Configuration spaces of points and their homotopy type

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Porto - June 26th 2019

A pre-example

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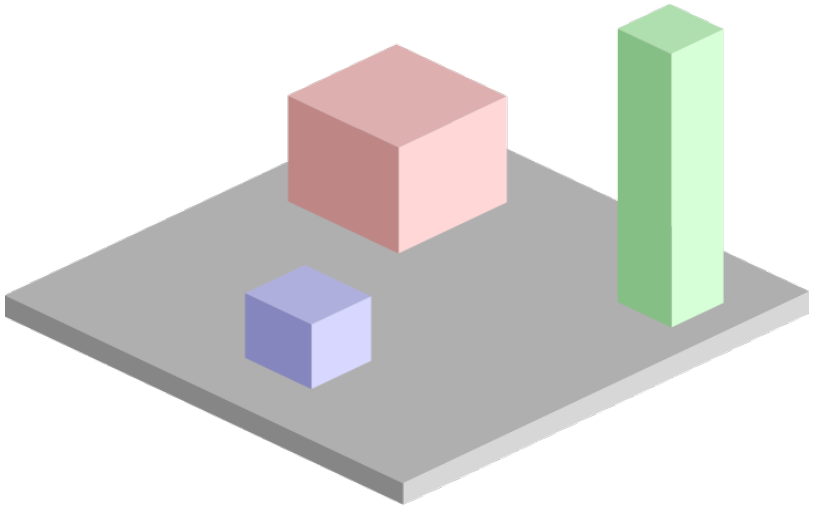

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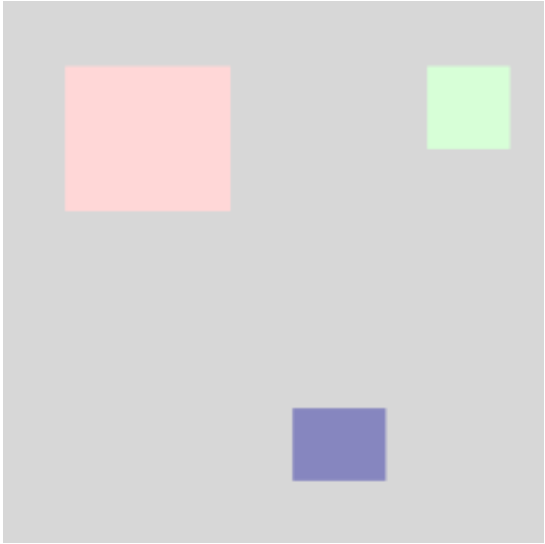

$$\text{Configuration space} = X \subset [0,1] \times [0,1].$$

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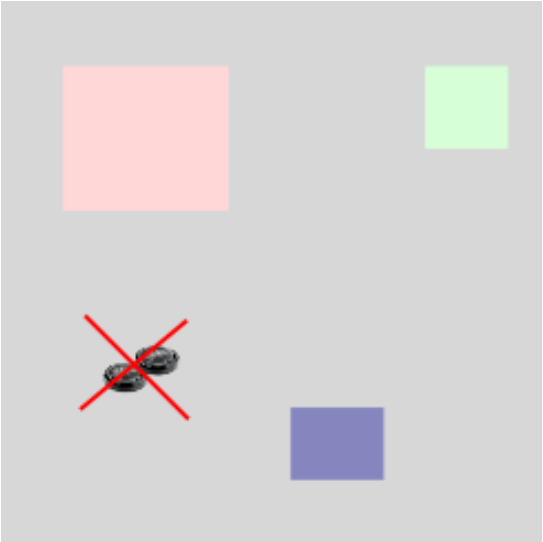
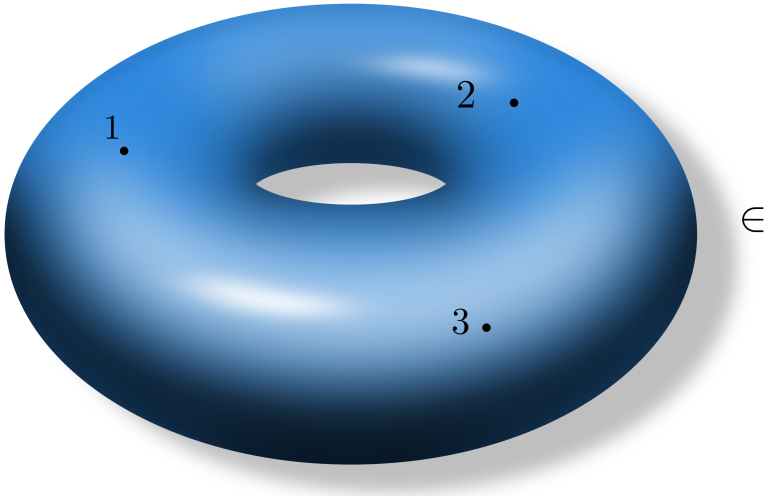


Configuration space $= X^n - \Delta$, $\qquad \Delta = \{(x_1, \ldots, x_n) | \exists x_i = x_j\}$.

