Symplectic geometry and knot invariants

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Outline

Knots and the Alexander polynomial Alex_K

Knot contact homology (KCH) and the augmentation polynomial Aug_K

Alex_K from Aug_K

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Knots

A knot is a simple closed curve in $\mathbb{R}^3$. A link is a finite collection of disjoint knots.

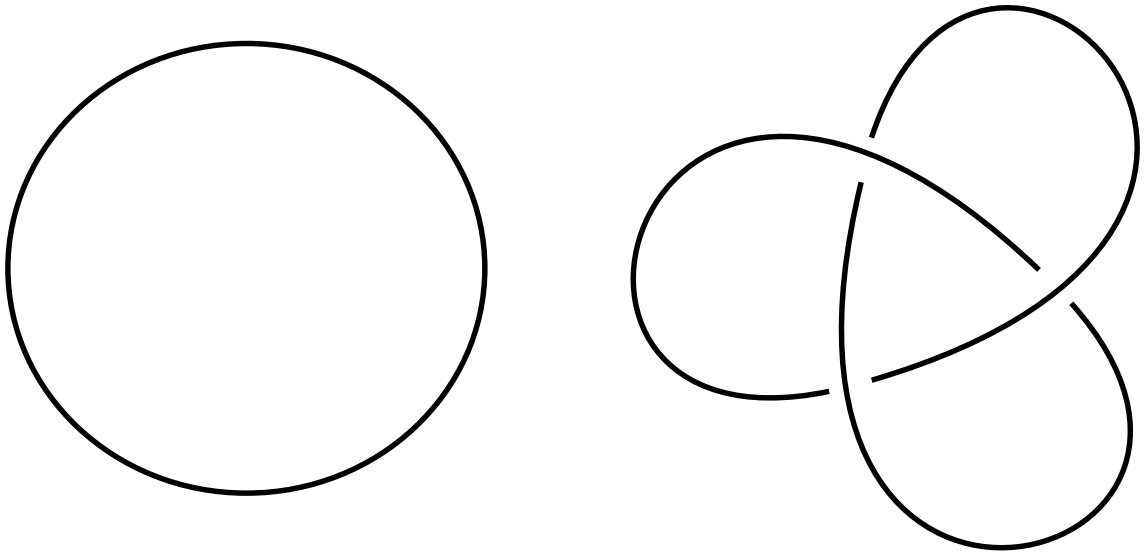


Figure: The unknot and the trefoil

Question: how to classify knots and links up to isotopy?

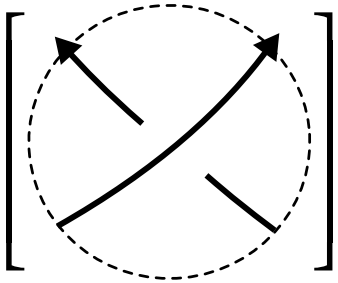
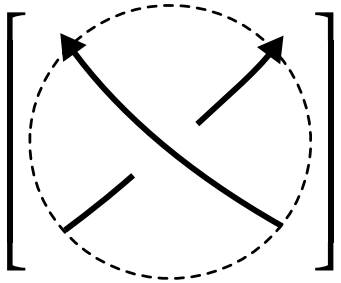
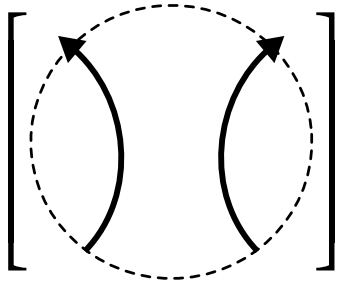
Alexander polynomial

This is a polynomial associated to a link, which is invariant under isotopy.

