hand side of (67) becomes

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)f^*(1-s)x^{-s}ds = 2\int_0^\infty (xt)^{1/4}e^{-(xt)^2}f(t)dt,$$

which is a modification of the Laplace transform. Moreover taking into account (68), we have the equalities

$$(L\ L^{-1}f)(x) = (L^{-1}\ Lf)(x) = \text{l.i.m.}_{N\to\infty}\frac{1}{2\pi i}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} f^*(s)x^{-s}ds = f(x).$$

Hence, in a similar manner we use (12), (65), (66) to derive the following Parseval identity for the modified Kontorovich-Lebedev transform and its conjugate

$$\int_0^\infty H(y,f)\hat{H}(y,f)dy = \int_0^\infty (Lf)(t)(L^{-1}f(t)dt = \int_0^\infty [f(t)]^2\,dt \tag{71}$$

and the equalities of norms

$$||H(\cdot,f)||_{L_2(\mathbb{R}_+)} = ||L^{-1}f||_{L_2(\mathbb{R}_+)},$$

$$||\hat{H}(\cdot,f)||_{L_2(\mathbb{R}_+)} = ||Lf||_{L_2(\mathbb{R}_+)}.$$

Theorem 5. *For the extended conjugate Watson kernels k, h in the sense of Definition* 3 *and any f_1, f_2, which satisfy condition* (70) *transformations* (35) *with kernels* (51), (56) *define almost everywhere functions $\omega_k(y,f_1)$, $\omega_h(y,f_2) \in L_2(\mathbb{R}_+)$ and the generalized Parseval equality*

$$\int_0^\infty \omega_k(y,f_1)\omega_h(y,f_2)dy = \int_0^\infty f_1(x)f_2(x)dx \tag{72}$$

holds. The reciprocal inversions for almost all $x > 0$ are transformations (33) *with kernels* (50), (55).

Proof. In fact, calling representations (60), (61), (62) the Parseval equality (71) and the properties of Watson's kernels we obtain

$$\int_0^\infty \omega_k(y,f_1)\omega_h(y,f_2)dy = \int_0^\infty g_{\rho_k}(t)g_{\theta_h}(t)dt = \int_0^\infty f_1(x)f_2(x)dx,$$

which proves (72). Moreover, with the boundedness properties of the Kontorovich-Lebedev and Watson transforms, relations (63), (64), (65), (66) we have the norm inequalities

$$[\sqrt{A}_h]^{-1}||L^{-1}f_1||_{L_2(\mathbb{R}_+)} \le ||\omega_k(\cdot,f_1)||_{L_2(\mathbb{R}_+)} \le \sqrt{A}_k||L^{-1}f_1||_{L_2(\mathbb{R}_+)}, \tag{73}$$

$$[\sqrt{A}_k]^{-1}||Lf_2||_{L_2(\mathbb{R}_+)} \le ||\omega_h(\cdot, f_2)||_{L_2(\mathbb{R}_+)} \le \sqrt{A}_h||Lf_2||_{L_2(\mathbb{R}_+)}, \tag{74}$$

where the norm $||Lf_2||_{L_2(\mathbb{R}_+)}$ is finite via conditions of the theorem.

In order to prove the inversion formulas we call again (60), (65) and inversion formula (10) for the Kontorovich-Lebedev transform. As a result we find the reciprocity

$$(L^{-1}g_{\rho_k})(x) = \frac{d}{dx}\int_0^\infty \hat{\mathcal{K}}(x,\tau)\omega_k(\tau, f)d\tau. \tag{75}$$

Taking a sequence $\{\chi_n(\tau)\}$ of smooth functions with compact support, which approximates $\omega_k(\tau, f)$ in L_2 we get via (73) that

$$u_n(x) = \frac{d}{dx}\int_0^\infty \hat{\mathcal{K}}(x,\tau)\chi_n(\tau)d\tau$$

has the limit $(L^{-1}g_{\rho_k})(x)$ in L_2. Hence differentiating with respect to x under the integral sign and integrating by parts we deduce (see (9), (11))

$$u_n(x) = -\int_0^\infty \mathcal{K}(x,\tau)\chi_n'(\tau)d\tau.$$

Hence taking the L-operator from both sides of the latter equality, we change the order of integration. Calculating the inner integral with respect to x we obtain (see (57))

$$(Lu_n)(x) = -\int_0^\infty \hat{\mathcal{H}}(x,\tau)\chi_n'(\tau)d\tau.$$

Now applying the conjugate Watson transform g_{θ_h}, using (55), (56) and integrating by parts, we find the composition of operators

$$(g_{\theta_h} \circ Lu_n)(x) = \frac{d}{dx}\int_0^\infty W_h(x,\tau)\chi_n(\tau)d\tau,$$

which yields the equality

$$\int_0^x (g_{\theta_h} \circ Lu_n)(x)dx = \int_0^\infty W_h(x,\tau)\chi_n(\tau)d\tau.$$