Configuration spaces of points

Goal : Understand configuration spaces of points

$$Conf_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j\}.$$



$\in Conf_3(\text{Torus})$

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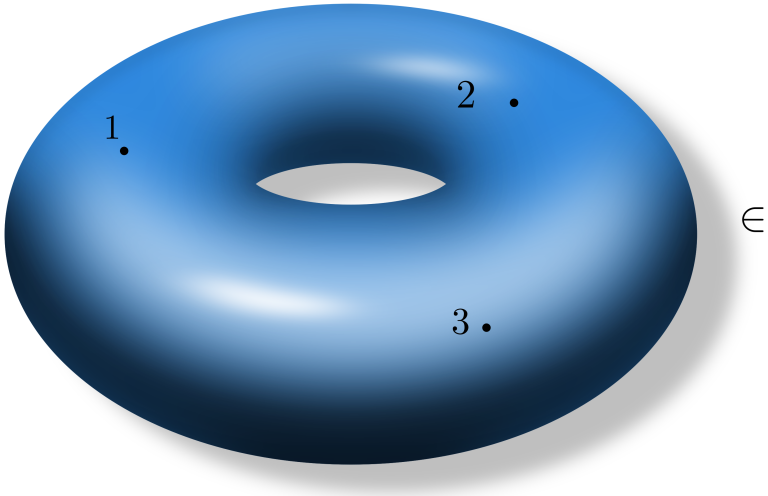

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Name dropping: • Braid groups, • Loop spaces,
• Particles in movement [physics] • Motion planning [robotics]

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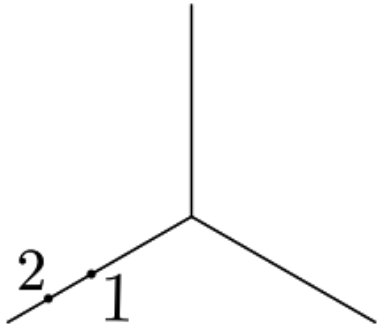
- 

