Graded-simple algebras and loop algebras

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Problem

How to reduce the study of graded-simple algebras to the study of graded and simple algebras?

The graded-central-simple algebras with split centroid were shown, by Allison, Berman, Faulkner and Pianzola, to be isomorphic to loop algebras of algebras graded by a quotient group that are central simple as ungraded algebras.

This is a very important reduction, as the graded-central-simple algebras may fail to be nice as ungraded algebras; for instance, they may fail to be simple or semisimple.

It will be shown here how to remove the restriction of the centroid being split, at the expense of allowing certain deformations of the loop algebra construction. These deformations will be based on a symmetric 2-cocycle on the grading group with values in the multiplicative group of the ground field.

1. Group-gradings
2. Loop algebras
3. Cocycle twists
4. Graded-central-simple algebras

Group-gradings

Definition

Let $\mathcal{A}$ be an algebra (over a field $\mathbb{F}$) and let G be an *abelian* group.

- A G-grading on $\mathcal{A}$ is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ such that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for any $g, h \in G$.
- The nonzero elements in $\mathcal{A}_g$ are said to be homogeneous of degree g.
- The support of Γ is the set $\{g \in G \mid \mathcal{A}_g \neq 0\}$.

Group-gradings

Example: group algebra

Given an abelian group G, the group algebra $\mathbb{F}G$ is endowed with a natural G-grading:

$$\mathbb{F}G = \bigoplus_{g \in G} \mathbb{F}g.$$

This is an example of a graded-field (commutative graded algebra where all homogeneous elements have an inverse).

Simple algebras

Let $\mathcal{B}$ be an algebra over $\mathbb{F}$:

- $\mathcal{B}$ is simple if it has no proper ideals and $\mathcal{B}^2 \neq 0$.
  In other words, $\mathcal{B}$ is simple if it is simple as a module for its multiplication algebra $\mathrm{Mult}(\mathcal{B})$.

- The centroid of $\mathcal{B}$ is the centralizer of $\mathrm{Mult}(\mathcal{B})$ in $\mathrm{End}_{\mathbb{F}}(\mathcal{B})$:

  $$C(\mathcal{B}) := \{f \in \mathrm{End}_{\mathbb{F}}(\mathcal{B}) : f(xy) = f(x)y = xf(y)\ \forall x, y \in \mathcal{B}\}.$$

  $C(\mathcal{B})$ is commutative if $\mathcal{B}^2 = \mathcal{B}$, and it is a field (an extension field of $\mathbb{F}$) if $\mathcal{B}$ is simple.

- $\mathcal{B}$ is central simple if it is simple and central: $C(\mathcal{B}) = \mathbb{F}\mathrm{id}$.

Any simple algebra is a central simple algebra, when considered as an algebra over its centroid.

Graded-simple algebras

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a G-graded algebra:

- $\mathcal{B}$ is graded-simple if it has no proper graded ideals and $\mathcal{B}^2 \neq 0$.

  Its centroid *inherits* a G-grading:

  $$C(\mathcal{B})_g := \{f \in C(\mathcal{B}) : f(\mathcal{B}_h) \subseteq \mathcal{B}_{gh} \ \forall h \in G\}.$$

- $\mathcal{B}$ is graded-central if $C(\mathcal{B})_e = \mathbb{F}\mathrm{id}$.

- $\mathcal{B}$ is graded-central-simple if it is graded-simple and graded-central.

Graded-simple algebras

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a graded-simple algebra, then:

- $C(\mathcal{B})$ is a graded-field.
- $\mathcal{B}$ is simple (ungraded) if and only if $C(\mathcal{B})$ is a field.
- $\mathbb{K} = C(\mathcal{B})_e$ is a field, and $\mathcal{B}$ is graded-central-simple considered as an algebra over $\mathbb{K}$.
- If $\mathcal{B}$ is graded-simple, and H is the support of the induced grading on $C(\mathcal{B})$, then H is a subgroup of G.
- If $\mathcal{B}$ is graded-central-simple, its centroid $C(\mathcal{B})$ is said to be split if it is isomorphic, as a graded algebra, to the group algebra $\mathbb{F}H$: $\quad C(\mathcal{B}) \simeq_G \mathbb{F}H$.