Hence passing to the limit when $n \to \infty$ and taking into account the boundedness and invertibility properties of the involved operators in L_2 we get

$$\lim_{n\to\infty}\int_0^x (g_{\theta_h} \circ Lu_n)(x)dx = \int_0^x (g_{\theta_h} \circ g_{\rho_k})(x)dx = \int_0^x f(x)dx$$

$$= \int_0^\infty W_h(x,\tau)\omega_k(\tau, f)d\tau$$

or for almost all $x > 0$ we prove the inversion formula (33) with the kernel (55)

$$f(x) = \frac{d}{dx} \int_0^\infty W_h(x,\tau)\omega_k(\tau, f) d\tau.$$

Analogously, calling representations (61), (66) and basing on estimations (74) in the same manner we establish the inversion formula

$$f(x) = \frac{d}{dx} \int_0^\infty W_k(x,\tau)\omega_h(\tau, f) d\tau$$

for the transform $\omega_h(\tau, f)$. Theorem 5 is proved.

Remark 2. Similar results we can prove for other index transformations with the extended Watson kernels.

4 Examples

In this final section we will pay our attention on the concrete cases of GKL-kernels and the corresponding index transforms. Among them we will find familiar classical index transformations.

For instance, taking the Watson kernel

$$k_c(x) = \sqrt{\frac{2}{\pi}} \sin x = \sqrt{\frac{2}{\pi}} \int_0^x \cos x \; dx,$$

we substitute it in (21). Then differentiating with respect to y under the integral sign we appeal to the relation (2.16.14.3) in [3, Vol. 2] and we get the following GKL-kernel

$$W_{k_c}(x,y) = \frac{2}{\pi|\Gamma(iy)|} \int_0^\infty K_{iy}(t) \frac{\sin xt}{t^{3/2}} dt = \frac{x}{\pi\sqrt{2}} \frac{\left|\Gamma\left(\frac{1}{4} + \frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|}$$

$$\times {}_2F_1\left(\frac{1}{4} + \frac{iy}{2}, \frac{1}{4} - \frac{iy}{2}; \frac{3}{2}; -x^2\right),$$

which is written in terms of the Gauss hypergeometric function [1, Vol. I]. Differentiating the kernel $W_{k_c}(x,y)$ with respect to x we find similarly

$$\frac{\partial}{\partial x} W_{k_c}(x,y) = \frac{2}{\pi|\Gamma(iy)|} \int_0^\infty K_{iy}(t) \frac{\cos xt}{t^{1/2}} dt = \frac{1}{\pi\sqrt{2}} \frac{\left|\Gamma\left(\frac{1}{4} + \frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|}$$

$$\times {}_2F_1\left(\frac{1}{4} + \frac{iy}{2}, \frac{1}{4} - \frac{iy}{2}; \frac{1}{2}; -x^2\right).$$

Thus calling the relation (22) of the kernel $\Omega_{k_c}(x,y)$, Theorem 2, Corollary 1 and equality (25) we obtain the following Watson- Plancherel type theorem for the Olevskii transform (c.f. [7]).

Theorem 6. *The index transform*

$$g_{k_c}(x) = l.i.m._{N\to\infty}\frac{1}{\pi\sqrt{2}}\int_{1/N}^{N}\frac{\left|\Gamma\left(\frac{1}{4}+\frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|}{}_2F_1\left(\frac{1}{4}+\frac{iy}{2},\ \frac{1}{4}-\frac{iy}{2};\ \frac{1}{2};-x^2\right)f(y)dy$$

is an isometric automorphism in $L_2(\mathbb{R}_+)$ *with the reciprocal inversion*

$$f(y) = l.i.m._{N\to\infty}\frac{1}{\pi\sqrt{2}}\frac{\left|\Gamma\left(\frac{1}{4}+\frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|}\int_{1/N}^{N}{}_2F_1\left(\frac{1}{4}+\frac{iy}{2},\ \frac{1}{4}-\frac{iy}{2};\ \frac{1}{2};-x^2\right)g_{k_c}(x)dx$$