$= Conf_2(Y)$

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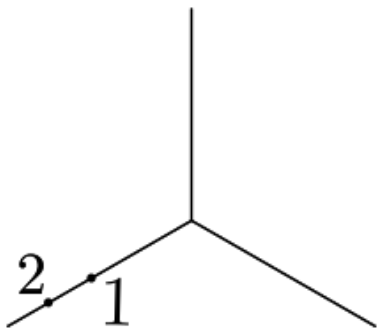
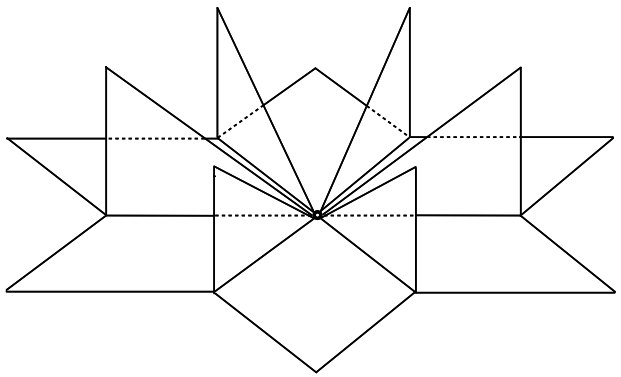
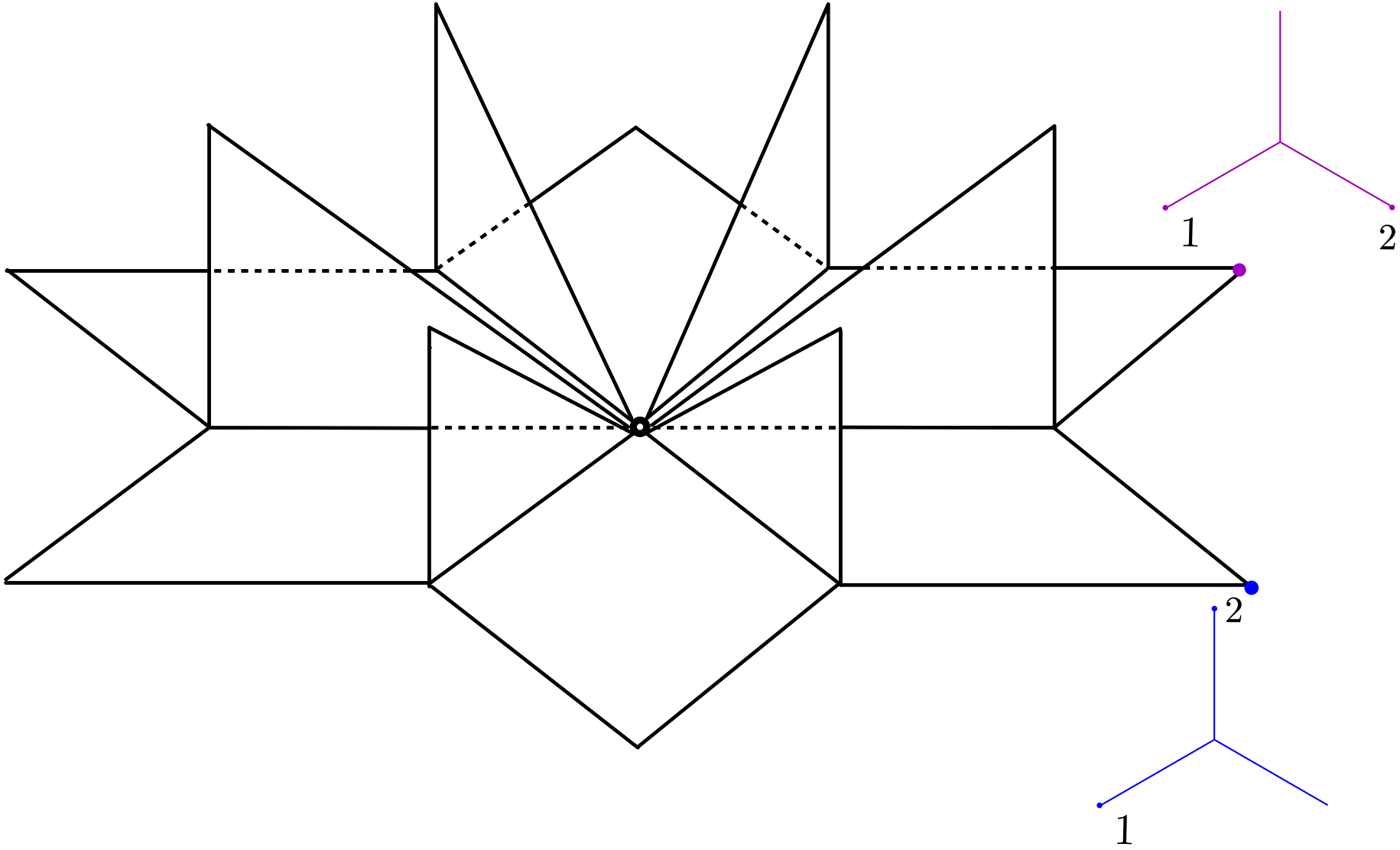
- 

