The geometry of noncommutative k-algebras

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Commutative deformation theory

- Consider a contravariant functor $F : \mathrm{Sch}_k \rightarrow \mathrm{Sets}$. Then a scheme X is called a fine moduli for F if $F \cong \mathrm{Hom}_k(-, X)$, and the element $\mathcal{F} \in F(X)$ corresponding to the identity $\mathrm{Id} \in \mathrm{Hom}_k(X, X)$ is called the *universal family*.

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- This is because we to each closed point $x \in X$ get the element $\mathcal{F}_x = F(\operatorname{Spec} k \xrightarrow{x} X)(\mathrm{Id}) \in F(\operatorname{Spec} k)$.

Commutative deformation theory

- A fine moduli is unique up to unique isomorphism, but it seldom exists. This leads e.g to Mumfords definition of a coarse moduli space. Another possible view is the infinitesimal, brought to light by M. Schlessinger in his famous article **Functors of Artin rings** from 1968.

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- We then consider the category $\underline{\ell}$ if local Artinian k-algebras

$$\begin{array}{ccc} k & \xrightarrow{\iota} & A \\ & \searrow^{\mathrm{Id}} & \downarrow \rho \\ & & k \end{array}$$

with residue field k.

Commutative deformation theory

- The functor F above defines a covariant functor $F : \underline{\ell} \longrightarrow \text{Sets}$ defined by $F(A) = F(\operatorname{Spec} A)$, and choosing one element $M \in \operatorname{Spec} k$, we arrive at the deformation functor $\operatorname{Def}_M : \underline{\ell} \longrightarrow \text{Sets}$ defined as the fibre functor $\operatorname{Def}_M(A) = \{M_A \in F(A) | M_{A,*} \cong M\}$.

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- To the category $\underline{\ell}$ we associate the procategory $\hat{\underline{\ell}}$ consisting of complete local k-algebras $\hat{A}$ s. t. $\hat{A}/\mathfrak{m}^n \in \underline{\ell}$ for all n, and we extend the functor Def_M to $\hat{\underline{\ell}}$ by $\operatorname{Def}_M(\hat{A}) = \varprojlim_{n \geq 1} \operatorname{Def}_M(\hat{A}/\mathfrak{m}^n)$.

Commutative deformation theory

- As is proved i Schlessingers article, there is a canonical isomorphism $\mathrm{Def}_M(\hat{A}) \cong \mathrm{Hom}(\mathrm{Hom}(\hat{A}, -), \mathrm{Def}_M))$, and if $\hat{\xi} \in \mathrm{Def}_M(\hat{A})$ induces an isomorphism, $\hat{A}$ is said to prorepresent Def_M, and $\hat{\xi}$ is called a proversal family.

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- The canonical example for a scheme X is to put $F = \mathrm{Hom}_k(-, X)$ and $M = k(x)$, the residuefield of the closed point x.
- Then $\mathrm{Def}_M(\hat{A}) = \{\mathcal{O}_X \otimes_k \hat{A}\text{- modules } M_{\hat{A}} | M_{\hat{A}} \text{ is flat over } \hat{A},\ M_{\hat{A},*} \cong M\}/\sim$, and by definition, this functor is prorepresented by $(\hat{\mathcal{O}}_{X,x}, \mathrm{Id})$.

Commutative deformation theory

- This example is one of the few where prorepresentablilty exists, so this also should be "coarsified". First of all, in the example above, $\mathrm{Def}_k(\hat{A}) = \mathrm{Hom}_X(\mathrm{Spec}\,\hat{A}, X) \cong \mathrm{Hom}_k(\hat{\mathcal{O}}_{X,x}, \hat{A})$ so that the tangent space of X in x is $(\mathfrak{m}/\mathfrak{m}^2)^* \cong \mathrm{Hom}(\hat{\mathcal{O}}_{X,x}, k[x]/(x^2)) \cong \mathrm{Def}(k[\varepsilon])$.