and the Parseval identity

$$\int_0^\infty |g_{k_c}(x)|^2dx = \int_0^\infty |f(y)|^2dy.$$

Moreover, this statement is equivalent to the following Watson equality

$$\frac{1}{2\pi^2}\int_0^\infty\frac{\left|\Gamma\left(\frac{1}{4}+\frac{iy}{2}\right)\right|^4}{|\Gamma(iy)|^2}{}_2F_1\left(\frac{1}{4}+\frac{iy}{2},\ \frac{1}{4}-\frac{iy}{2};\ \frac{3}{2};-\xi^2\right){}_2F_1\left(\frac{1}{4}+\frac{iy}{2},\ \frac{1}{4}-\frac{iy}{2};\ \frac{3}{2};-\eta^2\right)dy$$

$$= \min\left(\frac{1}{\xi},\ \frac{1}{\eta}\right),\qquad \xi,\ \eta>0.$$

Remark 2. According to relation (2.16.14.3) in [3, Vol. 2] GKL-kernel $W_{k_c}(x,y)$ can be reduced to the combination of the generalized Legendre functions. Namely, we have the representation

$$W_{k_c}(x,y) = \frac{\Gamma\left(iy-\frac{1}{2}\right)}{2|\Gamma(iy)|}\frac{(x^2+1)^{1/4}}{\cos\left((1+2iy)\frac{\pi}{4}\right)}\left[P_{-3/2}^{-iy}\left(-\frac{x}{\sqrt{1+x^2}}\right)-P_{-3/2}^{-iy}\left(\frac{x}{\sqrt{1+x^2}}\right)\right].$$

The Watson kernel

$$k_s(x) = \sqrt{\frac{2}{\pi}}(1-\cos x) = \sqrt{\frac{2}{\pi}}\int_0^x \sin x\ dx$$

via (22), (35) and a differentiation under the integral sign leads to the following Plancherel theorem.

Theorem 7. *The index transform*

$$\omega_{k_s}(y) = l.i.m._{N\to\infty}\frac{\sqrt{2}}{\pi}\frac{\left|\Gamma\left(\frac{3}{4}+\frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|}\int_{1/N}^{N}{}_2F_1\left(\frac{3}{4}+\frac{iy}{2},\ \frac{3}{4}-\frac{iy}{2};\ \frac{3}{2};-x^2\right)f(x)xdx$$

is an isometric automorphism in $L_2(\mathbb{R}_+)$ *with the reciprocal inversion*

$$f(x) = l.i.m._{N\to\infty} \frac{x\sqrt{2}}{\pi} \int_{1/N}^{N} \frac{\left|\Gamma\left(\frac{3}{4}+\frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|} {}_2F_1\left(\frac{3}{4}+\frac{iy}{2},\ \frac{3}{4}-\frac{iy}{2};\ \frac{3}{2}; -x^2\right) \omega_{k_s}(y)dy$$

and the Parseval equality

$$\int_0^\infty |\omega_{k_c}(y)|^2 dy = \int_0^\infty |f(x)|^2 dx.$$

On the other hand, calculating directly the kernel $W_{k_s}(x,y)$ by using a series representation for the cosine function and relation (2.16.2.1) in [3, Vol.2], we find the result

$$W_{k_s}(x,y) = \frac{2}{\pi|\Gamma(iy)|} \int_0^\infty K_{iy}(t) \frac{1-\cos xt}{t^{3/2}} dt = \frac{x^2}{\pi\sqrt{2}} \frac{\left|\Gamma\left(\frac{3}{4}+\frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|}$$

$$\times {}_3F_2\left(\frac{3}{4}+\frac{iy}{2},\ \frac{3}{4}-\frac{iy}{2},\ 1;\ \frac{3}{2},\ 2; -x^2\right)$$

in terms of the hypergeometric function ${}_3F_2$. Hence we have

Theorem 8. *The index transform*

$$g_{k_s}(x) = l.i.m._{N\to\infty} \frac{x^2}{\pi\sqrt{2}} \int_{1/N}^{N} \frac{\left|\Gamma\left(\frac{3}{4}+\frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|} {}_3F_2\left(\frac{3}{4}+\frac{iy}{2},\ \frac{3}{4}-\frac{iy}{2},\ 1;\ \frac{3}{2},\ 2; -x^2\right) f(y)dy$$

is an isometric automorphism in $L_2(\mathbb{R}_+)$ *with the reciprocal inversion*

$$f(y) = l.i.m._{N\to\infty} \frac{1}{\pi\sqrt{2}} \frac{\left|\Gamma\left(\frac{3}{4}+\frac{iy}{2}\right)\right|^2}{|\Gamma(iy)|} \int_{1/N}^{N} {}_3F_2\left(\frac{3}{4}+\frac{iy}{2},\ \frac{3}{4}-\frac{iy}{2},\ 1;\ \frac{3}{2},\ 2; -x^2\right) g_{k_s}(x)\, x^2 dx$$

