super Jordan triple systems

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Jordan triple systems

Example

Let $x, y, z \in M_{m,n}(\mathbb{C})$ and consider the trilinear product

$$(xyz) = xy^t z + zy^t x$$

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Definition (Jordan Triple Systems)

A Jordan triple system (JTS) is a vector space with a trilinear product $(\cdot, \cdot, \cdot)$ *satisfying:*

$$\begin{aligned} (xyz) &= (zyx) && \textit{Commutativity} \\ (uv(xyz)) &= && \textit{Principal identity} \\ = ((uvx)yz) - (x(vuy)z) + (xy(uvz)) & & & \end{aligned} \tag{1}$$

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If $L_{x,y}(z) = (xyz)$, the principal identity reads

$$L_{u,v}(xyz) = (L_{u,v}(x)yz) - (xL_{v,u}(y)z) + (xyL_{u,v}(z))$$

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Definition

A Kantor triple system (KTS) is a vector space with a ternary product satisfying the principal identity and such that, setting $K_{xz}(y) = (xyz) - (zyx)$,

$$K_{K_{uv}(x)y} = K_{(yxu)v} - K_{(yxv)u} \quad \text{Kantor identity} \tag{2}$$

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Note that if $K_{xy} = 0$ the Kantor identity is satisfied. Thus a JTS is a KTS.

Construction of KTS

Definition (Graded Lie algebras)

A Lie algebra $\mathfrak{g}$ is $\mathbb{Z}$-graded if $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
We say that $\mathfrak{g}$ is $(2k+1)$-graded if $\mathfrak{g}_i = 0$ for $|i| > k$.
If σ is an automorphism of $\mathfrak{g}$ such that $\sigma(\mathfrak{g}_i) = \mathfrak{g}_{-i}$ we will say that σ is grade-reversing.

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Proposition

Let $(\mathfrak{g}, \sigma)$ be a pair consisting of a 3-graded (resp. 5-graded) Lie algebra $\mathfrak{g}$ and a grade-reversing involution σ. Then $\mathfrak{g}_{-1}$ with triple product

$$(xyz) = [[x, \sigma(y)], z] \tag{3}$$

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Remark: It is possible to reverse this construction.

Tits-Kantor-Koecher construction (TKK)

$$\left\{ \begin{array}{l} \text{Simple Lie algebras} \\ \text{3-graded} \\ \text{grade-reversing involution} \end{array} \right\} \longleftrightarrow \{ \text{ simple JTS } \}$$

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TKK construction

Theorem (Cantarini, R., Santi-J. Alg, 2018)

There is a one-to-one correspondence between Kantor triple systems V and pairs $(\mathfrak{g}, \sigma)$, where $\mathfrak{g}$ is a simple 5-graded Lie algebra and σ a grade-reversing involution. Moreover, V is finite dimensional (resp. linearly-compact) if and only if $\mathfrak{g}$ is finite dimensional (resp. linearly-compact).

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Theorem (Cantarini, R., Santi)

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The infinite-dimensional linearly-compact simple Lie algebra of formal vector fields in n variables is

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$\mathfrak{g}_i \neq 0$ for infinitely-many i, i.e. the grading is not finite.

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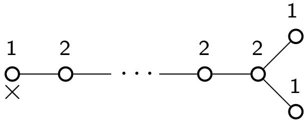
<table>
<tr><th>Type</th><th>Dynkin diagram</th><th>$\mathfrak{g}_{-1}$</th></tr>
<tr><td>D_n</td><td>
</td><td>$M_{2(n-1),1}(\mathbb{C})$</td></tr>
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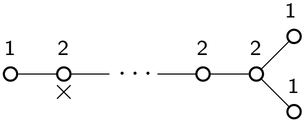
D_n		$M_{2(n-2),2}(\mathbb{C})$

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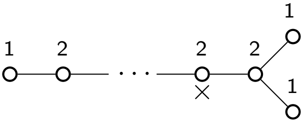
D_n		$M_{6,n-3}(\mathbb{C})$

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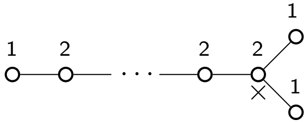
D_n		$M_{4,n-2}(\mathbb{C})$