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- So for each pointed (meaning that $F(k)$ is a one point set) covariant functor $F : \underline{\ell} \to \mathrm{Sets}$ we define $F(k[\varepsilon])$ as the tangent space of the functor. In the above example, $\mathrm{Def}(k[\varepsilon]) \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^* \cong \mathrm{Der}(\mathcal{O}_X, \mathrm{Hom}(k(x), k(x))) \cong \mathrm{Ext}^1_X(k(x), k(x))$.

Commutative deformation theory

Definition

$(\hat{H}, \hat{\xi})$ is called prorepresenting hull for $F : \hat{\underline{\ell}} \longrightarrow \mathsf{Sets}$ if the induced morphism $\phi : \mathrm{Hom}(\hat{H}, -) \to F$ is smooth and an isomorphism on the tangent level. A prorepresenting hull is unique up to (nonunique) isomorphism.

Commutative deformation theory

- Smoothness means that for *small morphisms* $0 \rightarrow I \rightarrow R \xrightarrow{\phi} S \rightarrow 0$ in $\underline{\ell}$, that is $I \cdot \mathfrak{m}_R = 0$, we can lift according to the diagram

$$\begin{array}{ccc} \exists \xi_R \in \mathrm{Hom}(\hat{H}, R) & \longrightarrow & F(R) \ni M_R \\ \downarrow & & \downarrow \\ \xi_S \in \mathrm{Hom}(\hat{H}, S) & \longrightarrow & F(S) \ni M_S \end{array}$$

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- To investigate existence of prorepresenting hulls, Schlessinger starts with the tangent space $F(k[\varepsilon])$, that is $\hat{H}/\mathfrak{m}^2 \cong k[[x_1, \ldots, x_d]]/\mathfrak{m}^2$. He then divides out with the smallest ideal that gives smoothness on the morphism $\hat{H}/\mathfrak{m}^{n+1} \to \hat{H}/\mathfrak{m}^n$ for each n.

Commutative deformation theory

O.A. Laudal defines an obstruction theory: If, in addition to a d-dimensional tangent space, there exists an r-dimensional obstruction space T^2 that is such that $M_S \in F(S)$ can be lifted to $M_R \in F(R)$ if and only if $0 = o(\phi, M_S) \in T^2 \otimes_k I$. Then there is an algorithm constructing

$$\hat{H} \cong k[[x_1, \dots, x_d]]/(f_1, \dots, f_r).$$

(In fact, my thesis was the formulation and application of this algorithm)

Commutative deformation theory

- So, if a global functor $F : \mathrm{Sch}_k \to \mathrm{Sets}$ is representable by a scheme X, we can use our algorithm to compute $\hat{\mathcal{O}}_{X,x}$ for each closed point, and generalizing, if we are searching for a moduli space for objects $\underline{c}$, we can compute the hull $\hat{H}(V)$ of the deformation functor and let this represent the completed local ring of the moduli space.

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- This is the reason for the name *local formal moduli*. If A is commutative k-algebra with only finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$, then $A \cong \prod_{i=1}^n A_{\mathfrak{m}_i}$. This is easy to see by looking into global sections, and is the standard Burnside theorem.

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- A general affine scheme is the sheafification of this fact. So if $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are maximal A-ideals, the ring $A_I = \prod_{i=1}^n A_{\mathfrak{m}_i}$ is a k-algebra with simple modules $V_i = A_{\mathfrak{m}_i}/\mathfrak{m}_i A_{\mathfrak{m}_i}$, $i = 1, \dots n$, and as such should be called a scheme for the family $V = \{V_i\}_{i=1}^n$.

Noncommutative deformation theory

- The algorithm constructing $\hat{H}(V)$ can clearly be generalized to right (or left) A-modules when A is not necessarily commutative. It can then even be generalized to construct a matrix algebra $(\hat{H}(V_i, V_j)) = (\hat{H}_{ij})$ with tangent space $(\mathrm{Ext}^1_A(V_i, V_j))$ for a finite set of right A-modules $V_1, \ldots, V_n$.

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- This is called the noncommutative local formal moduli. Our main reason for the interest of these hulls in the commutative situation was the trivial fact that a scheme X is the moduli of its closed points, or even that $\operatorname{Spec} A$ is a moduli for its simple modules.