and the Parseval identity

$$\int_0^\infty |g_{k_s}(x)|^2 dx = \int_0^\infty |f(y)|^2 dy.$$

The statement of the theorem is equivalent to the following Watson equality

$$\frac{1}{2\pi^2} \int_0^\infty \frac{\left|\Gamma\left(\frac{3}{4}+\frac{iy}{2}\right)\right|^4}{|\Gamma(iy)|^2} {}_3F_2\left(\frac{3}{4}+\frac{iy}{2},\ \frac{3}{4}-\frac{iy}{2},\ 1;\ \frac{3}{2},\ 2; -\xi^2\right)$$

$$\times {}_3F_2\left(\frac{3}{4}+\frac{iy}{2},\ \frac{3}{4}-\frac{iy}{2},\ 1;\ \frac{3}{2},\ 2; -\eta^2\right) dy = \frac{\min\,(\xi,\eta)}{\xi^2\eta^2}, \quad \xi,\ \eta > 0.$$

More general Watson kernel is given involving the Bessel function $J_\nu(x)$. Precisely, we have

$$k_\nu(x) = \int_0^x \sqrt{t}\ J_\nu(t)dt,\ \nu > -1.$$

Using the relation (1.8.1.1) in [3, Vol. 2] the latter integral can be expressed in terms of the hypergeometric function ${}_1F_2$

$$k_\nu(x) = \frac{x^{\nu+3/2}}{2^\nu(\nu+3/2)\Gamma(\nu+1)}\,{}_1F_2\left(\frac{3}{4}+\frac{\nu}{2};\ \frac{7}{4}+\frac{\nu}{2},\ \nu+1;\ -\frac{x^2}{4}\right).$$

Thus the corresponding GKL-kernel is related to the Olevskii transform and appealing to the relation (2.16.21.1) in [3, Vol. 2] we easily obtain the formula

$$\frac{\partial}{\partial x}W_{k_\nu}(x,y) = \sqrt{\frac{2x}{\pi}}\frac{1}{|\Gamma(iy)|}\int_0^\infty K_{iy}(t)J_\nu(xt)dt$$

$$= \frac{1}{\sqrt{2\pi}}\frac{x^{\nu+1/2}|\Gamma\left((1+\nu+iy)/2\right)|^2}{\Gamma(\nu+1)|\Gamma(iy)|}\,{}_2F_1\left(\frac{1+\nu+iy}{2},\frac{1+\nu-iy}{2};\nu+1;-x^2\right).$$

Therefore we get a generalization of Theorem 7 on an arbitrary $\nu > -1$ having

Theorem 9. *The index transform*

$$\omega_{k_\nu}(y) = l.i.m._{N\to\infty}\frac{1}{\sqrt{2\pi}}\frac{|\Gamma\left((1+\nu+iy)/2\right)|^2}{\Gamma(\nu+1)|\Gamma(iy)|}$$

$$\times\int_{1/N}^N {}_2F_1\left(\frac{1+\nu}{2}+\frac{iy}{2},\ \frac{1+\nu}{2}-\frac{iy}{2};\ 1+\nu;-x^2\right)f(x)x^{\nu+1/2}dx$$

is an isometric automorphism in $L_2(\mathbb{R}_+)$ *with the reciprocal inversion*

$$f(x) = l.i.m._{N\to\infty}\frac{x^{\nu+1/2}}{\Gamma(\nu+1)\sqrt{2\pi}}$$

$$\times\int_{1/N}^N \frac{|\Gamma\left((1+\nu+iy)/2\right)|^2}{|\Gamma(iy)|}\,{}_2F_1\left(\frac{1+\nu}{2}+\frac{iy}{2},\ \frac{1+\nu}{2}-\frac{iy}{2};\ 1+\nu;-x^2\right)\omega_{k_\nu}(y)dy$$

and the Parseval equality

$$\int_0^\infty |\omega_{k_\nu}(y)|^2dy = \int_0^\infty |f(x)|^2dx.$$

We note that Theorem 7 is an immediate consequence of the previous statement when $\nu = \frac{1}{2}$. Putting $\nu = 0$ we arrive at the Plancherel - Watson theorem for the Mehler-Fock transform [4, 7, 8].

