Symplectic geometry and knot invariants

Luís Diogo

Uppsala University & Universidade Federal Fluminense

Matemáticos Portugueses pelo Mundo 2019

Universidade do Porto, June 26, 2019
Outline

Knots and the Alexander polynomial $\text{Alex}_K$

Knot contact homology (KCH) and the augmentation polynomial $\text{Aug}_K$

$\text{Alex}_K$ from $\text{Aug}_K$
Outline

Knots and the Alexander polynomial $\text{Alex}_K$

Knot contact homology (KCH) and the augmentation polynomial $\text{Aug}_K$

$\text{Alex}_K$ from $\text{Aug}_K$
Knots

A knot is a simple closed curve in $\mathbb{R}^3$. A link is a finite collection of disjoint knots.

**Figure:** The unknot and the trefoil

**Question:** how to classify knots and links up to isotopy?
Alexander polynomial

This is a polynomial associated to a link, which is invariant under isotopy.

To define it, pick an orientation for the link and require

\[ \text{Alex}_{\text{unknot}} = 1 \]

and *skein relation* (near crossings of a projection of link to a generic plane in \( \mathbb{R}^3 \)):

\[
\begin{align*}
\text{Alex} & \begin{array}{c}
\includegraphics[width=2cm]{crossing1.png}
\end{array} - \text{Alex} & \begin{array}{c}
\includegraphics[width=2cm]{crossing2.png}
\end{array} = \left( \mu^{1/2} - \mu^{-1/2} \right) \text{Alex} & \begin{array}{c}
\includegraphics[width=2cm]{crossing3.png}
\end{array}
\end{align*}
\]

The knot is unchanged outside the neighborhood of the crossing.
Example: trefoil

\[
\text{Alex} \begin{bmatrix}
\begin{array}{c}
\text{trefoil}
\end{array}
\end{bmatrix} = \text{Alex} \begin{bmatrix}
\begin{array}{c}
\text{trefoil}
\end{array}
\end{bmatrix} + (\mu^{1/2} - \mu^{-1/2}) \text{Alex} \begin{bmatrix}
\begin{array}{c}
\text{Hopf link}
\end{array}
\end{bmatrix}
\]

\[= 1 + (\mu^{1/2} - \mu^{-1/2}) \text{Alex} \begin{bmatrix}
\begin{array}{c}
\text{Hopf link}
\end{array}
\end{bmatrix} \quad (\text{used } \text{Alex}_\text{unknot} = 1)\]

\[
\text{Alex} \begin{bmatrix}
\begin{array}{c}
\text{trefoil}
\end{array}
\end{bmatrix} = \text{Alex} \begin{bmatrix}
\begin{array}{c}
\text{trefoil}
\end{array}
\end{bmatrix} + (\mu^{1/2} - \mu^{-1/2}) \text{Alex} \begin{bmatrix}
\begin{array}{c}
\text{trefoil}
\end{array}
\end{bmatrix}
\]

\[= \mu^{1/2} - \mu^{-1/2} \quad (\text{Alex}_\text{unlink with 2 components} = 0)\]

\[\implies \text{Alex}_\text{trefoil} = 1 + (\mu^{1/2} - \mu^{-1/2})^2 = \mu - 1 + \mu^{-1}\]
**Dynamical definition of $\text{Alex}_K$**

**Simplifying assumption:** $K$ is *fibered*, i.e. there is a surjective map $f: S^3 \setminus K \rightarrow S^1$ without critical points.

$\nabla f$ is the gradient vector field of $f$ on $S^3 \setminus K$, i.e. $\langle \nabla f, . \rangle = df$.

Denote by $\gamma$ a flow loop of the vector field $\nabla f$. Its multiples are also flow loops.

**Milnor:** if $K$ is fibered, then

$$\text{Alex}_K(\mu) = (1 - \mu) \exp \left( \sum_{\gamma} \frac{\sigma(\gamma)}{m(\gamma)} \mu^d(\gamma) \right)$$

$(\sigma(\gamma) \in \{\pm 1\}$ sign, $m(\gamma) \in \mathbb{Z}$ multiplicity, $d(\gamma) \in \mathbb{Z} \cong H_1(S^3 \setminus K)$ homological degree).

Extended to general $K$ by Fried and Hutchings–Lee. Need also flow lines of $\nabla f$.

**Example: unknot**

$S^3 \setminus \text{unknot} \cong S^1 \times \mathbb{R}^2$.

We can arrange that the flow loops are the covers of the circle $S^1 \times \{0\}$. So,

$$\text{Alex}_{\text{unknot}}(\mu) = (1 - \mu) \exp \left( \sum_{k \geq 1} \frac{\mu^k}{k} \right) = (1 - \mu) \exp (- \log(1 - \mu)) = 1$$
Outline

Knots and the Alexander polynomial $\text{Alex}_K$

Knot contact homology (KCH) and the augmentation polynomial $\text{Aug}_K$

$\text{Alex}_K$ from $\text{Aug}_K$
Knot contact homology (KCH)

This is an invariant of knots defined using symplectic geometry. Cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$ is a symplectic manifold:

$$\omega = \sum_{i=1}^{3} dp_i \wedge dq_i$$

is a symplectic form ($d\omega = 0$ and $\omega^n$ is a volume form). $K \subset \mathbb{R}^3$ knot yields the conormal Lagrangian:

$$L_K := \{(q, p) \in T^*\mathbb{R}^3 \mid q \in K \text{ and } (\forall v \in T_qK)(p, v) = 0\}$$