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- The corresponding fact in the noncommutative situation is the generalized Burnside theorem: If A is finite dimensional, i.e. is complete and have the finite family of simple right-modules $V = \{V_1, \ldots, V_n\}$, then

$$A \cong (H_{ij} \otimes_k \operatorname{Hom}_k(V_i, V_j)).$$

Noncommutative deformation theory

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- For $s \in A$ we define the basisopen set $D(s) = \{V \in \mathrm{Simp}(A) | \rho(s) \text{ is invertible in } \mathrm{End}_k(V)\}$ where $\rho : A \rightarrow \mathrm{End}_k(V)$ is the structure morphism, the representation.

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- Notice that in the commutative case, we sheafify by saying that that each function $f : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ should be locally regular.
- In the noncommutative case, we sheafify by saying that each function $f : U \to \varprojlim_{\substack{V \subseteq U \\ \text{finite}}} (\hat{H}_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j))$ is locally regular.

Noncommutative deformation theory

Thus, in the commutative case we say that $X = \operatorname{Spec} A$ is a moduli for its closed points because $\hat{H}(A/\mathfrak{m}) \cong \hat{A}_{\mathfrak{m}}$ for each $\mathfrak{m}$. In the noncommutative situation, the set of finite subsets of $\operatorname{Simp} S$ is a scheme for A because $A \cong (\hat{H}_{ij} \otimes_k \operatorname{Hom}_k(V_i, V_j))$ for each finite subset.

Noncommutative deformation theory

For the classification theory, this is the following:

Definition

Let $\underline{c}$ be a set of right A-modules (or other objects). Then $\underline{c}$ is called a scheme for a k-algebra R if the simple right modules of R is in one-to-one correspondence with $\underline{c}$, and if for each finite subset V of simple modules in R, $(\hat{H}^R_{ij} \otimes_k \operatorname{Hom}_k(V_i, V_j)) \cong (\hat{H}^A_{ij})$ where c_i corresponds to V_i.

Example: Torsion free modules of rank one over the E_6 curve singularity

In the articles [6],[7] I prove that the morphism from $\mathrm{Def}_{\mathcal{F}}$ to Def_M where $M = \mathcal{F}_*$ is the localization in the singularity, is smooth. Thus I can use the theory of Def_M to compute the k-algebras covering the moduli space. Unfortunately, the internet connection at the hotel was too bad to upload the files, so I cannot give the example, but it is not too bad.

Example: A standard example

In the paper [8] i give the following example: Consider the 2-pointed k-algebra $A = \begin{pmatrix} k[t_{11}] & < t_{12} > /(t_{11}-1)t_{12} \\ 0 & k \end{pmatrix}$. This k-algebra has geometric points, i.e. simple A-modules, given by the line and the point respectively

$$V_1(a) = \begin{pmatrix} \mathbf{k}(\mathrm{a}) & 0 \\ 0 & 0 \end{pmatrix},\ V_2 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.$$

We are going to compute the local formal moduli $\hat{H}_V$, $V = \{V_1(a), V_2\}$ of the modules $V_1(a)$, V_2 for a fixed a, following the algorithm given in [2]. We start by computing the tangent spaces:

Example: A standard example

In general we have

$$Ext_A^1(V_i, V_j) = \mathrm{HH}^1(A, Hom_k(V_i, V_j)) = Der_k(A, Hom_k(V_i, V_j))/\,\mathrm{Ad},$$

where the bi-module structure on $Hom_k(V_i, V_j)$ is given by $a\phi(v_i) = \phi(av_i)$, $\phi \cdot a(v_i) = \phi(v_i)a$. Notice that by Ad we mean the trivial derivations ad_α, $\alpha \in Hom_k(V_i, V_j)$.