To define it, pick an orientation for the link and require

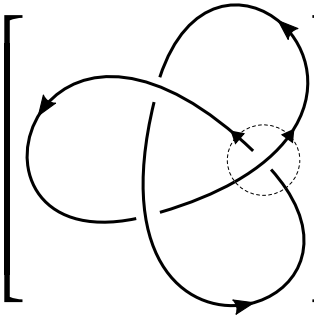
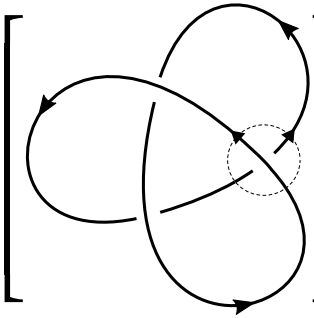
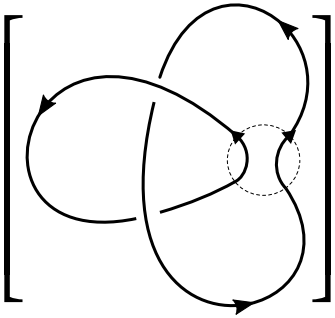
$$\mathrm{Alex}_{\mathrm{unknot}} = 1$$

and *skein relation* (near crossings of a projection of link to a generic plane in $\mathbb{R}^3$):

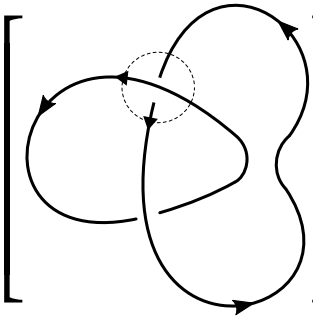
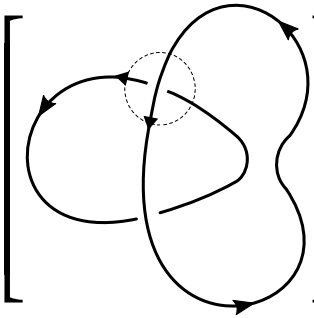
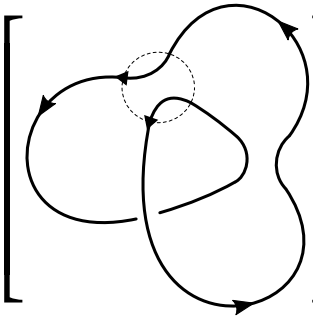
Alex  – Alex  $= (\mu^{1/2} - \mu^{-1/2})$ Alex 

The knot is unchanged outside the neighborhood of the crossing.

Example: trefoil

$\mathrm{Alex}\Big[$  $\Big] = \mathrm{Alex}\Big[$  $\Big] + (\mu^{1/2} - \mu^{-1/2})\,\mathrm{Alex}\Big[$  $\Big]$

$= 1 + (\mu^{1/2} - \mu^{-1/2})\,\mathrm{Alex}_{\text{Hopf link}}$ $\qquad$ (*used* $\mathrm{Alex}_{\text{unknot}} = 1$)

$\mathrm{Alex}\Big[$  $\Big] = \mathrm{Alex}\Big[$  $\Big] + (\mu^{1/2} - \mu^{-1/2})\,\mathrm{Alex}\Big[$  $\Big]$

$= \mu^{1/2} - \mu^{-1/2}$ $\qquad$ ($\mathrm{Alex}_{\text{unlink with 2 components}} = 0$)

$$\Longrightarrow \mathrm{Alex}_{\text{trefoil}} = 1 + (\mu^{1/2} - \mu^{-1/2})^2 = \mu - 1 + \mu^{-1}$$

Dynamical definition of Alex_K

Simplifying assumption: K is *fibered*, i.e. there is a surjective map $f\colon S^3 \setminus K \to S^1$ without critical points.
∇f is the gradient vector field of f on $S^3 \setminus K$, i.e. $\langle \nabla f, . \rangle = df$.
Denote by γ a flow loop of the vector field ∇f. Its multiples are also flow loops.

Milnor: if K is fibered, then

$$\mathrm{Alex}_K(\mu) = (1-\mu)\exp\left(\sum_\gamma \frac{\sigma(\gamma)}{m(\gamma)}\mu^{d(\gamma)}\right)$$

($\sigma(\gamma) \in \{\pm 1\}$ sign, $m(\gamma) \in \mathbb{Z}$ multiplicity, $d(\gamma) \in \mathbb{Z} \cong H_1(S^3 \setminus K)$ homological degree).
Extended to general K by Fried and Hutchings–Lee. Need also flow lines of ∇f.

