Ricci Flow and the Euclidean Gravitational Action

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Outline

- Ricci Flow
- Euclidean Gravitational Action
- Euclidean Schwarzschild
  - Headrick & Wiseman : hep-th/0606086
- Taub-NUT and Taub-Bolt
  - Holzegel, Schmelzer & CW : 0706.1694
- Further questions and future directions

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Ricci Flow

- Ricci flow is a geometric flow given by

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t))$$

- May combine with a diffeomorphism

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t)) + (\nabla\xi)_S$$

- For suitable ξ RHS is elliptic, hence have local existence and uniqueness for compact manifolds (de Turck trick)
- Tends to expand regions of negative curvature and contract regions of positive curvature

Ricci Flow

- For round S^n

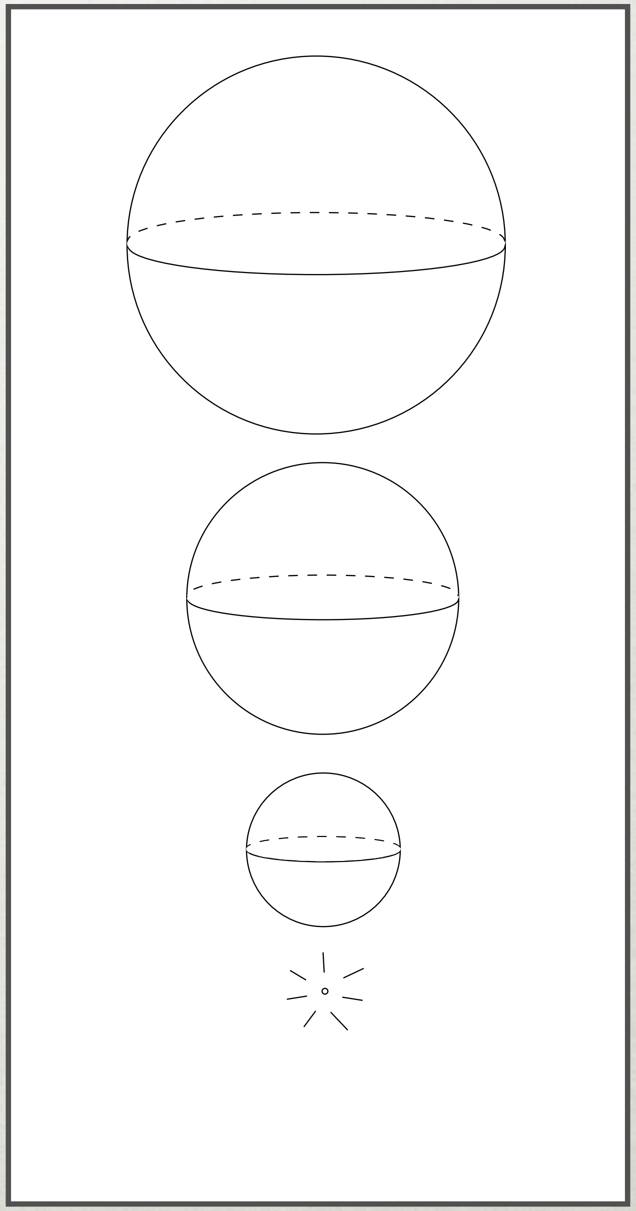
$$g(t) = r(t)^2 g_0$$

- Have $Ric(g) = (n-1)g_0$, so

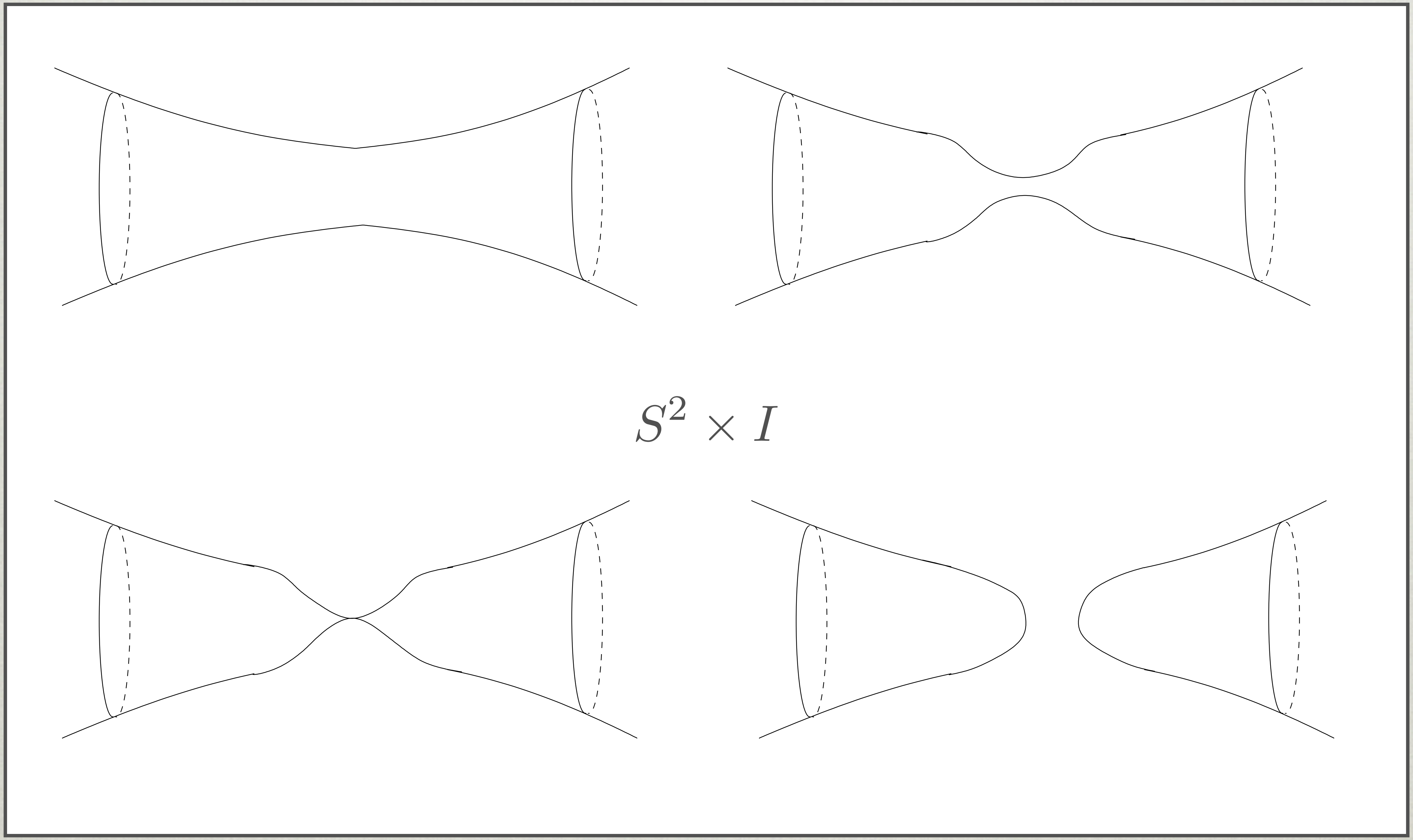
$$\frac{dr}{dt} = -\frac{n-1}{r}$$

- Solution is

$$r(t) = \sqrt{2(n-1)}\sqrt{T-t}$$



Ricci Flow



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Why Riemannian 4-manifolds?

- Propagators of a thermalized quantum field in flat space obey KMS condition

$$G(x-x', t-t'+i\beta) = G(x-x', t-t')$$

- Wick rotate, $t \to it$ and propagators are periodic with period β
- Can interpret as propagators on a flat Riemannian space with topology $S^1 \times \mathbb{R}^3$, 'hot flat space'
- For Schwarzschild find that Wick rotation only gives smooth manifold when it has period $\beta = 1/T_H$

Partition Function

- For a standard field theory, would define partition function to be

$$\mathcal{Z} = \int d[\phi] e^{-S[\phi]/\hbar}$$

Over periodic fields satisfying suitable b.c.s

- In the semi-classical limit $\hbar \to \infty$ this is dominated by critical points where

$$\frac{\delta S}{\delta \phi} = 0$$

Gravitational Action

- Idea of Euclidean quantum gravity is to investigate the Euclidean Einstein-Hilbert action

$$S[g] = -\frac{1}{16\pi G}\int_M \sqrt{g}R - \frac{1}{8\pi G}\int_{\partial M}\sqrt{\gamma}K$$

