~~To arrive to this formula, they re-cast the problem fully in K-theory.~~

~~- Ingredients of proof:~~

~~(I) Wrong-way functoriality:~~

~~If $i: X \hookrightarrow Y$ is an embedding, we get a map~~

E AS $\Rightarrow$ Gauss-Bonnet

Thm (Gauss - Bonnet)

If M is a compact orientable Riemannian manifold, $\dim M = 2$

then
$$\chi(M) = \frac{1}{2\pi}\int_M r \, dx$$

where

$\chi(M)$ = Euler characteristic = $\sum_j (-1)^j \dim H^j(M;\mathbb{R})$.

r = scalar curvature = $\sum_{l,m} R_{lmlm}$,

where $R_{ijkl} = \langle R(X_k, X_l) X_j, X_i \rangle$

- It is derived by applying AS to an appropriate differential operator:

Some Hodge theory: Let M: $\left\{\begin{array}{l}\text{closed}\\ \text{orientable}\\ \text{Riemannian}\end{array}\right\}$ mfd

- Consider the de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M)$$

- Define the operator $*: \Omega^p(M) \to \Omega^{n-p}(M)$, where:

$*\alpha :=$ the unique $(n-p)$-form s.t.

$$(\alpha, \beta) \cdot \mathrm{Vol} = \beta \wedge (*\alpha)$$

where (α, β) = pt/wise inner product of (α, β) (function on M)

e.g.) $M = \mathbb{R}^3$, $\mathrm{Vol} = dx\,dy\,dz$. Then:

$*dx = dy\,dz$, $*dy = -dx\,dz$, $*dx\,dy = dz$, $*dx\,dz = -dy$.

Lemma: $**\alpha = (-1)^{pn+p}\alpha$

Dfn: If $\alpha \in \Omega^p(M)$, define $\boxed{\delta\alpha = (-1)^{np+n+1} * d * \alpha}$

So $\delta: \Omega^p(M) \to \Omega^{p-1}(M)$ and $\delta^2 = 0$ because $d^2 = 0$.

- We'll show that δ is the adjoint of d:

Dfn: Let $\alpha, b \in \Omega^p(M)$. Define

$$\langle \alpha, b \rangle = \int_M (\alpha, b)\,\mathrm{Vol} = \int_M b \wedge *\alpha = \int_M \alpha \wedge *b.$$

$\rightarrow$

Lemma: $\langle \alpha, db \rangle = \langle \delta\alpha, b \rangle$ if $\alpha \in \Omega^p(M)$, $b \in \Omega^{p-1}(M)$.

Proof:

$$0 = \int d(b \wedge *\alpha) = \int db \wedge *\alpha + (-1)^{p-1} \int b \wedge d(*\alpha) =$$

$$= \langle db, \alpha \rangle + (-1)^{np+n} \langle b, *d*\alpha \rangle. \quad \square$$

Thm (Hodge) $\Omega^p(M) = \ker(d) \oplus \mathrm{Im}(d^*)$
$= \ker(d^*) \oplus \mathrm{Im}(d)$

Remark: The sums are direct, because:

$$\left.\begin{array}{l} \alpha \in \ker d \\ b \in d^*\gamma \in \mathrm{Im}\, d^* \end{array}\right\} \Rightarrow \langle \alpha, b \rangle = \langle \alpha, d^*\gamma \rangle = \langle d\alpha, \gamma \rangle = 0$$

Proof is a matter of PDEs:
Need to show that if $b \perp \ker d$ then $\exists$ solution of
$d^*\gamma = b$. $\square$

Dfn: $\boxed{D : \Omega^*(M) \to \Omega^*(M), \quad D = d + d^*}$

Lemma: $\ker(D) = \ker(d) \cap \ker(d^*)$.

Proof $D^2 = d^*d + dd^*$ because $d^2 = (d^*)^2 = 0$, so:

$$\|D\alpha\|^2 = \langle D^2\alpha, \alpha \rangle = \langle (d^*d + dd^*)\alpha, \alpha \rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2. \quad \square$$

Dfn: An $\alpha \in \Omega^p(M)$ (M: compact + Riemannian) is harmonic if $d\alpha = 0$ and $d^*\alpha = 0$.

Lemma: Every cohomological class of M contains exactly one harmonic form. So, $\ker(D) \cong \bigoplus_p H^p(M)$

Proof (i) Existence

Let $d\alpha = 0$. From Hodge thm $\Rightarrow \Omega^p(M) = \ker(d^*) \oplus \mathrm{Im}(d)$

So $\alpha = \alpha_1 + db$, where $d^*\alpha_1 = 0 \Rightarrow \alpha_1 = \alpha - db$

So $[\alpha_1] = [\alpha]$ and α_1 is harmonic

(ii) Uniqueness

Let α_1, α_2: harmonic.

If $[\alpha_1 - \alpha_2] = 0 \Rightarrow \alpha_1 - \alpha_2 \in \mathrm{Im}(d)$.

But we have $\alpha_1 - \alpha_2 \in \ker(d^*)$. because $\mathrm{Im}(d) \perp \ker(d^*)$

So we must have $\alpha_1 - \alpha_2 = 0$. ∎

Lemma: $\boxed{\text{Hodge thm} \Longleftrightarrow \Omega^*(M) = \ker(D) \oplus \mathrm{Im}(D)}$

Proof

$\longrightarrow$

$\Leftarrow$) Let $\alpha \in \underline{\Omega}^p(M)$. This will be written:

$$\alpha = \alpha_1 + (d\alpha_2 + d^*\alpha_3)$$

where $\alpha_1 \in \ker(D)$.

- We can assume $\left\{ \begin{array}{l} \alpha_1 \in \underline{\Omega}^p(M) \\ \alpha_2 \in \underline{\Omega}^{p-1}(M) \end{array} \right\}$ because:

$\ker(D) = \ker(d) \cap \ker(d^*)$, so if $\alpha_1 \in \underline{\Omega}^*(M)$ is a general element of $\ker(D)$ then its coordinates also belong to $\ker(D)$.

- Since $d^2 = 0$, $\Rightarrow \operatorname{Im}(d) \subseteq \ker(d)$, so:

$$\alpha = \underbrace{\alpha_1 + d\alpha_2}_{\substack{\cap \\ \ker(d)}} + \underbrace{d^*\alpha_3}_{\substack{\cap \\ \operatorname{Im}(d^*)}}.$$

$\Rightarrow$) Likewise $\blacksquare$

- The operator D maps $\underline{\Omega}^{even}(M) = \bigoplus_p \underline{\Omega}^{2p}(M) \to \underline{\Omega}^{odd}(M) = \bigoplus_p \underline{\Omega}^{2p+1}(M)$.

From Hodge thm and previous lemmas we have:

$$\operatorname{Ind}(D : \underline{\Omega}^{even} \to \underline{\Omega}^{odd}) = \sum_p (-1)^p \dim H^p(M) = \chi(M).$$

- If $\dim(M) = 2$ then AS $\Rightarrow$ Gauss-Bonnet.