$= Conf_2(Y) =$ 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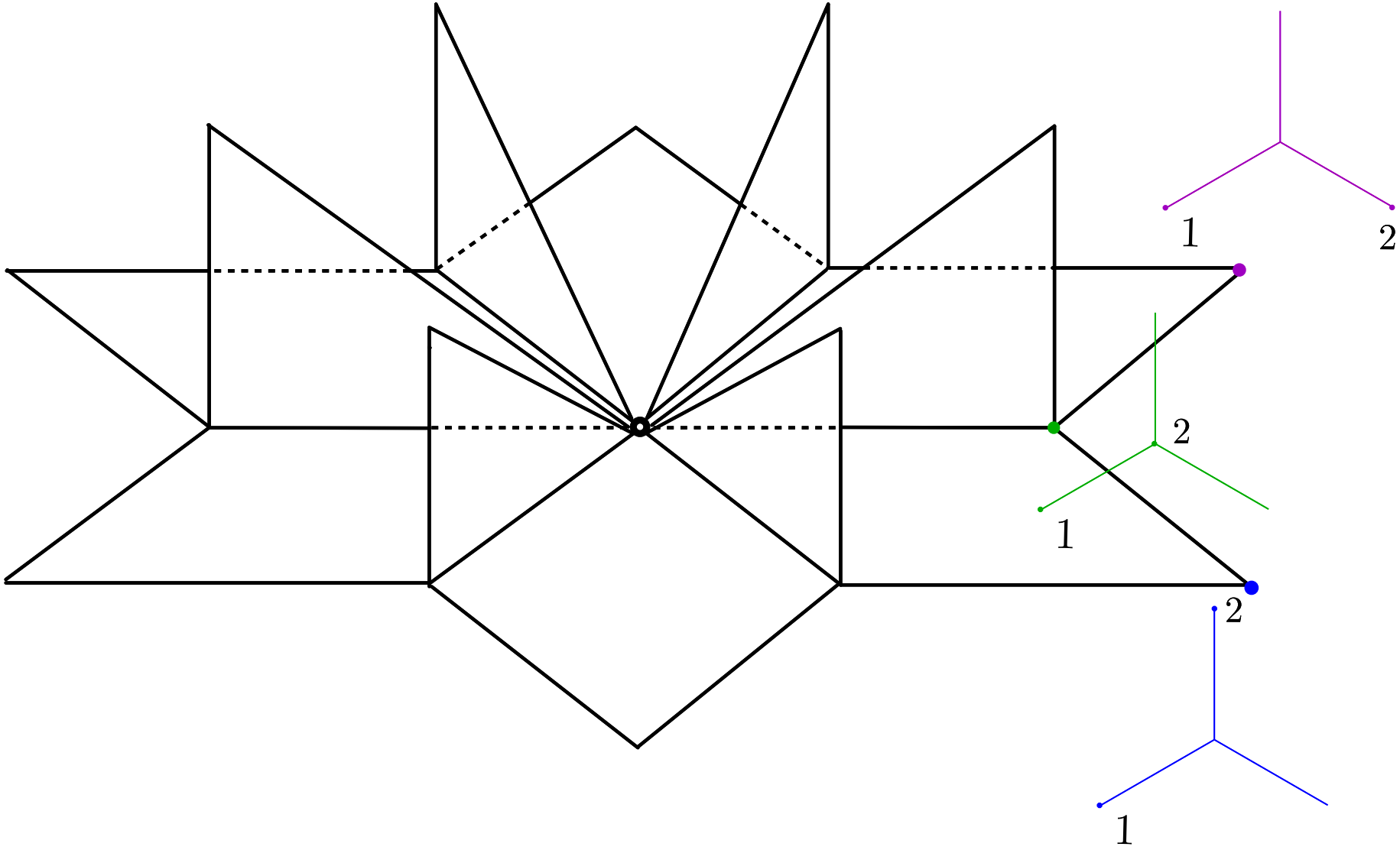
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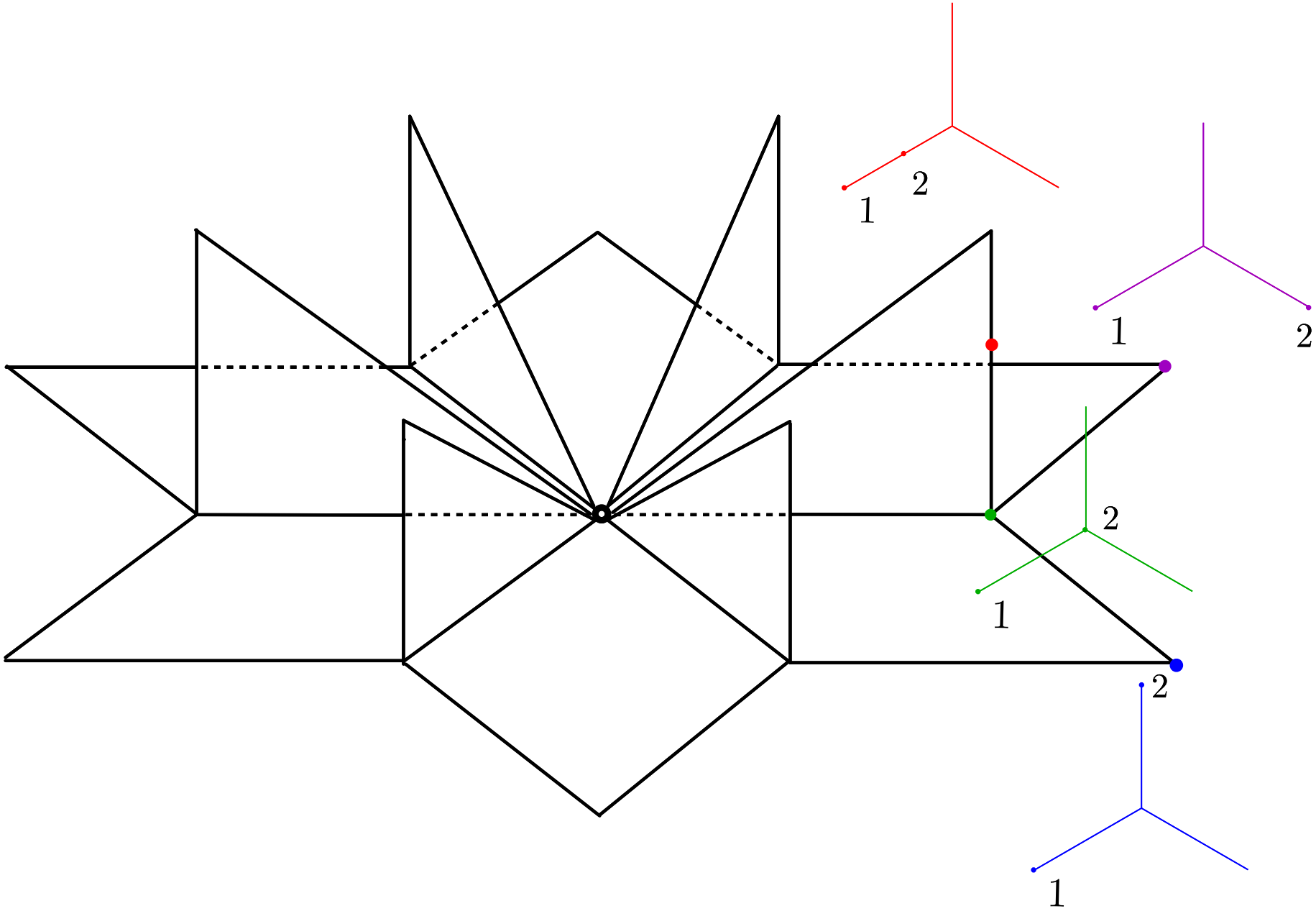