- Both hot flat space and Euclidean Schwarzschild are critical points

$$\frac{\delta S}{\delta g} = 0$$

- Can explore beyond critical points with a gradient flow

$$\frac{dg^A}{d\lambda} = -G^{AB}\frac{\delta S}{\delta g^B}$$

Ricci Flow as Gradient Flow

- G_{AB} is a metric on the space of metrics. Unique diffeo invariant metric with no derivatives is

$$G_{AB} dg^A dg^B = \frac{1}{32\pi} \int_M \left(dg^{\mu}{}_{\nu} + a (dg^{\mu}{}_{\mu})^2 \right) \sqrt{g}\, d^D x$$

- Gradient flow with respect to this metric is

$$\frac{dg_{\mu\nu}}{d\lambda} = -2R_{\mu\nu} + \frac{2a+1}{D+1} R g_{\mu\nu}$$

- Ricci flow when $a = -\frac{1}{2}$, G is not positive definite

Ricci Flow as RG flow

- Ricci flow arises as RG flow for a 2-D sigma model to 1 loop [Friedan]

$$S[\phi] = \frac{1}{4\pi\alpha'} \int d^2x \, g_{\mu\nu} \partial_i \phi^\mu(x) \partial^i \phi^\nu(x)$$

- To first order, RG flow is Ricci flow. With a dilaton:

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu} - 4\nabla_\mu \partial_\nu \Phi$$

$$\frac{\partial \Phi}{\partial t} = -\frac{D}{3\alpha'} + \nabla^2 \Phi - 2(\partial_\mu \Phi)^2$$

- Here $t = -\frac{1}{2}\alpha' \log \Lambda$, with Λ the energy scale

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Euclidean Schwarzschild

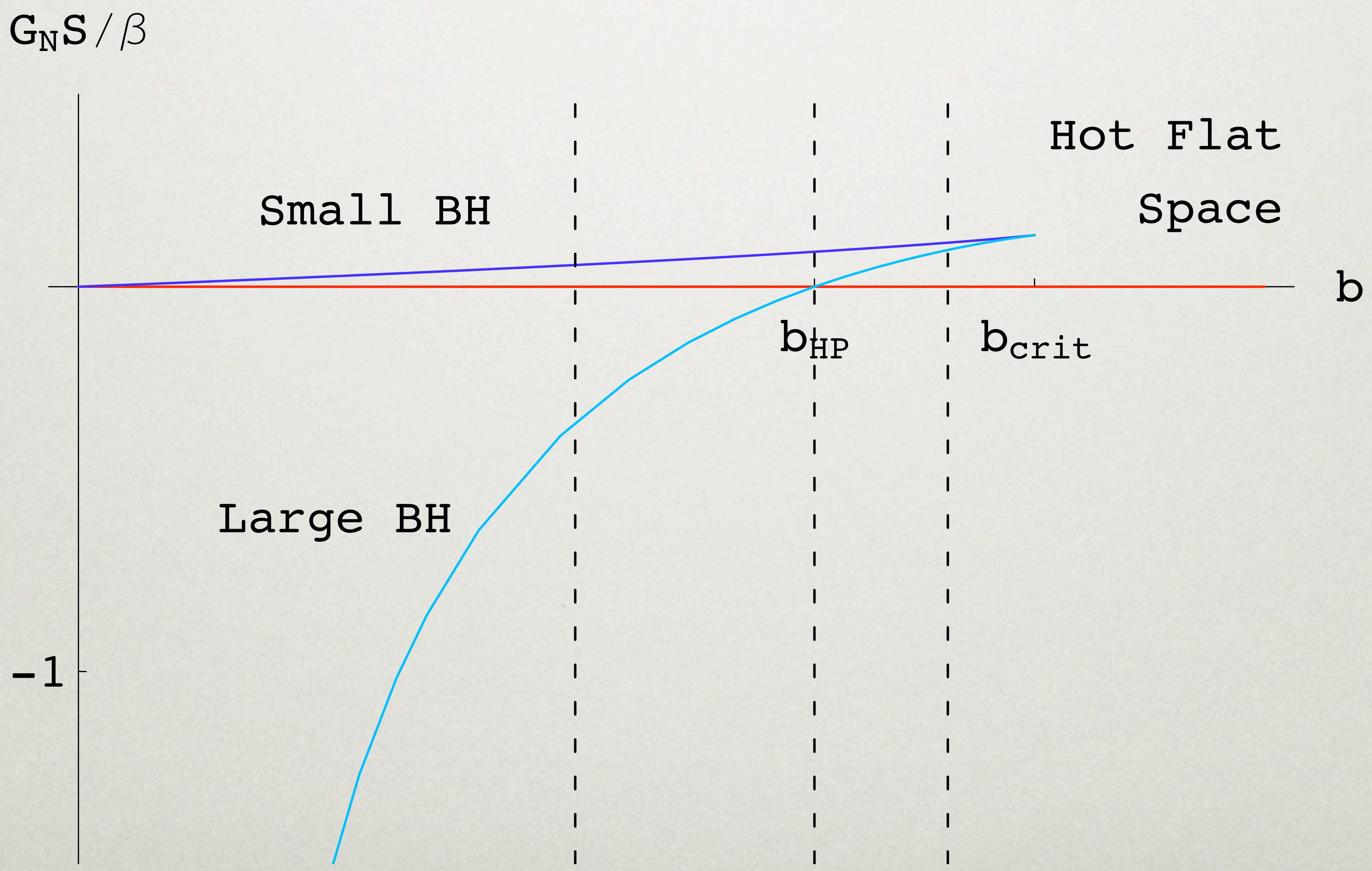
- 'Put it in a box': Seek a Ricci flat infilling $SO(3) \times U(1)$ invariant metric for a boundary $S^2 \times S^1$ with metric
$$ds^2 = R^2 \left(b^2 d\tau^2 + d\theta^2 + \sin^2\theta d\phi^2 \right)$$
- For any value of *b* have 'hot flat space', $S^1 \times \overline{B}^3_R$
- For $b < 4/3\sqrt{3}$ have two Schwarzschild metrics, both on $S^2 \times \overline{B}^2$ with metric