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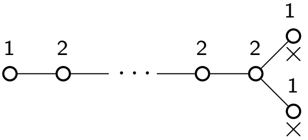
D_n		$M_{2,n-1}(\mathbb{C})$

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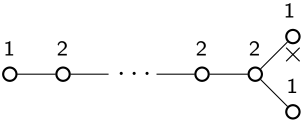
D_n		$Asym_n(\mathbb{C})$

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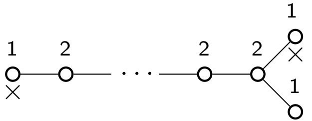
D_n		$Asym_{n-1}(\mathbb{C}) \oplus \mathbb{C}^{n-1}$

Grade-reversing involutions

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Theorem (Cantarini, R., Santi)

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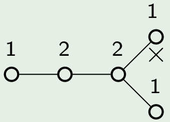
Solution: Given a 5-graded simple Lie algebra $\mathfrak{g}$, each non-isomorphic KTS structure of $\mathfrak{g}_{-1}$ corresponds exactly to one of the real forms $\mathfrak{g}^o$ compatible with the grading.

Compatible real forms

Example

The compatibility conditions of the real forms are easily shown in the diagram of D_5.

Type	Satake diagram	$\mathfrak{g}^o$ (Compatible)
D_5		$\mathfrak{so}(5,5)$ (Y)

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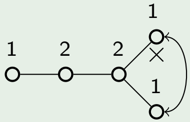
D_5		$\mathfrak{so}(4, 6)$ (N)

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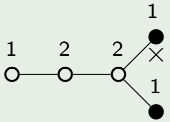
D_5		$\mathfrak{so}(3,7)$ (N)

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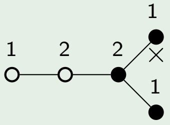
D_5		$\mathfrak{so}(2,8)$ (N)

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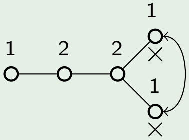
D_5		$\mathfrak{so}(4,6)$ (Y)

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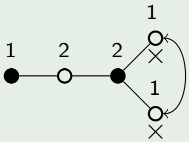
D_5		$\mathfrak{so}^*(10)$ (Y)

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(iii) of special type otherwise.

Exceptional KTS of Poincaré type F_4

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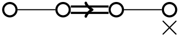

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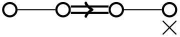

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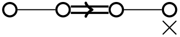

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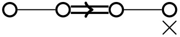

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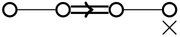

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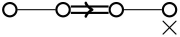

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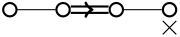

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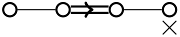

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- Ψ_B denote the bilinear map from $\mathbb{S} \otimes \mathbb{S} \to \mathfrak{so}(U)$ satisfying $\eta(\Psi_B(s, t)u, v) = (u \wedge v \circ s, t)_{\mathbb{S}}$.

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Let $p = 3, 7$. We fix an orthogonal decomposition $U = W \oplus W^{\perp}$ with $\dim(W) = p$ and denote by $\mathrm{vol}_p \in C\ell(U)$ the volume element of W and by r_W the orthogonal reflection across W.

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Theorem

The vector space $\mathbb{S}$ with triple product

$$(rst)_p = \Psi_B(r, \mathrm{vol}_p \circ s) \cdot t - \frac{1}{2}(r, \mathrm{vol}_p \circ s)_{\mathbb{S}}\, t \tag{5}$$

is a simple KTS with associated Lie algebra F_4 and derivation algebra $\mathfrak{so}(7, \mathbb{C})$ if $p = 7$ and $\mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(4, \mathbb{C})$ if $p = 3$.

JTS vs N=6

Example

Let $x, y, z \in M_{m,n}(\mathbb{C})$ and let

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Definition (N=6 3-algebra)

An N=6 3-algebra (N=6) is a vector space with a trilinear product satisfying

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N. Cantarini and V. Kac generalized the TKK to N=6 3-algebras.