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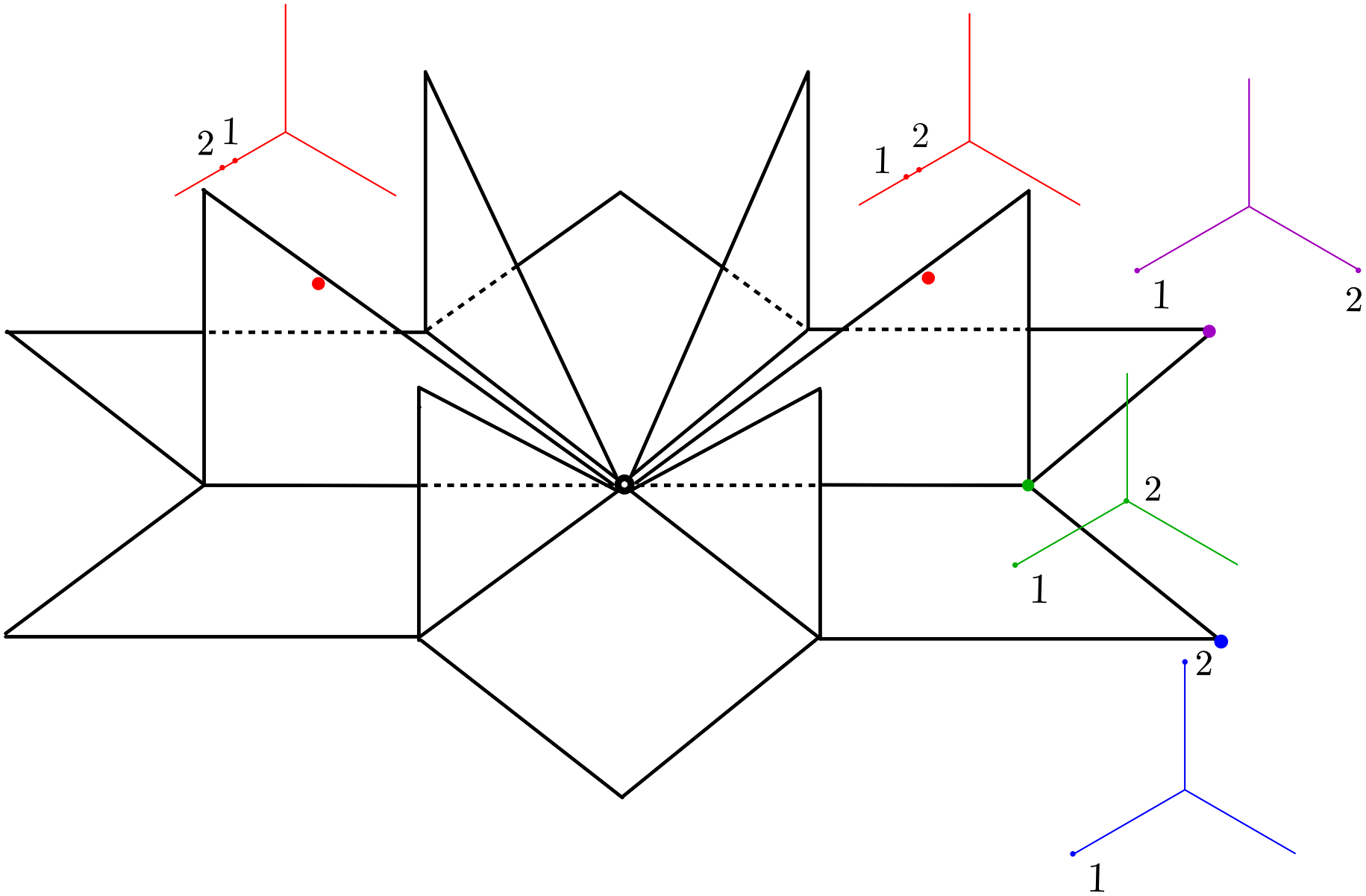
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Quick history of $Conf_n(X)$