$$ds^2 = 4r_0^2 \left(1 - \frac{r_0}{r}\right) d\tau^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

- Regular at r_0 if τ has period 2π - metric sphere
- *r* has range $(r_0, R]$ and two values of r_0 map to each *b*

Euclidean Schwarzschild

- Can plot action *S* against *b* (HW)



Euclidean Schwarzschild

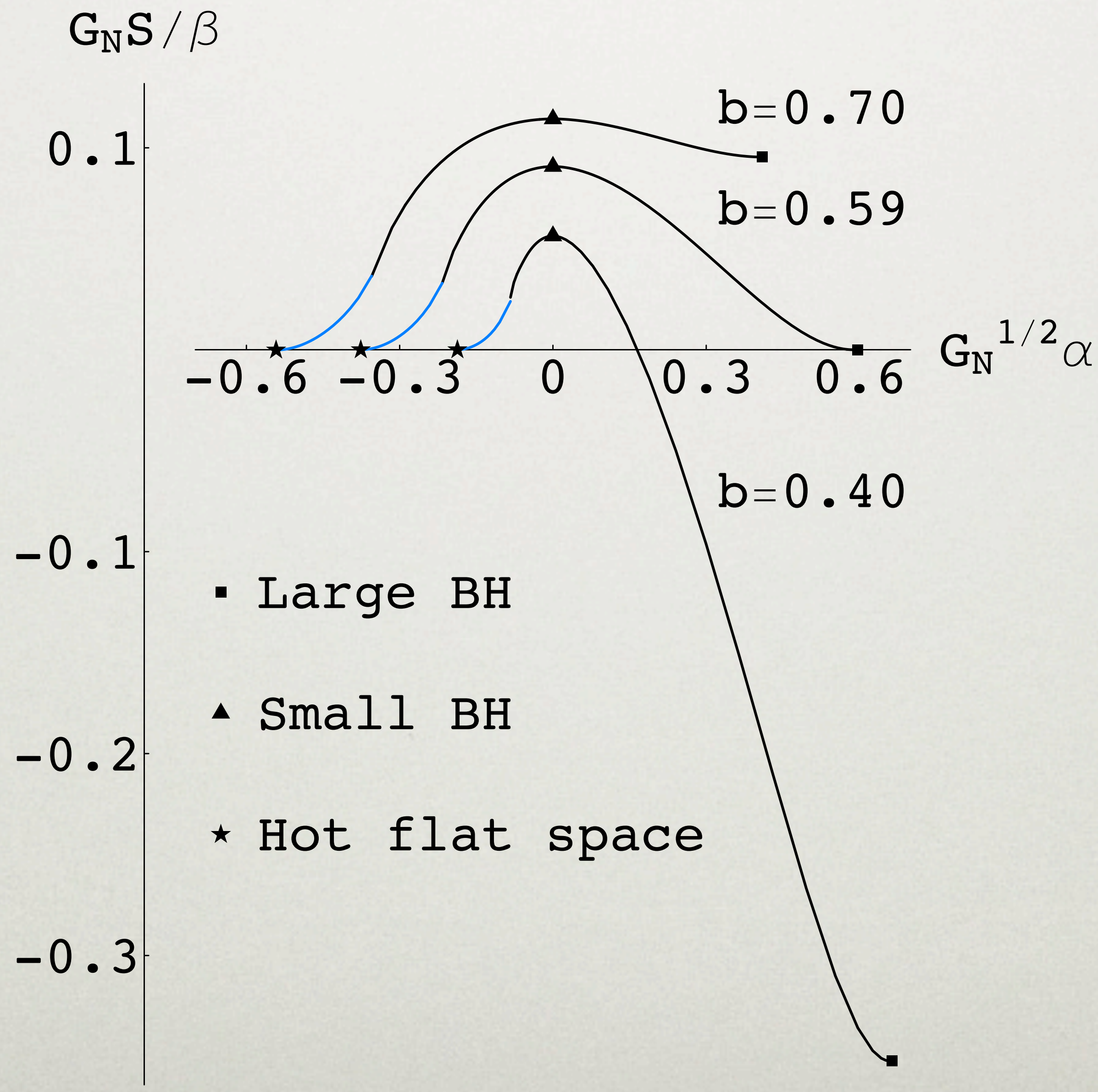
- Stability is determined by the eigenvalues of the second variation of the action:

$$\Delta^{A}{}_{B} = G^{AC} \frac{\delta^2 S}{\delta g^C \delta g^B}$$

- Fix gauge to eliminate zero modes (de Donder condition gives Lichnerowicz operator)
- Negative modes imply an instability
- Small black hole is unstable
- Flat space and large black hole are stable

Euclidean Schwarzschild

- Use Ricci flow to get a slice through space of metrics:



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Taub-NUT and Taub-Bolt

- Instead consider a homogeneously squashed S^3 boundary

$$ds^2 = \mu^2 \left(\sigma_1^2 + \sigma_2^2 + \tau^2 \sigma_3^2\right)$$

- Infilling metric takes the form

$$ds^2 = a(r)^2 dr^2 + b(r)^2 (\sigma_1^2 + \sigma_2^2) + c(r)^2 \sigma_3^2$$

- w.l.o.g $c(0) = 0$ and $a(r)^2 \sim a_0^2 + O(r^2)$

- Regular at $r = 0$ iff

$$b(0) \neq 0 \text{ and } c(r)^2 \sim \frac{1}{4} a_0^2 r^2 + O(r^4) \quad \text{(Bolt)}$$

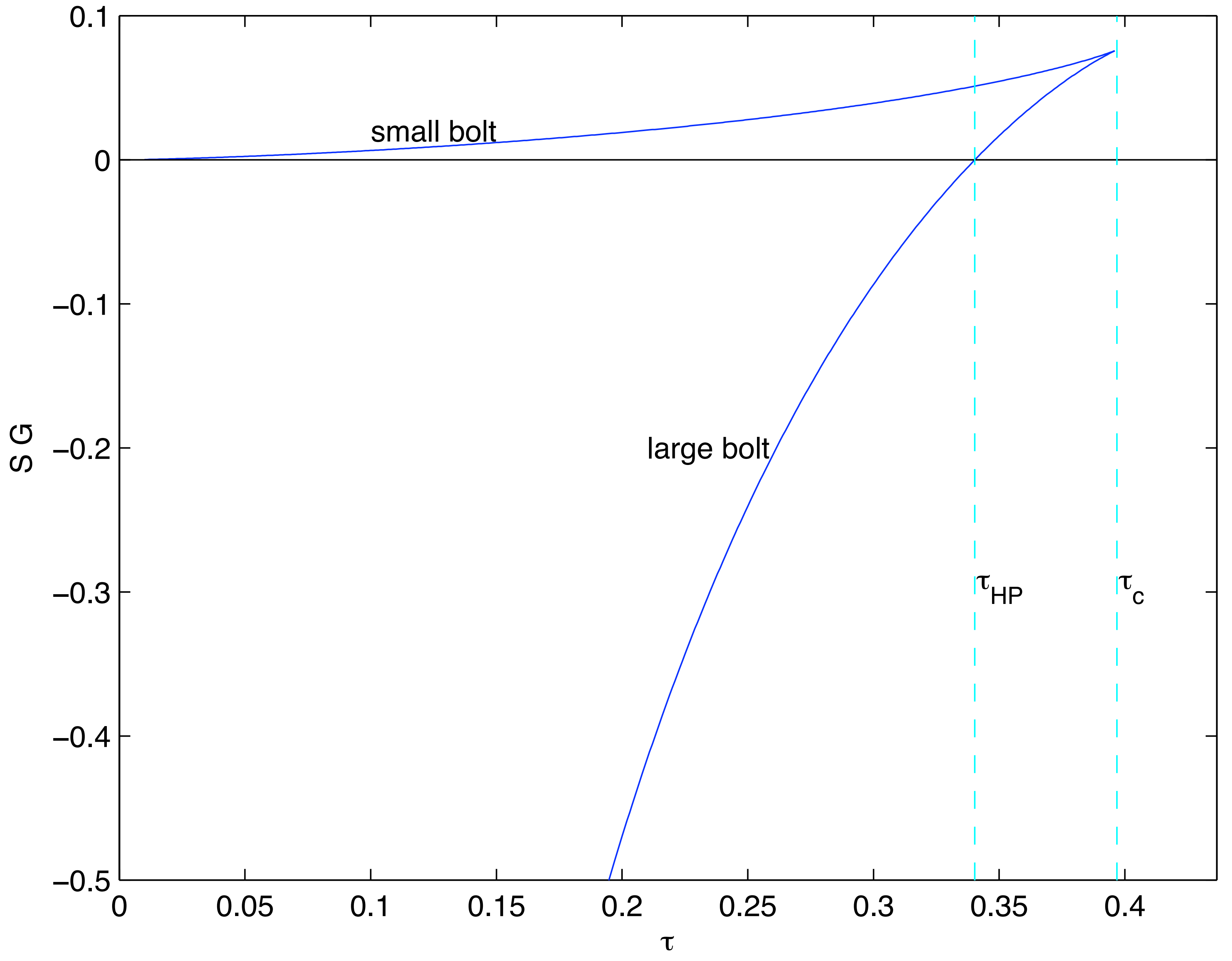
or

$$b(r)^2 \sim c(r)^2 \sim \frac{1}{4} a_0^2 r^2 + O(r^4) \qquad \text{(Nut)}$$

Taub-NUT and Taub-Bolt

- For $0 < \tau < 1$, have one infilling Ricci flat metric, Taub-NUT, containing a nut. Topology $\overline{B^4}$