Example: A standard example

Any derivation δ is determined on a generator set. In this particular example, we choose as generators

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; (t_{11}-a) = \begin{pmatrix} (t_{11}-a) & 0 \\ 0 & 0 \end{pmatrix}, \; t_{12} = \begin{pmatrix} 0 & t_{1\cdot} \\ 0 & 0 \end{pmatrix}$$

(1,1): $Ext^1_A(V_1(a), V_1(a))$:

$$\begin{aligned}
&\delta(e_1) = \delta(e_1^2) = e_1\delta(e_1) + \delta(e_1)e_1 = 2\delta(e_1) \Rightarrow \delta(e_1) = \delta(e_2) = 0 \\
&\delta(t_{11}-a) = \alpha \\
&\delta(t_{12}) = \delta(t_{12}e_2) = \delta(t_{12})e_2 + t_{12}\delta(e_2) = 0 \\
&\mathrm{ad}_\beta(t_{11}-a) = (t_{11}-a)\beta - \beta(t_{11}-a) = 0.
\end{aligned}$$

As basis for $Ext^1_A(V_1(a), V_1(a))$ we choose the one element set $\{\phi_{11} = (t_{11}-a)^\vee\}$.

Example: A standard example

(1,2): $Ext^1_A(V_1(a), V_2)$

$$\delta(e_1) = \alpha$$
$$\delta(e_2) = -\alpha$$
$$\delta(t_{11} - a) = \delta((t_{11} - a)e_1) = \delta(t_{11} - a)e_1 + (t_{11} - a)\delta(e_1) = 0$$
$$(a-1)\delta(t_{12}) = \delta(t_{11}t_{12} - t_{12}) = 0$$
$$\mathrm{ad}_\alpha(t_{11} - a) = 0$$
$$\mathrm{ad}_\alpha(e_1) = e_1\alpha - \alpha e_1 = \alpha$$
$$\mathrm{ad}_\alpha(e_2) = e_2\alpha - \alpha e_2 = -\alpha$$

Example: A standard example

Thus if $a = 1$ we choose as basis the one point set $\{\phi_{12} = t_{12}^{\vee}\}$, if $a \neq 1$, $Ext_A^1(V_1(a), V_2) = 0$.

(1,i): $Ext_A^1(V_2, V_i) = 0$, $i = 1, 2$ which is trivial.

For the rest we put $a = 1$, that is $V_1 = V_1(1)$ and we compute $\hat{H}_{\{V_1,V_2\}}$: Let $S = \begin{pmatrix} k[u_{11}] & < u_{12} > \\ 0 & k \end{pmatrix}$. Then the infinitesimal liftings are given by

$$\phi_2 = \begin{pmatrix} 1 \otimes \cdot a + u_{11} \otimes (t_{11} - 1)^{\vee} & u_{12} \otimes t_{12}^{\vee} \\ 0 & 1 \otimes \cdot a \end{pmatrix} : A \to (Hom_k(V_i, S_{2,ij} \otimes V_j)$$

Example: A standard example

Now $S_2 = S/rad^2$ and the obstruction for lifting to $R_3 = S/rad^3$ is

$$o = \begin{pmatrix} u_{11}^2 \otimes (t_{11}-1)^\vee (t_{11}-1)^\vee & u_{11}u_{12} \otimes (t_{11}-1)^\vee t_{12}^\vee \\ 0 & 0 \end{pmatrix}.$$

In general, $v^\vee w^\vee = (v \otimes w)^\vee = -d((vw)^\vee)$, so

$$(t_{11}-1)^\vee (t_{11}-1)^\vee = -d(((t_{11}-1)^2)^\vee).$$

But $(t_{11}-1)t_{12} = 0$ in A, thus $\overline{o} = \begin{pmatrix} 0 & u_{11}u_{12} \otimes o_{12} \\ 0 & 0 \end{pmatrix}$ with $o_{12} \neq 0$. Put $S_3 = S/(rad^3 + u_{11}u_{12})$. Then we can lift the A-module structure to S_3 by

$$\phi_3 = \begin{pmatrix} 1 \otimes \cdot a + u_{11} \otimes (t_{11}-1)^\vee + u_{11}^2 \otimes ((t_{11}-1)^2)^\vee & u_{12} \otimes t_{12}^\vee \\ 0 & 0 \end{pmatrix}.$$

We see that this ϕ_3 can be lifted to ϕ_n on $S_n = S_3/rad^n$, $n \geq 3$, giving the result

Example: A standard example

$$\hat{H} = \lim_{\leftarrow} S_n = \begin{pmatrix} k[[u_{11}]] & < u_{12} > / u_{11} u_{12} \\ 0 & k \end{pmatrix} \cong \lim_{\leftarrow} A / rad^n.$$

In general terms this says that A is a scheme for its 1-dimensional simple modules.

