Configuration spaces of points and their homotopy type

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A pre-example

[Image of a robot vacuum cleaner in a corner]

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Configuration space $= X \subset [0, 1] \times [0, 1]$. 
A pre-example

Configuration space $= X^n - \Delta$,  \[ \Delta = \{(x_1, \ldots, x_n) \mid \exists x_i = x_j \}. \]
Configuration spaces of points

Goal: Understand configuration spaces of points

\[ \text{Conf}_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j\}. \]
**Goal**: Understand configuration spaces of points

$$Conf_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j\}.$$
Examples

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## Quick history of $\text{Conf}_n(X)$

<table>
<thead>
<tr>
<th>Year</th>
<th>Reference</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>1969</td>
<td>[Arnold–Cohen]</td>
<td>Computation of $H^\bullet(\text{Conf}_n(\mathbb{R}^d))$</td>
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<tr>
<td>1978</td>
<td>[Cohen–Taylor]</td>
<td>Spectral sequence $\rightarrow H(\text{Conf}_n(M))$</td>
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<td>2004</td>
<td>[Lambrechts–Stanley]</td>
<td>Model of $\text{Conf}_2(M)$ for 2-connected manifolds</td>
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<td>1998</td>
<td>[Kontsevich]</td>
<td>Model $\text{Conf}_n(\mathbb{R}^d)$ using graphs</td>
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</table>
Homotopy theory: Studying spaces up to homotopy using algebraic invariants ((co)homology, homotopy groups, etc)

Reminders:

- \( f, g : X \to Y \) are homotopic (\( f \sim g \)) if \( \exists H_t : X \to Y \) with \( H_0 = f \) and \( H_1 = g \)
- \( f \) is is a homotopy equivalence if it is invertible up to \( \sim \)
- \( X \) and \( Y \) have the same homotopy type if they can be connected by homotopy equivalences.
An open question

Question

Does the homotopy type of $M$ determine the homotopy type of $Conf_n(M)$? How to “compute” the homotopy type of $Conf_n(M)$?
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### Non-compact manifolds

False: $Conf_2(\mathbb{R}^n) \sim S^{n-1}$.
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Closed manifolds

False: Salvatore and Longoni cook up a counter example from the lens spaces $L_{7,1}$ and $L_{7,2}$. 
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**Question**

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**Non-compact manifolds**

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**Closed manifolds**

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**For simply connected closed manifolds**

The homotopy invariance remains an open question...
A “simplification” - Rational Homotopy Theory

Idea: Usual (integral) homotopy type is given by maps $X \to Y$ that induce isomorphisms $\pi_i(X) \to \pi_i(Y) \forall i \geq 1$ (or $H(X, \mathbb{Z}) \to H(Y, \mathbb{Z})$)

Rational homotopy equivalence $\sim_{\mathbb{Q}}$: Maps such that $\pi_i^{\mathbb{Q}}(X) \to \pi_i^{\mathbb{Q}}(Y)$ (or $H(X, \mathbb{Q}) \to H(Y, \mathbb{Q})$)
A “simplification” - Rational Homotopy Theory

Idea: Usual (integral) homotopy type is given by maps $X \to Y$ that induce isomorphisms $\pi_i(X) \cong \pi_i(Y) \forall i \geq 1$ (or $H(X, \mathbb{Z}) \cong H(Y, \mathbb{Z})$)

Rational homotopy equivalence $\sim_Q$: Maps such that $\pi_i^Q(X) \cong \pi_i^Q(Y)$ (or $H(X, \mathbb{Q}) \cong H(Y, \mathbb{Q})$)

Magic (Quillen ’69, Sullivan ’77)

To understand the rational homotopy type of $X$, it is equivalent to understand the quasi-isomorphism type of a comm algebra $A(X)$. Think $A(X) = (C^*_{\text{sing}}(X, \mathbb{Q}), \cup, d_{\text{sing}})$ or $\Omega_{dR}(X)$ if $X$ is a smooth manifold.
Our contribution

Theorem (C.–Willwacher)

For simply connected closed smooth manifolds,

\[ M \sim_R N \Rightarrow \text{Conf}_n(M) \sim_R \text{Conf}_n(N) \]
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Theorem (C.–Willwacher)

For simply connected closed smooth manifolds,

\[ M \sim_{\mathbb{R}} N \Rightarrow \text{Conf}_n(M) \sim_{\mathbb{R}} \text{Conf}_n(N) \]

Theorem (C.–Idrissi–Lambrechts–Willwacher)

For simply connected compact manifolds with simply connected boundary, the real homotopy type of \( \text{Conf}_n(M) \) is determined by the real homotopy type of the inclusion \( \partial M \hookrightarrow M \). (\( \text{dim} \geq 7 \))
Strategy of proof

First step: Compactify $\text{Conf}_n(M) \rightsquigarrow \text{FM}_n(M)$, the Fulton–MacPherson compactification.
Let $M$ be a closed orientable manifold. We consider the differential graded $\mathbb{R}$-algebra $\text{Graphs}_n(M)$ to be spanned by graphs of the form

$$\omega_1 \omega_1 \omega_2 \omega_3 \in \text{Graphs}_4(M),$$

where $\omega_i \in H^*(M)$. 

![Graph diagram](image-url)
Let $M$ be a closed orientable manifold. We consider the differential graded $\mathbb{R}$-algebra $\text{Graphs}_n(M)$ to be spanned by graphs of the form

$$\omega_1 \cdot \omega_2 \cdot \omega_3 \cdot \omega_4 \in \text{Graphs}_4(M),$$

where $\omega_i \in H^\bullet(M)$. There is a differential $d = \delta + \Delta_{\text{extra}}$ and a product of graphs given by superposition of white vertices:
Theorem (C. – Willwacher)

Let $M$ be a closed, orientable manifold. Then, $\text{Graphs}_n(M)$ is a real model of $\text{Conf}_n(M)$, in the sense that there exists a map

$$f : \text{Graphs}_n(M) \xrightarrow{\sim} \Omega(FM_n(M)) \xrightarrow{\sim} \Omega(\text{Conf}_n(M))$$

of differential graded $\mathbb{R}$-algebras, inducing an isomorphism in homology.
More structure

Understood gluing with Idrissi, Lambrechts and Willwacher:
More structure

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Understand action from the framed little discs operad with Ducoulombier, Idrissi and Willwacher.
Obrigado!