1969 [Arnold–Cohen] Computation of $H^\bullet(Conf_n(\mathbb{R}^d))$

1978 [Cohen–Taylor] spectral sequence $\to H(Conf_n(M))$

2004 [Lambrechts–Stanley] Model of $Conf_2(M)$ for 2-connected manifolds

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1998 [Kontsevich] Model $Conf_n(\mathbb{R}^d)$ using graphs

Homotopy theory

Homotopy theory: Studying spaces up to homotopy using algebraic invariants ((co)homology, homotopy groups, etc)
Reminders:

- $f, g : X \to Y$ are homotopic ($f \sim g$) if $\exists H_t \colon X \to Y$ with $H_0 = f$ and $H_1 = g$
- f is is a homotopy equivalence if it is invertible up to $\sim$
- X and Y have the same homotopy type if they can be connected by homotopy equivalences.

An open question

Question

Does the homotopy type of M determine the homotopy type of $Conf_n(M)$? How to "compute" the homotopy type of $Conf_n(M)$?

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False: Salvatore and Longoni cook up a counter example from the lens spaces $L_{7,1}$ and $L_{7,2}$.

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For simply connected closed manifolds

The homotopy invariance remains an open question...

A “simplification” - Rational Homotopy Theory

Idea: Usual (integral) homotopy type is given by maps $X \to Y$ that induce isomorphisms $\pi_i(X) \xrightarrow{\sim} \pi_i(Y)\ \forall i \geq 1$ (or $H(X, \mathbb{Z}) \xrightarrow{\sim} H(Y, \mathbb{Z})$)

Rational homotopy equivalence $\sim_{\mathbb{Q}}$: Maps such that $\pi_i^{\mathbb{Q}}(X) \xrightarrow{\sim} \pi_i^{\mathbb{Q}}(Y)$ (or $H(X, \mathbb{Q}) \xrightarrow{\sim} H(Y, \mathbb{Q})$)

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Magic (Quillen ’69, Sullivan ’77)

To understand the rational homotopy type of X, it is equivalent to understand the quasi-isomorphism type of a comm algebra $A(X)$. Think $A(X) = (C^*_{\text{sing}}(X,\mathbb{Q}), \cup, d_{\text{sing}})$ or $\Omega_{dR}(X)$ if X is a smooth manifold.

Our contribution

Theorem (C.–Willwacher)

For simply connected closed smooth manifolds,

$$M \sim_{\mathbb{R}} N \Rightarrow Conf_n(M) \sim_{\mathbb{R}} Conf_n(N)$$

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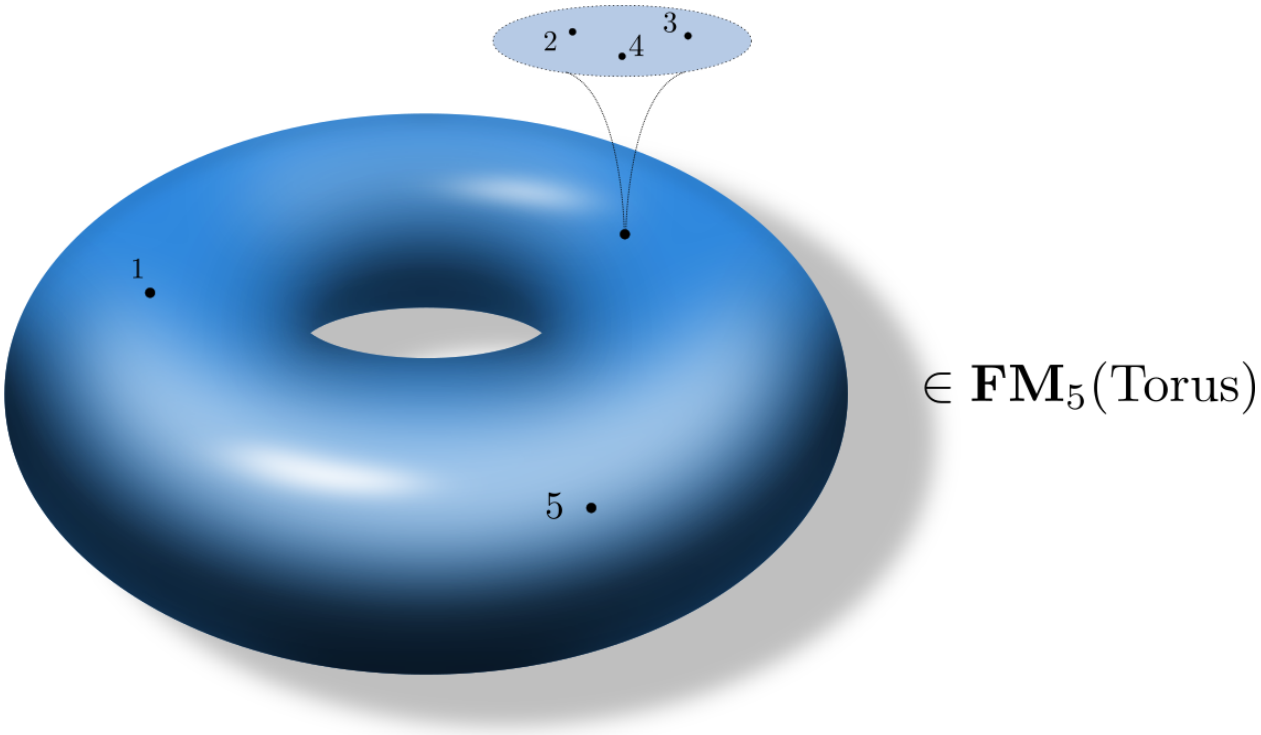
Theorem (C.–Idrissi–Lambrechts–Willwacher)