Example: 3-dimensional endomorphisms

Let

$$T = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} + (E_{ij})$$

where

$$(E_{ij})$$

is generated by

$$\begin{pmatrix} kt_{11}(1) + kt_{11}(2) + kt_{11}(3) & kt_{12}(1) + kt_{12}(2) + kt_{12}(3) & kt_{13}(1) + kt_{1} \\ 0 & kt_{22}(1) + kt_{22}(2) & kt_{23}(1) + \\ 0 & 0 & t_{33} \end{pmatrix}$$

Example: 3-dimensional endomorphisms

and let

$$f_{11}(1) = t_{11}(3)t_{11}(2) - t_{11}(2)t_{11}(3)$$
$$f_{11}(2) = t_{11}(3)t_{11}(1) - t_{11}(1)t_{11}(3)$$
$$f_{11}(3) = t_{11}(2)t_{11}(1) - t_{11}(1)t_{11}(2)$$

$$\begin{aligned} f_{12}(1) &= t_{11}(3)t_{12}(2) - t_{11}(2)t_{12}(3) - t_{12}(2)t_{22}(1) \\ &\quad - 3t_{12}(3)t_{22}^2(2) + 2t_{12}(3)t_{22}(1)t_{22}(2) \\ f_{12}(2) &= t_{11}(3)t_{12}(1) - t_{11}(1)t_{12}(3) - t_{12}(1)t_{22}(1) \\ &\quad + t_{12}(3)t_{22}(1)t_{22}^2(2) - 2t_{12}(3)t_{22}^3(2) \\ f_{12}(3) &= t_{11}(2)t_{12}(1) - t_{11}(1)t_{12}(2) - 2t_{12}(1)t_{22}(1)t_{22}(2) \\ &\quad + 3t_{12}(1)t_{22}^2(2) + t_{12}(2)t_{22}^2(2)t_{22}(1) - 2t_{12}(2)t_{22}^3(2) \end{aligned}$$

Example: 3-dimensional endomorphisms

$$f_{13}(1) = t_{11}(3)t_{13}(2) - t_{11}(2)t_{13}(3) - 3t_{13}(2)t_{33}(1) - t_{12}(2)t_{23}(1) - 3t_{12}$$
$$+ 3t_{13}(3)t_{33}^2(1) - 2t_{12}(1)t_{22}(2)t_{23}(1) - 2t_{12}(2)t_{22}(2)t_{23}(2)$$
$$f_{13}(2) = t_{11}(3)t_{13}(1) - t_{11}(1)t_{13}(3) - 3t_{13}(1)t_{33}(1) - t_{12}(1)t_{23}(1)$$
$$- t_{12}(3)t_{23}(2)t_{33}(1) - 2t_{12}(3)t_{22}(2)t_{23}(2) + t_{13}(3)t_{33}^3(1)$$
$$f_{13}(3) = t_{11}(2)t_{13}(1) - t_{11}(1)t_{13}(2) + 3t_{12}(1)t_{23}(2)$$
$$- t_{11}(3)t_{13}(1)t_{33}(1) + t_{11}(1)t_{13}(3)t_{33}(1) + t_{12}(1)t_{23}(1)t_{33}(1) - t_{12}$$
$$- 2t_{12}(1)t_{22}(2)t_{23}(1) - 2t_{12}(2)t_{22}(2)t_{23}(2)$$
$$\frac{1}{3}t_{11}(3)t_{13}(2)t_{33}^2(1) - \frac{1}{3}t_{11}(2)t_{13}(3)t_{33}^2(1)$$
$$- t_{12}(3)t_{23}(2)t_{33}^2(1) - \frac{1}{3}t_{12}(2)t_{23}(1)t_{33}^2(1) - 6t_{12}(3)t_{22}(2)t_{23}(2)t_{33}$$
$$- 2t_{12}(3)t_{22}(2)t_{23}(1)t_{33}^2(1)$$