- For $\tau < \tau_c$, have an additional 2 Ricci flat metrics containing bolts, Taub-Bolt. Topology $\mathbb{C}P^2 \setminus B^4$
- Taub-Bolt metrics distinguished by the size of the minimal S^2 between 'small' and 'large' bolt solutions
- Small bolt has an instability, while nut and large bolt are linearly stable
- Action diagram qualitatively similar to Schwarzschild case

Taub-NUT and Taub-Bolt



Taub-NUT and Taub-Bolt

- Simulate a Ricci flow starting at the small Taub-Bolt solution. Evolve the metric ansatz:

$$ds^2 = e^{2A(r,t)}dr^2 + e^{2B(r,t)}(\sigma_1^2 + \sigma_2^2) + r^2 e^{2C(r,t)}\sigma_3^2$$

- If bolt area increases initially, the flow smoothly approaches the large bolt solution
- If the bolt area decreases initially, get a singularity in the function B at $r = 0$ within finite flow time
- The minimal S^2 has collapsed. Requires surgery: use

$$ds^2 = e^{2\tilde{A}}\left(dr^2 + \frac{r^2}{4}e^{-2r^2\tilde{C}}\left(e^{2r^2\tilde{B}}(\sigma_1^2 + \sigma_2^2) + r^2\sigma_3^2\right)\right)$$