Loop algebras

Definition

Given an abelian group G, a subgroup H, the canonical projection

$$\pi : G \to \overline{G} = G/H, \quad g \mapsto \pi(g) = \bar{g},$$

and an algebra $\mathcal{A}$ graded by $\overline{G}$: $\mathcal{A} = \bigoplus_{\bar{g} \in \overline{G}} \mathcal{A}_{\bar{g}}$, the loop algebra $L_\pi(\mathcal{A})$ is the G-graded algebra

$$L_\pi(\mathcal{A}) := \bigoplus_{g \in G} \mathcal{A}_{\bar{g}} \otimes g$$

which is a subalgebra of the tensor product $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G$.

Graded-simple algebras and loop algebras

Theorem (Allison, Berman, Faulkner, and Pianzola)

Let G be an abelian group, H a subgroup of G, and $\pi : G \to \overline{G} = G/H$ the canonical projection.

1. *If $\mathcal{A}$ is a central simple algebra graded by $\overline{G}$, then the loop algebra $L_\pi(\mathcal{A})$ is a G-graded-central-simple algebra, and the map*

$$\begin{aligned} \mathbb{F}H &\longrightarrow C\big(L_\pi(\mathcal{A})\big) \\ h &\mapsto \big(x \otimes g \mapsto x \otimes hg\big) \end{aligned}$$

   for $g \in G$, $x \in \mathcal{A}_{\pi(g)}$, is an isomorphism of G-graded algebras. (Hence $C\big(L_\pi(\mathcal{A})\big) \simeq_G \mathbb{F}H$.)

Graded-simple algebras and loop algebras

Theorem (continued)

2. *Conversely, if $\mathcal{B}$ is a G-graded-central-simple algebra with split centroid: $C(\mathcal{B}) \simeq_G \mathbb{F}H$, then there exists a central simple and $\overline{G}$-graded algebra $\mathcal{A}$ such that $\mathcal{B} \simeq_G L_\pi(\mathcal{A})$.*

   This central simple algebra $\mathcal{A}$ may be obtained as $\mathcal{B}/\mathcal{I}\mathcal{B}$, where $\mathcal{I}$ is the augmentation ideal of $C(\mathcal{B}) \simeq_G \mathbb{F}H$.

Graded-simple algebras and loop algebras

Theorem (continued)

3. *If $\mathcal{A}$ and $\mathcal{A}'$ are central simple and $\overline{G}$-graded algebras, then $L_\pi(\mathcal{A}) \simeq_G L_\pi(\mathcal{A}')$ if and only if there is a character $\chi \in \mathrm{Hom}(H, \mathbb{F}^\times)$ such that $\mathcal{A}' \simeq_{\overline{G}} \mathcal{A}_\chi$.*

   The algebra $\mathcal{A}_\chi$ is defined on the same vector space as $\mathcal{A}$, but with new multiplication

$$x \cdot_\chi y = \chi\Big(s(\bar{g}_1)s(\bar{g}_2)s(\bar{g}_1\bar{g}_2)^{-1}\Big)xy$$

   for $\bar{g}_1, \bar{g}_2 \in \overline{G}$, $x \in \mathcal{A}_{\bar{g}_1}$, $y \in \mathcal{A}_{\bar{g}_2}$, where $s : \overline{G} \to G$ is an arbitrary section of the canonical projection $\pi : G \to \overline{G}$.

$\mathrm{Ext}(A, B)$ and extensions

Given two abelian groups A, B, the abelian group $\mathrm{Ext}(A, B)$ is the set of equivalence classes of extensions of A by B (in the category of abelian groups): $1 \to B \to E \to A \to 1$.

Two extensions

$$1 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow B \longrightarrow E' \longrightarrow A \longrightarrow 1$$

are equivalent if there is a homomorphism $\varphi : E \to E'$, necessarily bijective, such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 1 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 1 \end{array}$$

is commutative.