Theorem 10. *The index transform*

$$\mathcal{MF}(y) = l.i.m._{N\to\infty} \frac{\sqrt{\pi}}{\sqrt{2}\cosh(\pi y/2)|\Gamma(iy)|} \int_{1/N}^{N} P_{(iy-1)/2}(1+2x^2) f(x)\sqrt{x}\, dx$$

is an isometric automorphism in $L_2(\mathbb{R}_+)$ *with the reciprocal inversion*

$$f(x) = l.i.m._{N\to\infty} \sqrt{\frac{\pi x}{2}} \int_{1/N}^{N} \frac{P_{(iy-1)/2}(1+2x^2)}{\cosh(\pi y/2)|\Gamma(iy)|}\, \mathcal{MF}(y) dy$$

and the Parseval equality

$$\int_0^\infty |\mathcal{MF}(y)|^2 dy = \int_0^\infty |f(x)|^2 dx.$$

Further, a direct calculation of the kernel $W_{k_\nu}(x,y)$ yields

$$W_{k_\nu}(x,y) = \frac{\sqrt{2}}{\sqrt{\pi}|\Gamma(iy)|} \int_0^\infty K_{iy}(t) \frac{k_\nu(xt)}{t^{3/2}} dt = \frac{\sqrt{2}x^{3/2+\nu}}{\sqrt{\pi}(2\nu+3)\Gamma(1+\nu)} \frac{\left|\Gamma\left(\frac{\nu+1+iy}{2}\right)\right|^2}{|\Gamma(iy)|}$$

$$\times {}_3F_2\left(\frac{\nu+1+iy}{2}, \frac{\nu+1-iy}{2}, \frac{\nu}{2}+\frac{3}{4}; 1+\nu, \frac{\nu}{2}+\frac{3}{4}+1; -x^2\right).$$

Hence we have a generalization of Theorem 8 ($\nu = \frac{1}{2}$).

Theorem 11. *Let* $\nu > -1$. *The index transform*

$$g_{k_\nu}(x) = l.i.m._{N\to\infty} \frac{\sqrt{2}x^{3/2+\nu}}{\sqrt{\pi}(2\nu+3)\Gamma(1+\nu)}$$

$$\int_{1/N}^{N} \frac{\left|\Gamma\left(\frac{\nu+1+iy}{2}\right)\right|^2}{|\Gamma(iy)|} {}_3F_2\left(\frac{\nu+1+iy}{2}, \frac{\nu+1-iy}{2}, \frac{\nu}{2}+\frac{3}{4};\ 1+\nu, \frac{\nu}{2}+\frac{3}{4}+1; -x^2\right) f(y) dy$$

is an isometric automorphism in $L_2(\mathbb{R}_+)$ *with the reciprocal inversion*

$$f(y) = l.i.m._{N\to\infty} \frac{\sqrt{2}}{\sqrt{\pi}(2\nu+3)\Gamma(1+\nu)} \frac{\left|\Gamma\left(\frac{\nu+1+iy}{2}\right)\right|^2}{|\Gamma(iy)|}$$

$$\int_{1/N}^{N} {}_3F_2\left(\frac{\nu+1+iy}{2}, \frac{\nu+1-iy}{2}, \frac{\nu}{2}+\frac{3}{4};\ 1+\nu, \frac{\nu}{2}+\frac{3}{4}+1; -x^2\right) g_{k_\nu}(x)\ x^{3/2+\nu} dx$$

and the Parseval identity

$$\int_0^\infty |g_{k_\nu}(x)|^2 dx = \int_0^\infty |f(y)|^2 dy.$$

The statement of the theorem is equivalent to the following Watson equality

$$\frac{2}{\pi(2\nu+3)^2\Gamma^2(1+\nu)}\int_0^\infty \frac{\left|\Gamma\left(\frac{\nu+1+iy}{2}\right)\right|^4}{|\Gamma(iy)|^2}$$

$$\times {}_3F_2\left(\frac{\nu+1+iy}{2}, \frac{\nu+1-iy}{2}, \frac{\nu}{2}+\frac{3}{4};\ 1+\nu, \frac{\nu}{2}+\frac{3}{4}+1; -\xi^2\right)$$

$$\times {}_3F_2\left(\frac{\nu+1+iy}{2}, \frac{\nu+1-iy}{2}, \frac{\nu}{2}+\frac{3}{4};\ 1+\nu, \frac{\nu}{2}+\frac{3}{4}+1; -\eta^2\right) dy$$

$$= \frac{\min\,(\xi,\eta)}{(\xi\eta)^{3/2+\nu}}, \quad \xi,\ \eta > 0.$$

The case of the Wimp-Yakubovich index transform with the Meijer G-function (see [7, Ch. 7]) can be obtained by considering the Watson kernel of type