Lie superalgebras

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A Lie superalgebra is a $\mathbb{Z}_2$*-graded vector space,* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$*, endowed with a bracket which satisfies*

$$\begin{aligned} &[\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}] \subseteq \mathfrak{g}_{\bar{i}+\bar{j}} && \textit{grading compatible} \\ &[x, y] = -(-1)^{|x||y|}[y, x] && \textit{super anti-commutativity} \\ &[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] && \textit{super Jacobi identity} \end{aligned} \tag{7}$$

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If $\mathfrak{g}$ is $\mathbb{Z}$-graded, we say that the grading is consistent if

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Let $\epsilon \in \mathbb{Z}_2$. An automorphism of $\mathfrak{g}$ such that $\sigma^2(x) = (-1)^{\epsilon|x|}x$ is called an ϵ-involution.

TKK for N=6

$$\left\{ \begin{array}{l} \text{simple Lie superalgebras } \mathfrak{g} \\ \text{consistent 3-grading} \\ \text{grade-reversing } \bar{1}\text{-involution } \sigma \end{array} \right\} \longleftrightarrow \quad \left\{ \text{ simple N=6 } \right\}$$

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Theorem (N. Cantarini, V. Kac-Transformation groups, 2011)

The classification of simple N=6 3-algebras consists of

- *3 infinite families of finite-dimensional ones;*
- *1 infinite family and 5 exceptional infinite-dimensional linearly-compact ones.*

ϵ-super Jordan triple system

Definition (R.)

An ϵ-super Jordan triple system (ϵ-sJTS) is a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with a grading-compatible trilinear product satisfying

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with $\alpha(x, y, z) = |x||y| + |x||z| + |y||z|$,
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The even part of an ϵ-sJTS is a JTS. If $\epsilon = \bar{1}$, the odd part is N=6.

ϵ-sJTS and related structures

$$\text{Jordan superalgebras} \hookrightarrow \quad \bar{0} - sJTS \quad \begin{array}{cr} \swarrow & JTS(even) \\ \\ \nwarrow & anti - JTS(odd) \end{array}$$

$$\text{Jordan superalgebras} \hookrightarrow \quad \bar{1} - sJTS \quad \begin{array}{cr} \swarrow & JTS(even) \\ \\ \nwarrow & N = 6(odd) \end{array}$$

Main examples of ϵ-sJTS

Example

Let A be the associative superalgebra of supermatrices, $A = M_{(m|n)}(\mathbb{C}) \cong End(\mathbb{C}^{m|n})$, and let τ denote the supertransposition,

$$X^\tau = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)^\tau = \left(\begin{array}{c|c} a^t & -c^t \\ \hline b^t & d^t \end{array}\right)$$

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Remark: If we fix $Y = Id$ we obtain the Jordan superalgebra with product

$$(XZ) = XZ + (-1)^{|X||Z|} ZX$$

sJTS of rectangular supermatrices

Example

Let V be the super vector space of rectangular supermatrices, $V = M_{(m_1|m_2,n_1|n_2)}(\mathbb{C}) \cong Hom(\mathbb{C}^{m_1|m_2}, \mathbb{C}^{n_1|n_2})$ and let ϕ be one of the following maps from V to $V' = M_{(n_1|n_2,m_1|m_2)}(\mathbb{C})$:

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$$\epsilon = 0: \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{os\tau} = \begin{pmatrix} a^t & -c^t J \\ Jb^t & -Jd^t J \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\Pi\tau} = \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}$$

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Then, $sJ(V, \phi) := (V, (\cdot,\cdot,\cdot)_\phi)$, with triple product

$$(XYZ)_\phi = X\phi(Y)Z + (-1)^{\alpha(X,Y,Z)} Z\phi(Y)X$$

is an ϵ-sJTS.

TKK for ϵ-sJTS

Theorem (R.)

$$\left\{ \begin{array}{l} \textit{simple Lie superalgebra } \mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \\ \textit{3-graded } \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}_{1} \\ \textit{grade-reversing } \epsilon\textit{-involution} \end{array} \right\} \longleftrightarrow \left\{ \textit{ simple } \epsilon\textit{-sJTS } \right\}$$

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- If $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$ we get the TKK for JTS.