For simply connected compact manifolds with simply connected boundary, the real homotopy type of $Conf_n(M)$ is determined by the real homotopy type of the inclusion $\partial M \hookrightarrow M$. (dim ≥ 7)

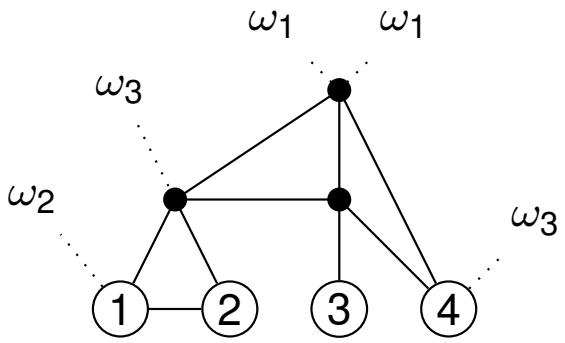
Strategy of proof

First step:
Compactify $Conf_n(M) \rightsquigarrow FM_n(M)$, the Fulton–MacPherson compactification.



Configuration spaces of points - A graph model

Let M be a closed orientable manifold. We consider the differential graded $\mathbb{R}$-algebra $Graphs_n(M)$ to be spanned by graphs of the form



$\in Graphs_4(M)$,

where $\omega_i \in H^\bullet(M)$.

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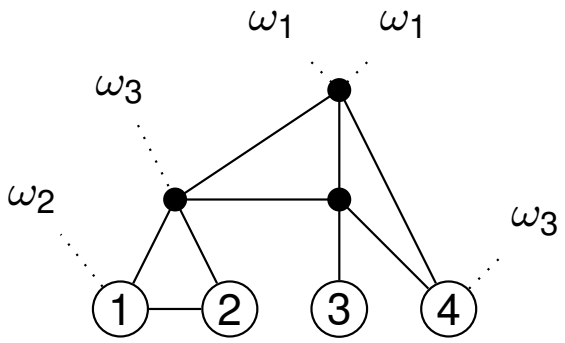

$\in Graphs_4(M)$,

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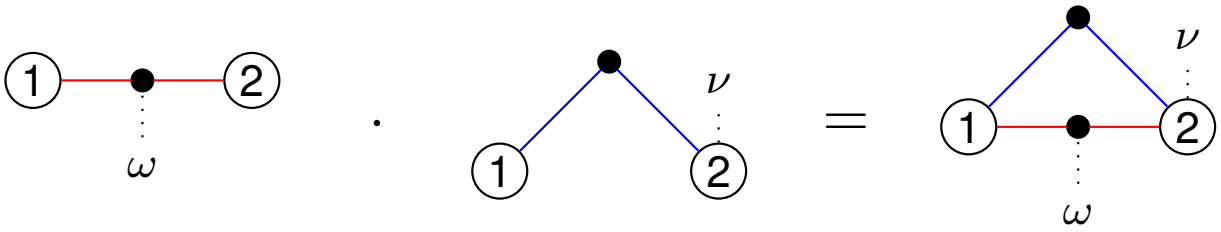
There is a differential $d = \underbrace{\delta}_{\text{contr.}} + \Delta_{\text{extra}}$ and a product of graphs given by superposition of white vertices:



Configuration spaces of points - A graph model

Theorem (C. – Willwacher)