$$k_G(x) = 2\int_0^x G^{m,n}_{2n,2m}\left(t^2 \left| \begin{matrix} (a_n+\frac{1}{2}), (-\overline{a}_n) \\ (b_m), (\frac{1}{2}-\overline{b}_m) \end{matrix} \right.\right) dt,$$

where

$$G^{m,n}_{2n,2m}\left(x^2 \left| \begin{matrix} \left(a_n+\frac{1}{2}\right), (-\overline{a}_n) \\ (b_m), \left(\frac{1}{2}-\overline{b}_m\right) \end{matrix} \right.\right) = \frac{1}{2\pi i}\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \frac{\prod_{k=1}^m \Gamma(b_k+s)\prod_{k=1}^n \Gamma\left(\frac{1}{2}-a_k-s\right)}{\prod_{k=1}^n \Gamma(-\overline{a}_k+s)\prod_{k=1}^m \Gamma\left(\frac{1}{2}+\overline{b}_k-s\right)} x^{-2s} ds$$

$$= \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\prod_{k=1}^m \Gamma\left(b_k+\frac{s}{2}\right)\prod_{k=1}^n \Gamma\left(\frac{1-s}{2}-a_k\right)}{\prod_{k=1}^n \Gamma\left(-\overline{a}_k+\frac{s}{2}\right)\prod_{k=1}^m \Gamma\left(\frac{1-s}{2}+\overline{b}_k\right)} x^{-s} ds.$$

Under the assumptions

$$\mathrm{Re}\ b_j > -\frac{\gamma}{2},\ j=1,\cdots,m; \quad \mathrm{Re}\ a_j < \frac{1-\gamma}{2},\ j=1,\cdots,n,\ 0\le\gamma<\frac{1}{2}$$

we see that the latter integrand has no poles in the strip $-\gamma \le \mathrm{Re}\ s \le 1-\gamma$. Moreover, when $m-n \ge \left[\frac{1}{\frac{1}{2}-\gamma}\right]+1$, then the contour of integration $\left(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty\right)$ can be replaced by the vertical line $(\gamma-i\infty, \gamma+i\infty)$ and we obtain the representation

$$G^{m,n}_{2n,2m}\left(x^2 \left| \begin{matrix} \left(a_n+\frac{1}{2}\right), (-\overline{a}_n) \\ (b_m), \left(\frac{1}{2}-\overline{b}_m\right) \end{matrix} \right.\right) = \frac{1}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod_{k=1}^m \Gamma\left(b_k+\frac{s}{2}\right)\prod_{k=1}^n \Gamma\left(\frac{1-s}{2}-a_k\right)}{\prod_{k=1}^n \Gamma\left(-\overline{a}_k+\frac{s}{2}\right)\prod_{k=1}^m \Gamma\left(\frac{1-s}{2}+\overline{b}_k\right)} x^{-s} ds,$$

where the latter integral converges absolutely. Thus via (9), (21) and (30) we find the expression for the partial derivative with respect to x of the GKL- kernel $W_{k_G}(x,y)$, namely

$$\frac{\partial}{\partial x}W_{k_G}(x,y) = 2\sqrt{\frac{2}{\pi}}\frac{1}{|\Gamma(iy)|}\int_0^\infty K_{iy}(t)G^{m,n}_{2n,2m}\left((xt)^2 \left| \begin{matrix} \left(a_n+\frac{1}{2}\right), (-\overline{a}_n) \\ (b_m), \left(\frac{1}{2}-\overline{b}_m\right) \end{matrix} \right.\right)\frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2\sqrt{\pi}\;|\Gamma(iy)|}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod_{k=1}^{m}\Gamma\left(b_k+\frac{s}{2}\right)\prod_{k=1}^{n}\Gamma\left(\frac{1-s}{2}-a_k\right)}{\prod_{k=1}^{n}\Gamma\left(-\overline{a}_k+\frac{s}{2}\right)\prod_{k=1}^{m}\Gamma\left(\frac{1-s}{2}+\overline{b}_k\right)}$$

$$\times\Gamma\left(\frac{1-s}{2}-\frac{1}{4}+\frac{iy}{2}\right)\Gamma\left(\frac{1-s}{2}-\frac{1}{4}-\frac{iy}{2}\right)(2x)^{-s}ds$$

$$= \frac{1}{\sqrt{\pi}\;|\Gamma(iy)|}G^{m,n+2}_{2(n+1),2m}\left(4x^2 \left| \begin{matrix} \frac{1}{4}+\frac{iy}{2}, \frac{1}{4}-\frac{iy}{2}, \; \left(a_n+\frac{1}{2}\right), (-\overline{a}_n) \\ (b_m), \left(\frac{1}{2}-\overline{b}_m\right) \end{matrix} \right.\right).$$