$\mathrm{Ext}(A, B)$ and extensions

For a homomorphism $f : A' \to A$, the natural homomorphism

$$f^* : \mathrm{Ext}(A, B) \longrightarrow \mathrm{Ext}(A', B)$$
$$[\xi] \mapsto [\xi f]$$

is obtained by means of the commutative diagram:

$$\begin{array}{lccccccccc}
\xi f : & 1 & \longrightarrow & B & \xrightarrow{j} & \tilde{E} & \xrightarrow{\tilde{p}_2} & A' & \longrightarrow & 1 \\
& & & \| & & \downarrow{\scriptstyle \tilde{p}_1} & & \downarrow{\scriptstyle f} & & \\
\xi : & 1 & \longrightarrow & B & \xrightarrow{i} & E & \xrightarrow{p} & A & \longrightarrow & 1
\end{array}$$

where $\tilde{E}$ is the pull-back of p and f.

$\mathrm{Ext}(A, B) \simeq \mathrm{H}^2_{\mathrm{sym}}(A, B)$

On the other hand, the set of equivalence classes of central extensions, in the category of groups, of the group A by the abelian group B:

$$1 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 1$$

with $i(B)$ central in E, can be identified with the second cohomology group $\mathrm{H}^2(A, B) = \mathrm{Z}^2(A, B)/\mathrm{B}^2(A, B)$, where

$$\mathrm{Z}^2(A, B) = \{\sigma : A \times A \to B \mid \sigma(a_1, a_2)\sigma(a_1 a_2, a_3) = \sigma(a_1, a_2 a_3)\sigma(a_2, a_3)\ \forall a_1, a_2, a_3 \in A\}$$

is the set of 2-cocycles, and $\mathrm{B}^2(A, B) = \{\mathrm{d}\gamma \mid \gamma : A \to B \text{ a map}\}$ is the set of 2-coboundaries:

$$\mathrm{d}\gamma(a_1, a_2) = \gamma(a_1)\gamma(a_2)\gamma(a_1 a_2)^{-1}.$$

$\mathrm{Ext}(A, B) \simeq \mathsf{H}^2_{\mathrm{sym}}(A, B)$

The element in $\mathsf{H}^2(A, B)$ that corresponds to the central extension

$$\xi : 1 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 1$$

is obtained by fixing a section $s : A \to E$ of p. Then for any $a_1, a_2 \in A$, there is a unique element $\sigma(a_1, a_2) \in B$ such that

$$i\big(\sigma(a_1, a_2)\big) = s(a_1)s(a_2)s(a_1a_2)^{-1},$$

and $\sigma : A \times A \to B$ is a 2-cocycle whose equivalence class $[\sigma]$ is the element in $\mathsf{H}^2(A, B)$ that corresponds to the equivalence class of ξ.

Moreover, for A and B abelian, ξ is an abelian extension if and only if σ is symmetric.

$\mathrm{Ext}(A, B) \simeq \mathsf{H}^2_{\mathrm{sym}}(A, B)$

Denote by $\mathsf{Z}^2_{\mathsf{sym}}(A, B)$ the subgroup of $\mathsf{Z}^2(A, B)$ of the symmetric 2-cocycles, and note that $\mathsf{B}^2(A, B)$ is contained in $\mathsf{Z}^2_{\mathsf{sym}}(A, B)$. Then, for A and B abelian, $\mathrm{Ext}(A, B)$ can be identified with the quotient

$$\mathsf{H}^2_{\mathsf{sym}}(A, B) = \mathsf{Z}^2_{\mathsf{sym}}(A, B)/\mathsf{B}^2(A, B).$$

The map $f^* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$ becomes:

$$\begin{aligned} f^* : \mathsf{H}^2_{\mathsf{sym}}(A, B) &\longrightarrow \mathsf{H}^2_{\mathsf{sym}}(A', B) \\ [\sigma] &\mapsto [\sigma \circ (f \times f)] \end{aligned}$$

A long exact sequence

Given an abelian group G, a subgroup H, and the associated quotient group $\overline{G} = G/H$, consider the corresponding short exact sequence

$$\zeta : \; 1 \longrightarrow H \xhookrightarrow{\iota} G \xrightarrow{\pi} G/H \longrightarrow 1.$$

For any abelian group F, this induces a long exact sequence:

$$\begin{aligned} 1 \to \mathrm{Hom}(G/H, F) &\xrightarrow{\pi^*} \mathrm{Hom}(G, F) \xrightarrow{\iota^*} \mathrm{Hom}(H, F) \\ &\xrightarrow{\delta} \mathrm{H}^2_{\mathrm{sym}}(G/H, F) \xrightarrow{\pi^*} \mathrm{H}^2_{\mathrm{sym}}(G, F) \xrightarrow{\iota^*} \mathrm{H}^2_{\mathrm{sym}}(H, F) \to 1 \end{aligned}$$

the *connecting homomorphism* $\delta : \mathrm{Hom}(H, F) \to \mathrm{H}^2_{\mathrm{sym}}(G/H, F)$ being given by

$$\delta(f) = [f \circ \sigma],$$

with $\sigma(\bar{g}_1, \bar{g}_2) = s(\bar{g}_1)s(\bar{g}_2)s(\bar{g}_1\bar{g}_2)^{-1}$.