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- If $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$ we get the TKK for JTS.
- If $\epsilon = \bar{1}$ and $\mathfrak{g}$ has a consistent grading we have the TKK for N=6.

Classification of ϵ-sJTS

Through the TKK we can classify simple finite-dimensional and infinite-dimensional linearly-compact ϵ-sJTS by classifying the 3-gradings of simple Lie superalgebras and their grade-reversing ϵ-involution.

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Remark: If $\mathfrak{g}$ satisfies the requirements for the TKK then $\mathfrak{g}_{\bar{0}}$ is a semisimple 3-graded Lie algebra and $\sigma_{|\mathfrak{g}_{\bar{0}}}$ is a grade-reversing involution of $\mathfrak{g}_{\bar{0}}$.

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Thus, using the theory developed for KTS, we can classify the grade-reversing ϵ-involution by their action on $\mathfrak{g}_{\bar{0}}$ and then extend it to the whole Lie superalgebra. When $\mathfrak{g}_{\bar{1}}$ is an irreducible $\mathfrak{g}_{\bar{0}}$-module this extension is unique. Note that this is not a trivial matter when dealing with the exceptional cases.

Classification of finite-dimensional ϵ-sJTS

Theorem (R.)

The list of simple finite-dimensional ϵ-sJTS over $\mathbb{C}$, up to isomorphism, consists of 21 infinite families and 4 exceptional cases.

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sJTS with Lie superalgebra $F(4)$

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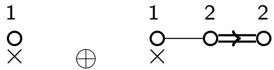
$\mathfrak{g}_{\bar{0}} \cong$ 

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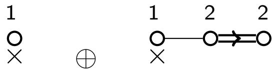

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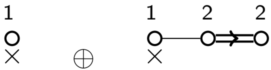
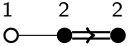
$\mathfrak{g}_{\bar{0}} \cong$ 

The real forms of $\mathfrak{so}(7,\mathbb{C})$ are: $\mathfrak{so}(1,6)$ 

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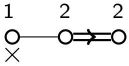
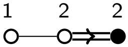
$\oplus$ 

The real forms of $\mathfrak{so}(7,\mathbb{C})$ are: $\mathfrak{so}(2,5)$ 

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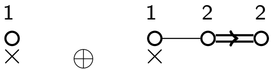
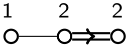
$\mathfrak{g}_{\bar{0}} \cong$ 

The real forms of $\mathfrak{so}(7,\mathbb{C})$ are: $\mathfrak{so}(3,4)$ 

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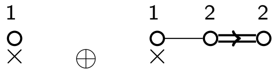
$\mathfrak{g}_{\bar{0}} \cong$ 

The real forms of $\mathfrak{so}(7,\mathbb{C})$ are: $\mathfrak{so}(p, 7-p)$

Since $\mathfrak{g}_{\bar{1}}$ is irreducible we get 3 non-isomorphic ϵ-sJTS.

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We write the generic element of $\mathfrak{g}_{-1}$ in vector form $v=(a_v,u_v,s_v)$, with $a_v\in\mathbb{C}, u_v\in U, s_v\in\mathbb{S}_5$.

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$$(xyz) = \ -\begin{pmatrix} 2a_x a_y a_z \\ (u_x, r_W(u_y))u_z + (u_z, r_W(u_y))u_x - r_W(u_y)(u_x, u_z) \\ \frac{4}{3}(\Psi_B(s_x, (\mathrm{vol}_p \circ s_y)) \cdot s_z - \frac{1}{2}(s_x, (\mathrm{vol}_p \circ s_y))_{\mathbb{S}}\, s_z) \end{pmatrix} + r \tag{8}$$

Theorem

$V = \mathbb{C} \oplus U \oplus \mathbb{S}_5$ with triple product (8) is a simple $\bar{0}$-sJTS if $p = 1, 2$ and $\bar{1}$-sJTS if $p = 3$ with associated Lie superalgebra $F(4)$.

Infinite-dimensional linearly-compact Lie superalgebras

Example

Let $m, n > 0$, $x_1, \ldots, x_m$ be commuting variables, $\xi_1, \ldots, \xi_n$ anticommuting variables and set $|x_i| = \bar{0}, |\xi| = \bar{1}$.