Theorem 12. *The index transform*

$$\omega_{k_G}(y) = l.i.m._{N\to\infty}\frac{1}{\sqrt{\pi}\;|\Gamma(iy)|}$$

$$\int_{1/N}^{N} G^{m,n+2}_{2(n+1),2m}\left(4x^2 \left| \begin{matrix} \frac{1}{4}+\frac{iy}{2}, \frac{1}{4}-\frac{iy}{2}, \; \left(a_n+\frac{1}{2}\right), (-\overline{a}_n) \\ (b_m), \left(\frac{1}{2}-\overline{b}_m\right) \end{matrix} \right.\right) f(x)dx$$

is an isometric automorphism in $L_2(\mathbb{R}_+)$ with the reciprocal inversion

$$f(x) = l.i.m._{N\to\infty}\frac{1}{\sqrt{\pi}}\int_{1/N}^{N} G^{m,n+2}_{2(n+1),2m}\left(4x^2 \left| \begin{matrix} \frac{1}{4}+\frac{iy}{2}, \frac{1}{4}-\frac{iy}{2}, \; \left(a_n+\frac{1}{2}\right), (-\overline{a}_n) \\ (b_m), \left(\frac{1}{2}-\overline{b}_m\right) \end{matrix} \right.\right)\frac{\omega_{k_G}(y)}{|\Gamma(iy)|}dy$$

and the Parseval equality

$$\int_0^\infty |\omega_{k_G}(y)|^2 dy = \int_0^\infty |f(x)|^2 dx.$$

Now we will consider an example of index transforms involving the extended Watson kernels. This transform is associated with Whittaker's functions and was investigated in [12]. In fact, following Definition 3, we take

$$k^*(s) = \left[\Gamma\left(\frac{3}{4}-\frac{s}{2}-i\lambda\right)\right]^{-1}, \quad h^*(s) = \Gamma\left(\frac{1}{4}+\frac{s}{2}-i\lambda\right), \; \lambda\in\mathbb{R}.$$

These kernels evidently satisfy equation (7) and k^* is unbounded on the line Re $s = \frac{1}{2}$. Hence the corresponding GKL- kernels (49), (54) can be written in the form (cf. [12])

$$\frac{d}{dy}\Omega_k(x,y) = \frac{\partial^2}{\partial y \partial x}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Phi(s,y)}{\Gamma\left(\frac{1}{4}+\frac{s}{2}-i\lambda\right)}\frac{x^s}{s}ds = \frac{\sqrt{2x}}{\sqrt{\pi}|\Gamma(iy)|}e^{-(8x^2)^{-1}}W_{i\lambda,iy}\left(\frac{1}{4x^2}\right),$$

$$\frac{d}{dy}\Omega_h(x,y) = \frac{\partial^2}{\partial y \partial x}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s,y)\Gamma\left(\frac{3}{4}-\frac{s}{2}-i\lambda\right)\frac{x^s}{s}ds$$

$$= \frac{\sqrt{2x}\Gamma\left(\frac{1}{2}-i(\lambda+y)\right)\Gamma\left(\frac{1}{2}-i(\lambda-y)\right)}{\sqrt{\pi}|\Gamma(iy)|}e^{(8x^2)^{-1}}W_{i\lambda,iy}\left(\frac{1}{4x^2}\right),$$

where $W_{\mu,\nu}(z)$ is Whittaker's function [1, Vol. II].

Theorem 13. *Let f_1, f_2 satisfy condition (70). Index transforms with Whittaker's function*

$$g_1(y) = l.i.m._{N\to\infty}\frac{\sqrt{2}}{\sqrt{\pi}|\Gamma(iy)|}\int_{1/N}^{N} e^{-(8x^2)^{-1}}W_{i\lambda,iy}\left(\frac{1}{4x^2}\right)f_1(x)\sqrt{x}\,dx,$$

$$g_2(y) = l.i.m._{N\to\infty}\frac{\sqrt{2}\Gamma\left(\frac{1}{2}-i(\lambda+y)\right)\Gamma\left(\frac{1}{2}-i(\lambda-y)\right)}{\sqrt{\pi}|\Gamma(iy)|}$$

$$\times\int_{1/N}^{N} e^{(8x^2)^{-1}}W_{i\lambda,iy}\left(\frac{1}{4x^2}\right)f_2(x)\sqrt{x}\,dx,$$