A long exact sequence

Proposition

Let G and F be abelian groups, H a subgroup of G, and $\tau' : H \times H \to F$ a symmetric 2-cocycle. Then there is a symmetric 2-cocycle $\tau \in \mathrm{Z}^2_{\mathrm{sym}}(G, F)$ that extends τ' (i.e., $\tau' = \tau|_{H\times H}$).

Proof

$\iota^* : \mathrm{H}^2_{\mathrm{sym}}(G, F) \to \mathrm{H}^2_{\mathrm{sym}}(H, F)$ surjective

$\Rightarrow \exists \tilde{\tau} \in \mathrm{Z}^2_{\mathrm{sym}}(G, F)$ such that $[\tilde{\tau}|_{H\times H}] = \iota^*([\tilde{\tau}]) = [\tau']$,

$\Rightarrow \exists \gamma : H \to F$ such that $\tau' = (\tilde{\tau}|_{H\times H})(\mathrm{d}\gamma)$. That is,

$$\tau'(h_1, h_2) = \tilde{\tau}(h_1, h_2)\gamma(h_1)\gamma(h_2)\gamma(h_1 h_2)^{-1}.$$

Extend γ to a map $\tilde{\gamma} : G \to F$. Then $\tau = \tilde{\tau}(\mathrm{d}\tilde{\gamma})$ satisfies $\tau|_{H\times H} = \tau'$.

Cocycle twists

Definition

Let G be an abelian group, let $\mathcal{A}$ be an algebra over $\mathbb{F}$ endowed with a G-grading: $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$, and let $\tau : G \times G \to \mathbb{F}^\times$ be a symmetric 2-cocycle. Define a new multiplication on $\mathcal{A}$ by the formula

$$x * y := \tau(g_1, g_2)xy$$

for $g_1, g_2 \in G$, $x \in \mathcal{A}_{g_1}$, $y \in \mathcal{A}_{g_2}$.

The new algebra thus defined will be called the τ-twist of $\mathcal{A}$, and will be denoted by $\mathcal{A}^\tau$.

For any abelian group G and symmetric 2-cocycle $\tau \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$, the τ-twist $\big(\mathbb{F}G\big)^\tau$ of the group algebra $\mathbb{F}G$ is denoted traditionally by $\mathbb{F}^\tau G$ (a twisted group algebra).

Any G-graded-field $\mathcal{F}$ with $\mathcal{F}_e = \mathbb{F}$ is isomorphic to $\mathbb{F}^\tau H$, for some subgroup H of G and some $\tau \in \mathsf{Z}^2_{\mathsf{sym}}(H, \mathbb{F}^\times)$.

Cocycle twists

Example: $\mathcal{A}_\chi$

Let G be an abelian group, H a subgroup, and $\overline{G} = G/H$ the corresponding quotient. Let $\mathcal{A}$ be a $\overline{G}$-graded algebra and let $\chi \in \mathrm{Hom}(H, \mathbb{F}^\times)$ (a character on H).

The algebra $\mathcal{A}_\chi$, considered by Allison et al., defined on $\mathcal{A}$ by

$$x \cdot_\chi y = \chi\Big(s(\overline{g_1})s(\overline{g_2})s(\overline{g_1 g_2})^{-1}\Big)xy$$

for a section $s : \overline{G} \to G$, coincides with the $(\chi \circ \sigma)$-twist $\mathcal{A}^{\chi\circ\sigma}$. ($\sigma \in \mathrm{Z}^2_{\mathrm{sym}}(\overline{G}, H)$ is given by $\sigma(\bar{g}_1, \bar{g}_2) = s(\bar{g}_1)s(\bar{g}_2)s(\bar{g}_1\bar{g}_2)^{-1}$.)