Example: 3-dimensional endomorphisms

$$f_{22}(1) = t_{22}(2)t_{22}(1) - t_{22}(1)t_{22}(2)$$

$$\begin{aligned} f_{23}(1) = & -t_{22}(1)t_{23}(2) + 3t_{23}(2)t_{33}(1) \\ & + t_{23}(1)t_{33}^2(1) - 2t_{22}(2)t_{23}(1)t_{33}(1) + t_{22}^2(2)t_{23}(1) \end{aligned}$$

Example: 3-dimensional endomorphisms

Proposition

The noncommutative local formal moduli of the modules corresponding to the closure of the orbits of

$$M_1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \; M_2 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \; \textit{and } M_3 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

under the action of $\mathrm{Gl}(3)$ *is*

$$T/(f_{ij}(l))$$

Example: 3-dimensional endomorphisms

The case with two different eigenvalues, that is matrices of the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix},$$

gives the parametric surface

$$\begin{aligned} s_1 &= 2\lambda_1 + \lambda_2 \\ s_2 &= -2\lambda_1\lambda_2 - \lambda_1^2 \Rightarrow 4s_1^3 s_3 - s_1^2 s_2^2 + 18 s_1 s_2 s_3 - 4 s_2^3 + 27 s_3^2 = 0. \\ s_3 &= \lambda_1^2 \lambda_2 \end{aligned} \tag{28}$$

Example: 3-dimensional endomorphisms

The case with all eigenvalues equal, that is matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

gives the parametric curve

$$\begin{array}{rcl} s_1 & = & 3\lambda \\ s_2 & = & -3\lambda^2 \\ s_3 & = & \lambda^3 \end{array} \Rightarrow s_2 = -\frac{1}{3}s_1^2 \wedge s_3 = \frac{1}{27}s_1^3.$$

Example: 3-dimensional endomorphisms

Lead by our geometric intuition, the geometric picture should show three generic points. The case with all three eigenvalues different is well known to be parameterized by the points in affine 3-space. A point in this affine 3-space, on the surface, represents a new 3-dimensional affine space. This is glued onto this point. A point on the curve on the surface represents a new 3-dimensional affine space which is glued onto this point. Outside the curve and the surface, all points are identified. In this section, we will use some effort to understand how the geometric picture fits in with the local computations.

Example: 3-dimensional endomorphisms

Consider the local formal moduli $\hat{H}(V_1, V_2, V_3)$, that is the local formal moduli of the worst case, the case corresponding to all eigenvalues equal. Let $H = H(V_1, V_2, V_3)$ be an algebraization. Necessary conditions for this k-algebra to give the moduli, i.e. the affine ring for $M_3(k)/\operatorname{Gl}_3(k)$, is that the simple modules of this ring are in one to one correspondence with the orbits, and that it is closed under forming local moduli (of the simple modules). In particular, the Ext^1-dimensions must be the same. The simple H-modules corresponds to the quotients $V_1(*)$, $V_2(*)$ and $V_3(*)$ of the rings on the diagonal of H by their maximal ideals. Recalling that

$$\operatorname{Ext}^1_H(V_i, V_j) \cong HH^1(H, \operatorname{Hom}_k(V_i, V_j)) \cong \operatorname{Der}_k(H, \operatorname{Hom}_k(V_i, V_j))),$$

we can compute the tangent space dimensions $\operatorname{Ext}^1_H(V_i, V_j)$ by looking at k-derivations δ. We see that this dimension drops if $\delta(f) \neq 0$ for some relation f.

Example: 3-dimensional endomorphisms

This means that regarding tangent space dimensions, we can work with the relations as commutative polynomials. Let $V_1(t_{11}(1), t_{11}(2), t_{11}(3))$, $V_2(t_{22}(1), t_{22}(2))$ and $V_3 = t_{33}(1)$ be three points (simple modules) on the diagonal of H. Then the constant ext^1-locus is given as follows:

Example: 3-dimensional endomorphisms

$$
\begin{aligned}
f_{12}(1) &= t_{12}(3)(-t_{11}(2) - 3t_{22}^2(2) + 2t_{22}(1)t_{22}(2)) \\
&\quad + t_{12}(2)(t_{11}(3) - t_{22}(1)) = 0 \\
f_{12}(2) &= t_{12}(1)(t_{11}(3) - t_{22}(1)) + t_{12}(3)(-t_{11}(1) + t_{22}(1)t_{22}^2(2) - 2t_{22}^3(2 \\
f_{12}(3) &= t_{12}(1)(t_{11}(2) - 2t_{22}(1)t_{22}(2) + 3t_{22}^2(2)) \\
&\quad + t_{12}(2)(-t_{11}(1) + t_{22}(1)t_{22}^2(2) - 2t_{22}^3(2)) = 0
\end{aligned}
$$

Example: 3-dimensional endomorphisms

We put

$$t_{11}(1) = s_3,\ \ t_{11}(2) = s_2,\ \ t_{11}(3) = s_1,\ \ t_{22}(1) = \lambda_2,\ \ t_{22}(2) = \lambda_1$$

and we get the equations

$$\begin{aligned} s_1 &= \lambda_2 \\ s_2 &= 2\lambda_1\lambda_2 - 3\lambda_1^2 \\ s_3 &= \lambda_1^2\lambda_2 - 2\lambda_1^3 \end{aligned}$$

Example: 3-dimensional endomorphisms

which is exactly the point

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 - 2\lambda_1 \end{pmatrix}$$

on the surface (28).

Example: 3-dimensional endomorphisms

$$f_{13}(1) = t_{13}(2)(t_{11}(3) - 3t_{33}(1)) + t_{13}(3)(-t_{11}(2) + 3t_{33}^2(1)) = 0$$

$$f_{13}(2) = t_{13}(1)(t_{11}(3) - 3t_{33}(1)) + t_{13}(3)(-t_{11}(1) + t_{33}^3(1)) = 0$$

$$f_{13}(3) = t_{13}(1)(t_{11}(2) - t_{11}(3)t_{33}(1)) + t_{13}(2)(-t_{11}(1) + \frac{1}{3}t_{11}(3)t_{33}^2(1) + t_{13}(3)(t_{11}(1)t_{33}(1) - \frac{1}{3}t_{11}(2)t_{33}^2(1)) = 0.$$

Example: 3-dimensional endomorphisms

We put

$$t_{11}(1) = s_3,\ t_{11}(2) = s_2,\ t_{11}(3) = s_1,\ t_{33}(1)\lambda_1,$$

and we get the following equations:

$$\begin{array}{rcl} s_1 & = & 3\lambda_1 \\ s_2 & = & 3\lambda_1^2 \\ s_3 & = & \lambda_1^3 \\ s_2 & = & s_1\lambda_1 \\ s_3 & = & \frac{1}{3}s_1\lambda_1^2 \\ s_3\lambda_1 & = & \frac{1}{3}s_2\lambda_1^2 \end{array} \Leftrightarrow \begin{array}{rcl} s_1 & = & 3\lambda_1 \\ s_2 & = & 3\lambda_1^2 \\ s_3 & = & \lambda_1^3 \end{array}$$

Example: 3-dimensional endomorphisms

This gives the points on the curve

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

(2,3)

$$f_{23}(1) = t_{23}(1)(t_{33}^2(1) - 2t_{22}(2)t_{33}(1) + t_{22}^2(2)) \\ + t_{23}(2)(-t_{22}(1) + 3t_{33}(1)).$$

On the curve, the above chosen parameters corresponds to

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & 3\lambda_1 - 2\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 - 2\lambda_1 \end{pmatrix},$$

Example: 3-dimensional endomorphisms

that is

$$t_{22}(1) = 3\lambda_1, \; t_{22}(2) = \lambda_1, \; t_{33}(1) = \lambda_1.$$

This is true for both equations above:

Example: 3-dimensional endomorphisms

$$t_{22}(1) = 3t_{33}(1) \Leftrightarrow 3\lambda_1 = 3\lambda_1$$
$$2t_{22}(2)t_{33}(1) = t_{33}^2(1) + t_{22}^2(2) \Leftrightarrow 2\lambda_1^2 = 2\lambda_1^2.$$

Thus the constant ext^1-locus is preserved on the curve.