Taub-NUT and Taub-Bolt

- Surgery changes action *S* by an infinitesimal amount
- After surgery, metric smoothly approaches Taub-NUT solution
- Topology change

$$\mathbb{C}P^2 \setminus B^4 \longrightarrow \overline{B^4}$$

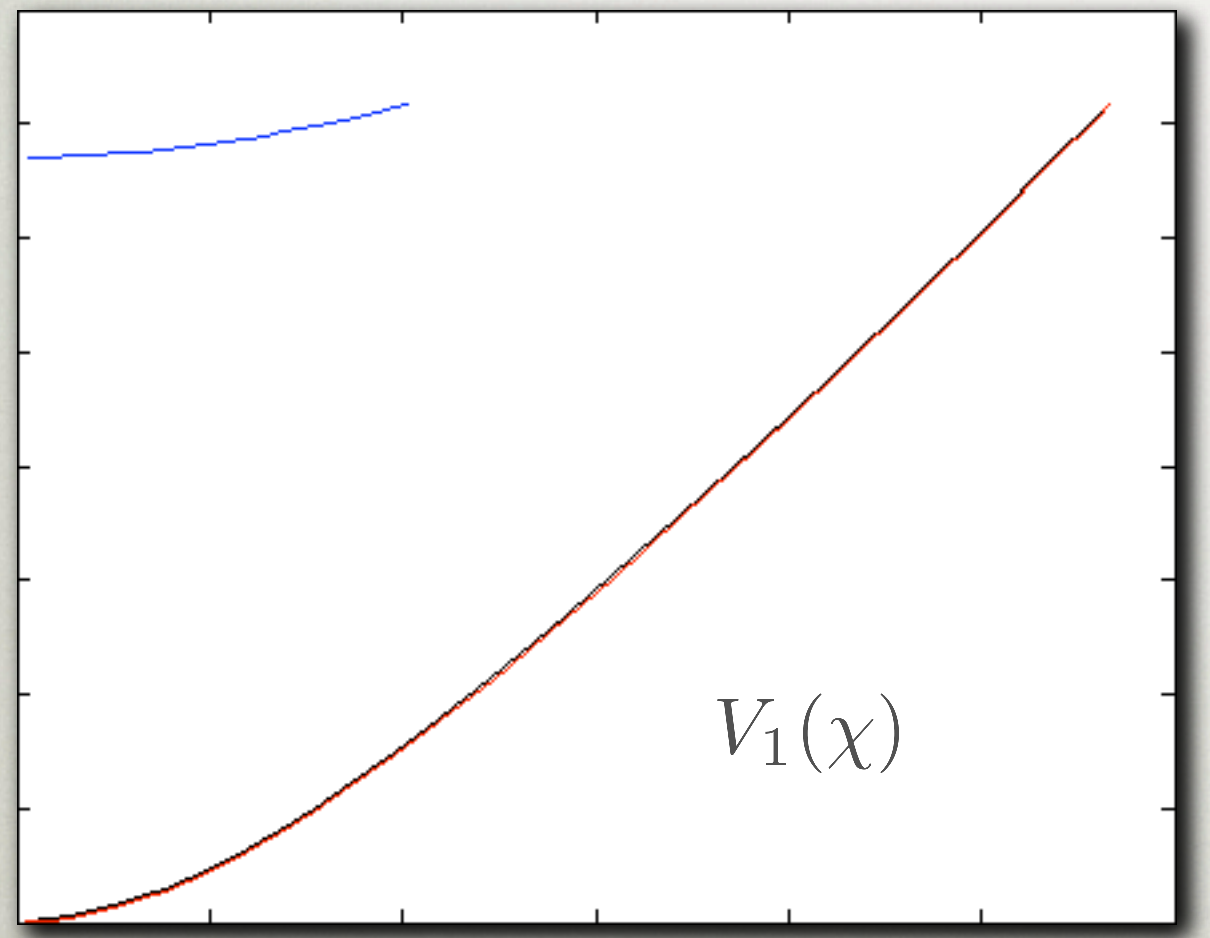
- Compare to Headrick and Wiseman

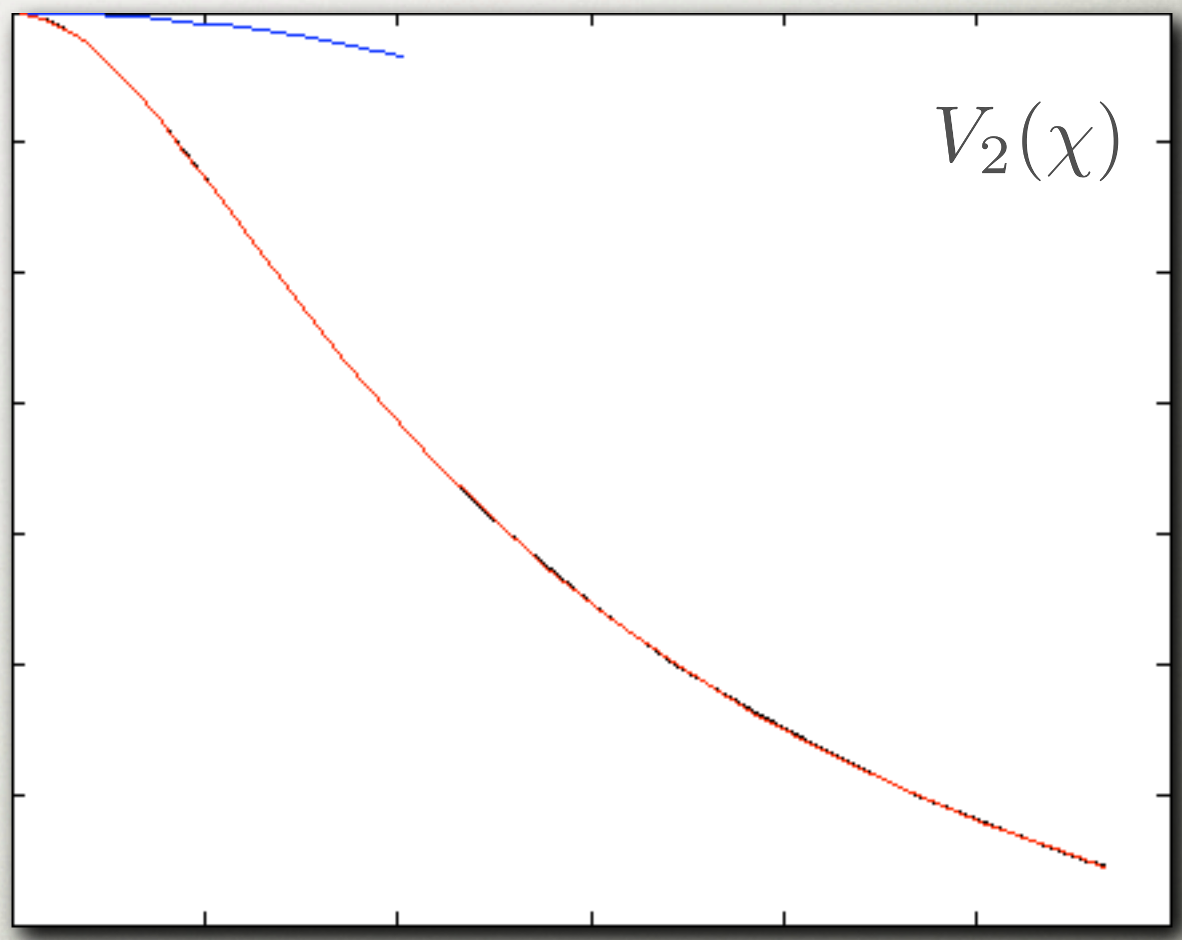
$$S^2 \times \overline{B^2} \longrightarrow S^1 \times \overline{B^3}$$

Taub-NUT and Taub-Bolt

- Can put metric into gauge invariant form

$$ds^2 = d\chi^2 + V_1(\chi)^2(\sigma_1^2 + \sigma_2^2) + \chi^2 V_2(\chi)^2 \sigma_3^2$$





Future Directions and Open Questions

- General existence and uniqueness for the Dirichlet boundary value problem
- Possible surgeries in 4-dimensions
- Is surgery necessary / possible for RG flow
- Long term properties of the flow in these specific cases
  - Do all initial data on $\overline{B^4}$ converge to Taub-NUT?
  - Do all flows within this symmetry class reach Taub-NUT or Taub-Bolt after at most one surgery?