Note that $[\chi \circ \sigma] = \delta(\chi)$, where δ is the connecting homomorphism for $F = \mathbb{F}^\times$.

Let G be an abelian group and let $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a G-graded algebra over $\mathbb{F}$.

1. For $\sigma, \tau \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$, $\big(\mathcal{A}^\sigma\big)^\tau = \mathcal{A}^{\sigma\tau}$.
2. If $\tau \in \mathsf{B}^2(G, \mathbb{F}^\times)$, then $\mathcal{A}^\tau \simeq_G \mathcal{A}$. More generally, if $\tau_1, \tau_2 \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$ and $[\tau_1] = [\tau_2]$ in $\mathsf{H}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$, then $\mathcal{A}^{\tau_1} \simeq_G \mathcal{A}^{\tau_2}$. In other words, for $\tau \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$, the G-graded isomorphism class of $\mathcal{A}^\tau$ depends only on $[\tau] \in \mathsf{H}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$.
3. If $\overline{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}$ and $\tau \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$, then $\mathcal{A}^\tau \otimes_{\mathbb{F}} \overline{\mathbb{F}} \simeq_G \mathcal{A} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. In particular, if $\mathcal{A}$ is an asociative, alternative, Lie, linear Jordan, ..., algebra, so is $\mathcal{A}^\tau$.
4. If $\mathcal{A}$ is graded-simple, so is $\mathcal{A}^\tau$, and $C(\mathcal{A}^\tau) \simeq_G C(\mathcal{A})^\tau$.

Twisted loop algebras

Given an abelian group G, a subgroup H, a $\overline{G} = G/H$-graded algebra algebra $\mathcal{A}$, and a symmetric 2-cocycle $\tau \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$, the τ-twist $\big(L_\pi(\mathcal{A})\big)^\tau$ will be denoted by $L^\tau_\pi(\mathcal{A})$ and called a cocycle twisted loop algebra.

Remark

$L^\tau_\pi(\mathcal{A})$ is the subalgebra $\bigoplus_{g\in G} \mathcal{A}_{\overline{g}} \otimes g$ of the tensor product $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}^\tau G$, where $\mathbb{F}^\tau G$ is the twisted group algebra.

Main result

Theorem

Let G be an abelian group.

1. *Let H be a subgroup of G, $\mathcal{A}$ a central simple and $\overline{G}$-graded algebra, and let $\tau \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$. Then $L^\tau_\pi(\mathcal{A})$ is G-graded-central-simple and $C\big(L^\tau_\pi(\mathcal{A})\big) \simeq_G \mathbb{F}^{\tau'} H$, where $\tau' = \tau|_{H\times H}$.*
2. *Conversely, if $\mathcal{B}$ is a G-graded-central-simple algebra, then there is a subgroup H of G, a central simple and $\overline{G} = G/H$-graded algebra $\mathcal{A}$, and a symmetric 2-cocycle $\tau \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$ such that $\mathcal{B} \simeq_G L^\tau_\pi(\mathcal{A})$.*

Main result

Theorem (continued)

3. *For $i = 1, 2$, let H_i be a subgroup of G, $\mathcal{A}_i$ a central simple and G/H_i-graded algebra, $\tau_i \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$. Denote by $\pi_i : G \to \overline{G_i} = G/H_i$ the canonical projection, $i = 1, 2$. Then $L^{\tau_1}_{\pi_1}(\mathcal{A}_1) \simeq_G L^{\tau_2}_{\pi_2}(\mathcal{A}_2)$ if and only if the following conditions are satisfied:*
    - $H_1 = H_2 =: H$, *so* $\pi_1 = \pi_2 =: \pi : G \to \overline{G} = G/H$.
    - $\iota^*([\tau_1]) = \iota^*([\tau_2])$ *in* $\mathsf{H}^2_{\mathsf{sym}}(H, \mathbb{F}^\times)$, *where* $\iota : H \hookrightarrow G$ *is the inclusion, and*
    - *there is a 2-cocycle* $\mu \in \mathsf{Z}^2_{\mathsf{sym}}(\overline{G}, \mathbb{F}^\times)$ *such that* $[\tau_1] = \pi^*([\mu])[\tau_2]$ *in* $\mathsf{H}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$ *and* $\mathcal{A}_1^\mu \simeq_{\overline{G}} \mathcal{A}_2$.