Example: unknot

$S^3 \setminus \text{unknot} \cong S^1 \times \mathbb{R}^2$.
We can arrange that the flow loops are the covers of the circle $S^1 \times \{0\}$. So,

$$\mathrm{Alex}_{\text{unknot}}(\mu) = (1-\mu)\exp\left(\sum_{k\geq 1}\frac{\mu^k}{k}\right) = (1-\mu)\exp\left(-\log(1-\mu)\right) = 1$$

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Knot contact homology (KCH)

This is an invariant of knots defined using symplectic geometry.
Cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$ is a symplectic manifold:
$\omega = \sum_{i=1}^{3} dp_i \wedge dq_i$ is a symplectic form ($d\omega = 0$ and $\omega^{\wedge n}$ is a volume form).
$K \subset \mathbb{R}^3$ knot yields the *conormal Lagrangian*:

$$L_K := \{(q,p) \in T^*\mathbb{R}^3 \,|\, q \in K \text{ and } (\forall v \in T_qK)\langle p, v\rangle = 0\}$$

Lagrangian condition: $\omega|_{L_K} \equiv 0$. L_K is diffeomorphic to $S^1 \times \mathbb{R}^2$.
Write $S^*\mathbb{R}^3 \cong S^2 \times \mathbb{R}^3$ for the unit sphere cotangent bundle (a contact manifold).
The *conormal Legendrian* is the 2-torus

$$\Lambda_K := L_K \cap S^*\mathbb{R}^3.$$

Reeb chords of Λ_K are trajectories of the geodesic flow that start and end on Λ_K.
The differential graded algebra (DGA) of K is

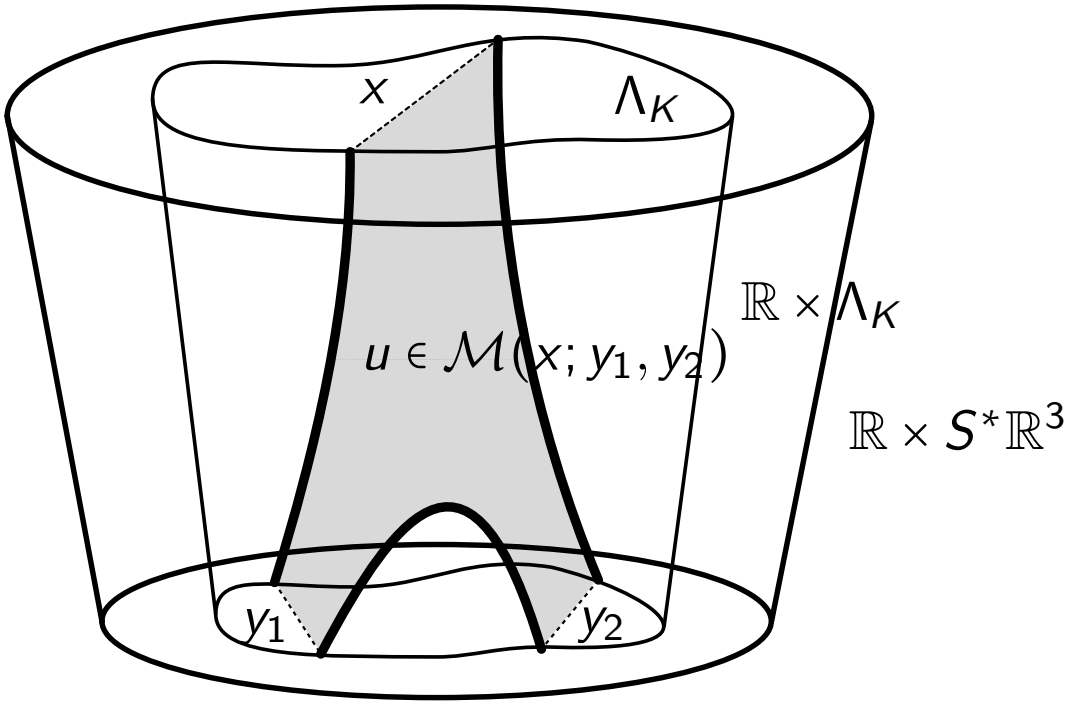
$$\mathcal{A}_K := \text{tensor algebra freely generated over some ring } R \text{ by Reeb chords in } \Lambda_K$$

Differential

Given a Reeb cord x,

$$\partial x = \sum_{\text{Reeb chords } y_1,\ldots,y_k} \left(\sum_{u \in \mathcal{M}(x;y_1,\ldots,y_k)} c(u) \right) y_1 \otimes \ldots \otimes y_k$$

where u is a *J-holomorphic curve*, i.e. for some almost complex structure J compatible with ω, we have $\boxed{\overline{\partial}_J u = 0}$



$c(u) \in R$ keeps track of the homology class of u.