are defined in $L_2(\mathbb{R}_+)$ and reciprocal inversion formulas hold

$$f_1(x) = l.i.m._{N\to\infty}\sqrt{\frac{2x}{\pi}}\int_{1/N}^{N}\frac{\Gamma\left(\frac{1}{2}-i(\lambda+y)\right)\Gamma\left(\frac{1}{2}-i(\lambda-y)\right)}{|\Gamma(iy)|}$$

$$\times e^{(8x^2)^{-1}}W_{i\lambda,iy}\left(\frac{1}{4x^2}\right)g_1(y)\,dy,$$

$$f_2(x) = l.i.m._{N\to\infty}\sqrt{\frac{2x}{\pi}}\int_{1/N}^{N} e^{-(8x^2)^{-1}}W_{i\lambda,iy}\left(\frac{1}{4x^2}\right)g_2(y)\,\frac{dy}{|\Gamma(iy)|}$$

as well as the generalized Parseval equality (72).

Finally, as an application we take two Watson kernels $k_{G_1},\ k_{G_2}$ as Meijer's G-functions (see above)

$$k_{G_1}(x) = 2\int_0^x G^{m_1,n_1}_{2n_1,2m_1}\left(t^2\left|\begin{matrix}(a_{n_1}+\frac{1}{2}),(-\overline{a}_{n_1})\\(b_{m_1}),(\frac{1}{2}-\overline{b}_{m_1})\end{matrix}\right.\right)dt,$$

$$k_{G_2}(x) = 2\int_0^x G^{m_2,n_2}_{2n_2,2m_2}\left(t^2 \left| \begin{matrix} (c_{n_2}+\frac{1}{2}), (-\overline{c}_{n_2}) \\ (d_{m_2}), (\frac{1}{2}-\overline{d}_{m_2}) \end{matrix} \right.\right) dt,$$

and we use the Parseval equalities for the Kontorovich-Lebedev and the Mellin transform. As a result we come out with the value of a general index integral of the product of two G-functions, which involves many particular formulas containing, for example, in [3, Vol. 2]

$$\frac{1}{4\pi^2}\int_0^\infty \tau \sinh \pi\tau G^{m_1,\ n_1+2}_{2(n_1+1),\ 2m_1}\left(4x^2 \left| \begin{matrix} \frac{1}{4}+\frac{i\tau}{2}, \frac{1}{4}-\frac{i\tau}{2},\ \left(a_{n_1}+\frac{1}{2}\right), (-\overline{a}_{n_1}) \\ (b_{m_1}), \left(\frac{1}{2}-\overline{b}_{m_1}\right) \end{matrix} \right.\right)$$

$$\times G^{m_2,\ n_2+2}_{2(n_2+1),\ 2m_2}\left(4y^2 \left| \begin{matrix} \frac{1}{4}+\frac{i\tau}{2}, \frac{1}{4}-\frac{i\tau}{2},\ \left(c_{n_2}+\frac{1}{2}\right), (-\overline{c}_{n_2}) \\ (d_{m_2}), \left(\frac{1}{2}-\overline{d}_{m_2}\right) \end{matrix} \right.\right) d\tau$$

$$= \int_0^\infty G^{m_1,\ n_1}_{2n_1,\ 2m_1}\left((xt)^2 \left| \begin{matrix} \left(a_{n_1}+\frac{1}{2}\right), (-\overline{a}_{n_1}) \\ (b_{m_1}), \left(\frac{1}{2}-\overline{b}_{m_1}\right) \end{matrix} \right.\right) G^{m_2,\ n_2}_{2n_2,\ 2m_2}\left((yt)^2 \left| \begin{matrix} \left(c_{n_2}+\frac{1}{2}\right), (-\overline{c}_{n_2}) \\ (d_{m_2}), \left(\frac{1}{2}-\overline{d}_{m_2}\right) \end{matrix} \right.\right) dt$$

$$= \frac{1}{4y} G^{m_1+n_2,\ m_2+n_1}_{2(m_2+n_1),\ 2(m_1+n_2)}\left(\left(\frac{x}{y}\right)^2 \left| \begin{matrix} \left(a_{n_1}+\frac{1}{2}\right), \left(\frac{1}{2}-d_{m_2}\right), (-\overline{a}_{n_1}), (\overline{d}_{m_2}) \\ (b_{m_1}),\ (-c_{n_2}),\ \left(\frac{1}{2}-\overline{b}_{m_1}\right),\ \left(\overline{c}_{n_2}+\frac{1}{2}\right) \end{matrix} \right.\right).$$

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