The constant ext^1-locus for the local formal moduli for a point on the surface, that is the case with exactly two different eigenvalues, is given by the equations (for simplicity we put $\lambda = 1$)

Example: 3-dimensional endomorphisms

$$\begin{aligned} f_{12}(1) &= t_{12}(3)(-t_{11}(2) - 2t_{22}(2) + 2t_{22}(1)t_{22}(2) - 3t_{22}^2(2)) \\ &\quad + t_{12}(2)(t_{11}(3) - t_{22}(1)) \\ f_{12}(2) &= t_{12}(3)(-t_{11}(1) - t_{22}^2(2) + t_{22}(1)t_{22}^2(2) - 2t_{22}^3(2)) \\ &\quad + t_{12}(1)(t_{11}(3) - t_{22}(1)) \\ f_{12}(3) &= t_{12}(2)(-t_{11}(1) - t_{22}^2(2) + t_{22}(1)t_{22}^2(2) - 2t_{22}^3(2)) \\ &\quad + t_{12}(1)(t_{11}(2) + 2t_{22}(2) - 2t_{22}(1)t_{22}(2) + 3t_{22}^2(2)). \end{aligned}$$

We let

$$t_{11}(3) = s_1+1,\ t_{11}(2) = s_2,\ t_{11}(1) = s_3,\ t_{22}(2) = \lambda_1,\ t_{22}(1) = \lambda_2.$$

Then we get the equations

Example: 3-dimensional endomorphisms

$$\begin{aligned} s_1 &= \lambda_2 - 1 \\ s_2 &= -2\lambda_1 + 2\lambda_1\lambda_2 - 3\lambda_1^2 \\ s_3 &= -\lambda_1^2 + \lambda_1^2\lambda_2 - 2\lambda_1^3, \end{aligned}$$

which is the surface

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 - 1 - 2\lambda_1 \end{pmatrix}.$$

Example: 3-dimensional endomorphisms

All in all, we have proved:

Theorem

The simple modules of $\mathrm{M}_3(k)^{\mathrm{Gl}_3(k)} =$

$$\begin{pmatrix} k[s_1,s_2,s_3] & < t_{12}(1), t_{12}(2), t_{12}(3) > & < t_{13}(1), t_{13}(2), t_{13}(3) > \\ 0 & k[t_1,t_2] & < t_{23}(1), t_{23}(2) > \\ 0 & 0 & k[u] \end{pmatrix} / \mathfrak{b}$$

where $\mathfrak{b}$ *is the two-sided ideal generated by the relations in the case with all eigenvalues equal, are in one to one correspondence with the* $\mathrm{Gl}_3(k)-$*orbits of* $\mathrm{M}_3(k)$. *Letting a finite family* V *of* $A-G$*-modules correspond to the simple modules* $V^{\mathrm{Gl}_3(k)}$, *the formal moduli* $\hat{H}_V$ *of* $A-G$*-modules is isomorphic to the formal moduli* $\hat{H}_{V^{\mathrm{Gl}_3(k)}}$ *of* $M_3(k)^{\mathrm{Gl}_3(k)}$*-modules. As such* $\mathrm{M}_3(k)^{\mathrm{Gl}_3(k)} = \mathcal{O}^{\mathrm{Gl}_3(k)}$.

Example: 3-dimensional endomorphisms

This gives the picture of the moduli for $\mathrm{Gl}_3(k)$ as the affine 3-space, the affine 2-space and the curve. Notice that the affine 2-space in the middle is the blowup of the surface along the curve.

On Lie algebras of dimension 3

Proposition

The noncommutative moduli $\underline{\text{Lie}}(3)$ *is*

$$\mathbb{M} = \begin{pmatrix} k[C] & k[C]_{(C-\frac{1}{2})} < t_{12} > & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 & k \end{pmatrix}$$

where the two first rows correspond to the Lie-algebras $\mathfrak{g}(C)$ *and* l_1 *respectively, and where the four last rows corresponds to* $\underline{sl}_2(k)$, $\underline{n}_3$, l_{-1}, *and* $ab = \mathfrak{g}_0$ *(respectively).*

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