Coefficient ring

Group ring $R := \mathbb{C}[H_2(S^*\mathbb{R}^3, \Lambda_K; \mathbb{Z})] \cong \mathbb{C}[\lambda^{\pm 1}, \mu^{\pm 1}, Q^{\pm 1}]$, after choice of splitting of short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \overset{t\in\mathbb{Z}}{\cong} H_2(S^*\mathbb{R}^3; \mathbb{Z}) \arrow[r] & H_2(S^*\mathbb{R}^3, \Lambda_K; \mathbb{Z}) \arrow[r] & \overset{x,p\in\mathbb{Z}^2}{\cong} H_1(\Lambda_K; \mathbb{Z}) \arrow[r] \arrow[l, bend right, dashed] & 0 \end{tikzcd}$$

Denote $\lambda = e^x, \mu = e^p, Q = e^t$.

Knot contact homology(KCH)

is the homology of the DGA $(\mathcal{A}_K, \partial)$.

Ekholm–Ng–Shende: (An enhancement of) KCH (replacing Λ_K with its union with a cotangent fiber sphere $S^*_{pt}\mathbb{R}^3$) is a complete knot invariant.
This means that two knots have isomorphic (enhanced) KCH iff they are isotopic.

Augmentations

$(\mathcal{A}_K, \partial)$ is a very big chain complex (tensor algebra on Reeb chords).

Definition

An *augmentation* is a DGA map $\varepsilon\colon \mathcal{A}_K \to \mathbb{C}$.

Can use ε to *linearize* KCH. Get chain complex on the *vector space* generated by chords (much smaller and more manageable than $\mathcal{A}_K$).

Augmentation variety

V_K is the (union of maximal dimensional irreducible components of the Zariski closure of the) set

$$\left\{(\varepsilon(\lambda), \varepsilon(\mu), \varepsilon(Q)) \in (\mathbb{C}^*)^3 \,|\, \varepsilon \text{ is an augmentation}\right\}$$

(this is the collection of values that augmentations take on the variables λ, μ, Q).

Augmentation polynomial

$\mathrm{Aug}_K(\lambda, \mu, Q)$ is an irreducible Laurent polynomial such that $V_K = V(\mathrm{Aug}_K)$.

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Examples

$$\mathrm{Aug}_{\mathrm{unknot}}(\lambda, \mu, Q) = 1 - \lambda - \mu + \lambda\mu Q$$

$$\mathrm{Aug}_{\mathrm{trefoil}}(\lambda, \mu, Q) = \lambda^2(\mu - 1) + \lambda(\mu^4 - \mu^3 Q + 2\mu^2 Q^2 - 2\mu^2 Q - \mu Q^2 + Q^2) + (\mu^3 Q^4 - \mu^4 Q^3)$$

Buzzwords

Aug_K should agree with the *Q-deformed A-polynomial* of K, introduced by Aganagic–Vafa in the study of mirror symmetry for the resolved conifold.

Outline

Getting Alex_K from Aug_K

Theorem (D.–Ekholm)

$$\mathrm{Alex}_K(\mu) = (1-\mu)\exp\left(\int -\frac{\partial_Q \mathrm{Aug}_K}{\mu\,\partial_\lambda \mathrm{Aug}_K}|_{\lambda=Q=1}\, d\mu\right)$$

if $\partial_\lambda \mathrm{Aug}_K|_{\lambda=Q=1}$ is not identically zero.

Idea of the proof

Reinterpret gradient flow lines and loops in dynamical definition of Alex_K as J-holomorphic curves.
Relate to KCH via several cobordisms of moduli spaces of J-holomorphic curves.

THANK YOU!!