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Let $\Lambda(m, n)$ denote the associative superalgebra $\mathbb{C}[[x_1, \ldots, x_m]] \otimes \Lambda(\xi_1, \ldots, \xi_n)$ and let $W(m, n) = Der(\Lambda(m, n))$.

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$W(m, n)$ is infinite-dimensional linearly-compact, simple and it consists of linear operators of the form:

$$\sum_i P_i \frac{\partial}{\partial x_i} + \sum_j Q_j \frac{\partial}{\partial \xi_j}, \quad P_i, Q_j \in \Lambda(m, n)$$

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A $\mathbb{Z}$-grading of $W(m, n)$ is defined by an $(m+n)$-tuple of integers $(a_1, \dots, a_m | b_1, \dots, b_n)$ and setting

$$\deg(x_i) = a_i = -deg(\tfrac{\partial}{\partial x_i}), \quad \deg(\xi_j) = b_j = -deg(\tfrac{\partial}{\partial \xi_i})$$

Classification of infinite-dimensional ϵ-sJTS

Theorem (N. Cantarini, V. G. Kac)

A simple 3-graded infinite-dimensional linearly-compact Lie superalgebras is isomorphic to one of the following

- $H(2m|n): \ (0,\dots,0|1,0,\dots,0,-1)$
- $K(2m+1|n): \ (0,\dots,0|1,0,\dots,0,-1)$
- $E(1|6): \ (0|1,0,\dots,0,-1)$
- $S(1|2): \ (0|1,0)$
- $S(1|2): \ (0|1,1)$
- $SHO(3|3): \ (0,0,0|1,1,1)$
- $SKO(2|3;1): \ (0,0|1,1,1)$

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The 3-gradings are induced by $\mathbb{Z}$-gradings of $W(m,n)$, since the simple Lie superalgebras are all embedded in $W(m,n)$.

ϵ-sJTScon Lie superalgebra $H(2m|n+2)$

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Let $P(2m, n)$ denote the Poisson superalgebra, i.e. $\Lambda(2m, n)$ with the Poisson bracket $\{,\}$:

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Then $P(2m, n)$ with triple product:

$$(fgh)_\phi = \{f, \phi(g)\}h + (-1)^{\alpha(f,g,h)}\{h, \phi(g)\}f - (-1)^{|f||g|}\{\phi(g), f\}h$$

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Remark: Different ϕ with different spectra give rise to non-isomorphic ϵ-sJTS.

Classification of infinite-dimensional ϵ-sJTS

Conjecture (R.)

The list of simple infinite-dimensional linearly-compact ϵ-sJTS over $\mathbb{C}$, up to isomorphism, consists of 4 infinite families and 10 exceptional cases.

Future developments

Proposition

Let $(\mathfrak{g}, \sigma)$ be a pair consisting of a $(2k+1)$-graded Lie superalgebra $\mathfrak{g}$ and a grade-reversing ϵ-involution σ. Then $\mathfrak{g}_{-1}$ with triple product

$$(xyz) = [[x, \sigma(y)], z] \tag{9}$$

satisfies the principal identity.

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- For $k = 2$ we obtain a super-version of Kantor triple systems, with Kantor tensor $K_{xz}(y) = (xyz) - (-1)^{\alpha(xyz)}(zyx)$ (new realizations of $G(3)$).
- Moreover, this construction gives an infinite series of supersymmetric structures, which are the super analogue of *generalized Jordan triple systems*.

The End

Antonio Ricciardo
University of Bologna

sJTS with Lie superalgebra $F(4)$

The Lie superalgebra $\mathfrak{g} = F(4)$ is defined by

$$\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(7, \mathbb{C}) \,, \quad \mathfrak{g}_{\bar{1}} \cong V \otimes Spin_7 \,,$$

with $V \cong \mathbb{C}^2$, resp. $Spin_7$, the standard representation of $\mathfrak{sl}(2, \mathbb{C})$, resp. the spin representation of $\mathfrak{so}(7, \mathbb{C})$.