Lagrangian condition: $\omega|_{L_K} \equiv 0$. $L_K$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

Write $S^*\mathbb{R}^3 \cong S^2 \times \mathbb{R}^3$ for the unit sphere cotangent bundle (a contact manifold). The conormal Legendrian is the 2-torus

$$\Lambda_K := L_K \cap S^*\mathbb{R}^3.$$ 

Reeb chords of $\Lambda_K$ are trajectories of the geodesic flow that start and end on $\Lambda_K$. The differential graded algebra (DGA) of $K$ is

$$\mathcal{A}_K := \text{tensor algebra freely generated over some ring } R \text{ by Reeb chords in } \Lambda_K$$
Differential

Given a Reeb cord $x$,

$$\partial x = \sum_{\text{Reeb chords } y_1, \ldots, y_k} \left( \sum_{u \in \mathcal{M}(x; y_1, \ldots, y_k)} c(u) \right) y_1 \otimes \ldots \otimes y_k$$

where $u$ is a $J$-holomorphic curve, i.e. for some almost complex structure $J$ compatible with $\omega$, we have $\overline{\partial}_J u = 0$

$c(u) \in R$ keeps track of the homology class of $u$. 

Symplectic geometry and knot invariants

June 26, 2019
Coefficient ring

Group ring $R := \mathbb{C}[H_2(S^*\mathbb{R}^3, \Lambda_K; \mathbb{Z})] \cong \mathbb{C}[\lambda^{\pm1}, \mu^{\pm1}, Q^{\pm1}]$, after choice of splitting of short exact sequence

\[ 0 \longrightarrow H_2(S^*\mathbb{R}^3; \mathbb{Z}) \longrightarrow H_2(S^*\mathbb{R}^3, \Lambda_K; \mathbb{Z}) \longrightarrow H_1(\Lambda_K; \mathbb{Z}) \longrightarrow 0 \]

Denote $\lambda = e^x$, $\mu = e^p$, $Q = e^t$.

Knot contact homology (KCH) is the homology of the DGA $(\mathcal{A}_K, \partial)$.

Ekholm–Ng–Shende: (An enhancement of) KCH (replacing $\Lambda_K$ with its union with a cotangent fiber sphere $S^*_\text{pt}\mathbb{R}^3$) is a complete knot invariant.

This means that two knots have isomorphic (enhanced) KCH iff they are isotopic.
Augmentations

$(\mathcal{A}_K, \partial)$ is a very big chain complex (tensor algebra on Reeb chords).

**Definition**

An *augmentation* is a DGA map $\varepsilon: \mathcal{A}_K \to \mathbb{C}$.

Can use $\varepsilon$ to *linearize* KCH. Get chain complex on the vector space generated by chords (much smaller and more manageable than $\mathcal{A}_K$).

**Augmentation variety**

$V_K$ is the (union of maximal dimensional irreducible components of the Zariski closure of the) set

$$\{(\varepsilon(\lambda), \varepsilon(\mu), \varepsilon(Q)) \in (\mathbb{C}^*)^3 | \varepsilon \text{ is an augmentation}\}$$

(this is the collection of values that augmentations take on the variables $\lambda, \mu, Q$).

**Augmentation polynomial**

$\text{Aug}_K(\lambda, \mu, Q)$ is an irreducible Laurent polynomial such that $V_K = V(\text{Aug}_K)$. 
Augmentation polynomial

Augmentation polynomial
$\text{Aug}_K(\lambda, \mu, Q)$ is an irreducible Laurent polynomial such that $V_K = V(\text{Aug}_K)$.

Examples

$\text{Aug}_{\text{unknot}}(\lambda, \mu, Q) = 1 - \lambda - \mu + \lambda \mu Q$

$\text{Aug}_{\text{trefoil}}(\lambda, \mu, Q) = \lambda^2(\mu - 1) + \lambda(\mu^4 - \mu^3 Q + 2\mu^2 Q^2 - 2\mu^2 Q - \mu Q^2 + Q^2) + (\mu^3 Q^4 - \mu^4 Q^3)$

Buzzwords

$\text{Aug}_K$ should agree with the $Q$-deformed A-polynomial of $K$, introduced by Aganagic–Vafa in the study of mirror symmetry for the resolved conifold.
Outline

Knots and the Alexander polynomial $Alex_K$

Knot contact homology (KCH) and the augmentation polynomial $Aug_K$

$Alex_K$ from $Aug_K$
Getting $\text{Alex}_K$ from $\text{Aug}_K$

**Theorem (D.–Ekholm)**

$$\text{Alex}_K(\mu) = (1 - \mu) \exp \left( \int - \frac{\partial Q \text{Aug}_K}{\mu \partial_\lambda \text{Aug}_K} \bigg|_{\lambda=Q=1} d\mu \right)$$

if $\partial_\lambda \text{Aug}_K \big|_{\lambda=Q=1}$ is not identically zero.

**Idea of the proof**

Reinterpret gradient flow lines and loops in dynamical definition of $\text{Alex}_K$ as $J$-holomorphic curves.
Relate to KCH via several cobordisms of moduli spaces of $J$-holomorphic curves.

**THANK YOU!!**