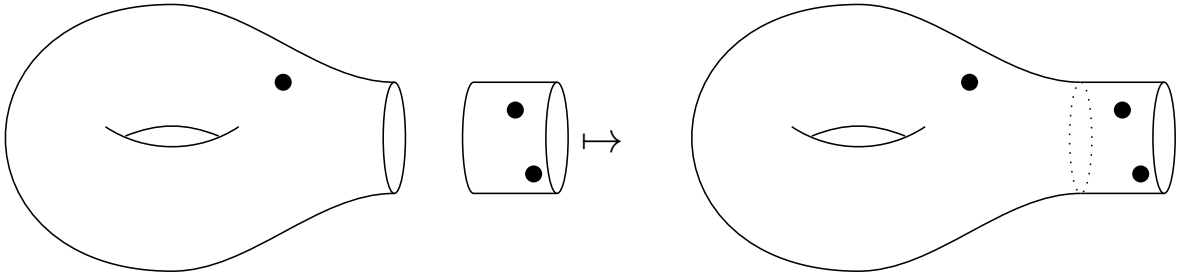
Let M be a closed, orientable manifold. Then, $Graphs_n(M)$ is a real model of $Conf_n(M)$, in the sense that there exists a map

$$f\colon Graphs_n(M) \xrightarrow{\sim} \Omega(FM_n(M)) \xrightarrow{\sim} \Omega(Conf_n(M))$$

of differential graded $\mathbb{R}$-algebras, inducing an isomorphism in homology.

More structure

Understood gluing with Idrissi, Lambrechts and Willwacher:



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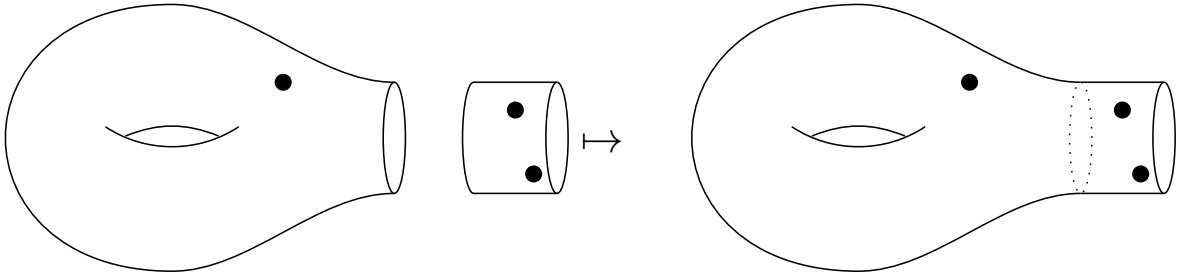
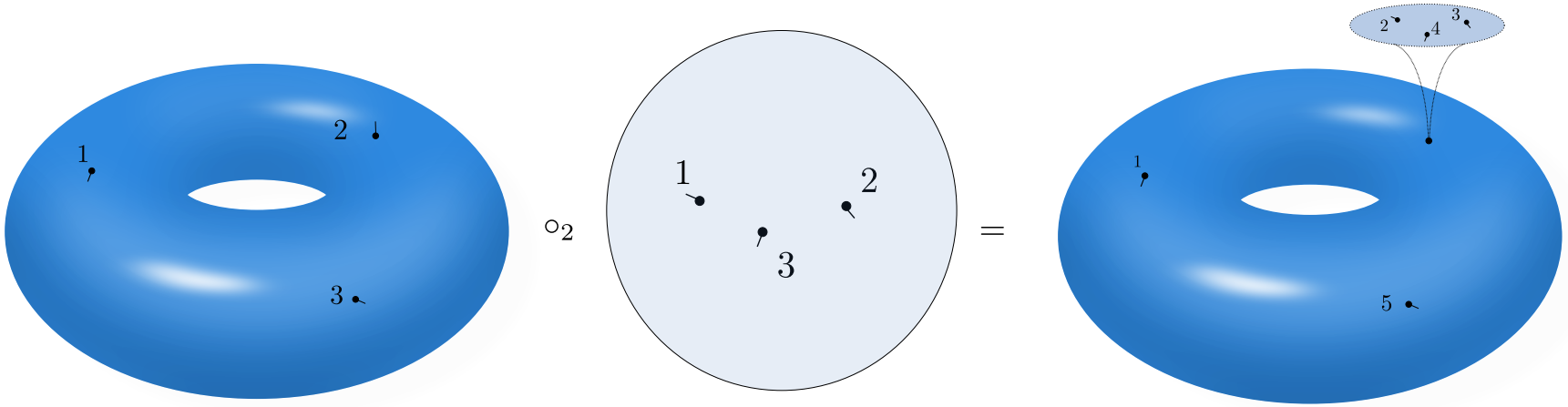


Understand action from the framed little discs operad with Ducoulombier, Idrissi and Willwacher.



$\in \mathsf{FM}^{fr}_{\Sigma_1}(3)$ $\in \mathsf{FM}^{fr}_{\mathbb{R}^2}(3)$ $\in \mathsf{FM}^{fr}_{\Sigma_1}(5)$

The end

Obrigado!