Main result

Sketch of proof

If $\mathcal{B}$ is a G-graded-central-simple algebra, and H is the support of its centroid, then $C(\mathcal{B})$ is a twisted group algebra:

$$C(\mathcal{B}) \simeq_G \mathbb{F}^{\tau'} H, \text{ for a 2-cocycle } \tau' \in \mathsf{Z}^2_{\mathsf{sym}}(H, \mathbb{F}^\times).$$

Then there is a 2-cocycle $\tau \in \mathsf{Z}^2_{\mathsf{sym}}(G, \mathbb{F}^\times)$ such that $\tau|_{H \times H} = \tau'$, and

$$C(\mathcal{B}^{\tau^{-1}}) \simeq_G C(\mathcal{B})^{\tau^{-1}} \simeq_G \left(\mathbb{F}^{\tau'} H\right)^{\tau^{-1}} = \mathbb{F}H,$$

so $C(\mathcal{B}^{\tau^{-1}})$ is split, and we are in the situation studied by Allison et al.

Graded-central-simple algebras

Denote by

- $\overline{\mathfrak{B}}(G, \mathbb{F})$ the set of isomorphism classes of G-graded-central-simple algebras, being $[\mathcal{B}]$ the class of an algebra $\mathcal{B}$.
- $\overline{\mathfrak{A}}(G, \mathbb{F})$ the set consisting of the equivalence classes of triples $(H, [\tau], \mathcal{A})$, where H is a subgroup of G, $[\tau] \in \mathrm{H}^2_{\mathrm{sym}}(G, \mathbb{F}^\times)$, and $\mathcal{A}$ is a central simple and G/H-graded algebra, the equivalence relation being given by

$$(H_1, [\tau_1], \mathcal{A}_1) \sim (H_2, [\tau_2], \mathcal{A}_2)$$

if $H_1 = H_2 (=: H)$, $\iota^*([\tau_1]) = \iota^*([\tau_2])$, and if there is a $\mu \in \mathrm{Z}^2_{\mathrm{sym}}(G/H, \mathbb{F}^\times)$ such that $[\tau_1] = \pi^*([\mu])[\tau_2]$ and $\mathcal{A}_1^\mu \simeq_{G/H} \mathcal{A}_2$.

Graded-central-simple algebras

Corollary

The map

$$\begin{aligned} \overline{\mathfrak{A}}(G,\mathbb{F}) &\longrightarrow \overline{\mathfrak{B}}(G,\mathbb{F}) \\ [(H,[\tau],\mathcal{A})] &\mapsto [L_\pi^\tau(\mathcal{A})] \end{aligned}$$

is a bijection.

Graded-central-simple algebras

In order to reduce the freedom in choosing τ above we may fix, for all subgroups H of G, a section $\xi_H : \mathrm{H}^2_{\mathrm{sym}}(H, \mathbb{F}^\times) \to \mathrm{H}^2_{\mathrm{sym}}(G, \mathbb{F}^\times)$ of ι^*, and consider the set

- $\overline{\mathfrak{A}}'(G, \mathbb{F})$ of triples $(H, [\tau'], [\mathcal{A}])$, where $H \leq G$, $[\tau'] \in \mathrm{H}^2_{\mathrm{sym}}(H, \mathbb{F}^\times)$, and $[\mathcal{A}]$ is the equivalence class of a central simple and G/H-graded algebra $\mathcal{A}$, under the equivalence relation being given by $\mathcal{A}_1 \sim \mathcal{A}_2$ if there is a character $\chi \in \mathrm{Hom}(H, \mathbb{F}^\times)$ such that $\left(\mathcal{A}_1\right)_\chi \simeq_{G/H} \mathcal{A}_2$.

Corollary

The map

$$\overline{\mathfrak{A}}'(G, \mathbb{F}) \longrightarrow \overline{\mathfrak{B}}(G, \mathbb{F}), \quad (H, [\tau'], [\mathcal{A}]) \mapsto [L^\tau_\pi(\mathcal{A})]$$

where τ is any 2-cocycle in $\mathrm{Z}^2_{\mathrm{sym}}(G, \mathbb{F}^\times)$ such that $[\tau] = \xi_H([\tau'])$, is a bijection.

References

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