The bracket between even elements is the Lie bracket of $\mathfrak{g}_{\bar{0}}$, while the bracket between even elements and odd elements is given by the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ as a representation.

Let $(,)_J$ and $(,)_{\mathbb{S}}$ denote, respectively, the standard symplectic form on $\mathbb{C}^2$ and the invariant symmetric form on $Spin_7$. We let $\mathcal{B}$ be a basis of $\mathfrak{so}(7, \mathbb{C})$ and define the following maps:

$$\begin{array}{ll} \Psi_J : S^2 V \to sl(2, \mathbb{C}) : & \Psi_J(v_1, v_2)(v) = (v_2, v)_J v_1 - (v, v_1)_J v_2 \,, v \in \mathbb{C}^2 \\ \Psi_{\mathbb{S}} : \Lambda^2 Spin_7 \to so(7, \mathbb{C}) : & \Psi_{\mathbb{S}}(s_1, s_2) = \sum_{x \in \mathcal{B}} (s_1, -x^t \cdot s_2)_{\mathbb{S}} x \,. \end{array}$$

The bracket between odd elements $v_i \otimes s_i \in \mathfrak{g}_{\bar{1}}$, $i = 1, 2$, is defined by

$$[v_1 \otimes s_1, v_2 \otimes s_2] = -\frac{4}{3}(v_1, v_2)_J \Psi_{\mathbb{S}}(s_1, s_2) + (s_1, s_2)_{\mathbb{S}} \Psi_J(v_1, v_2) \,.$$

Jordan algebras

Definition

A *Jordan algebra* is a vector space J with a symmetric commutative product, $(,)$, which satisfies the Jordan identity

$$(xy)x^2 = x(yx^2)$$

Remark: The Jordan identity and commutativity imply power associativity (just take $x = y$ in the Jordan identity).

Example

The basic example of Jordan algebra consists of $M_n(\mathbb{C})$, the space of complex matrices of order n, with product

$$(xy) = xy + yx$$

TKK detail

Let V be a centerless KTS and define the operators, for $x, y, z \in V$:

$$L_{x,y}(z) := (xyz)\,, \quad \varphi_x(y) := L_{y,x}\,, \quad D_{x,y}(z) := -\varphi_{K_{x,y}(z)}\,.$$

$$\begin{array}{ccccccccccc} \mathfrak{g} = \mathfrak{g}(V) & = & \langle K_{x,y} \rangle & \oplus & V & \oplus & \langle L_{x,y} \rangle & \oplus & \langle \varphi_x \rangle & \oplus & \langle D_{x,y} \rangle \\ & & -2 & & -1 & & 0 & & 1 & & 2 \end{array}$$

The Lie bracket is defined by

$$[x, y] := K_{x,y}\,, \qquad [A, x] := A(x)$$

for A either $L_{x,y}$, ϕ_x or $D_{x,y}$ and extended using

- transitivity (if $A \in \mathfrak{g}_i$, $i \geq 0$ and $[A, x] = 0$ $\forall x \in \mathfrak{g}_{-1}$ then $A = 0$)
- Jacobi identity, since $\mathfrak{g}$ is fundamental ($\mathfrak{g}_{-1}$ generates $\mathfrak{g}_{-2}$).

TKK for ϵ-sJTS in detail

Let V be a ϵ-sJTS and let us define the following operators of V

$$L_{x,y}(z) = (xyz)\,,\ \phi_y(x,z) = -(-1)^{|y||x|}(xyz)\,.$$

We let $Lie(V)$ be the 3-graded vector space

$$Lie(V)_{-1} = V,\ Lie(J)_0 = \langle L_{x,y}\rangle,\ Lie(J)_1 = \langle \phi_z \rangle$$

and define on $Lie(V)$ the following Lie superalgebra bracket:

$$[x,y] = 0\,, [L_{x,y}, z] = (xyz)\,, [\phi_x, y] = -(-1)^{|x||y|}L_{y,x}\,,$$
$$[L_{u,v}, L_{x,y}] = L_{(uvx),y} - (-1)^{\beta(u,v,x,y)+\epsilon|u|}L_{x,(yuv)}\,,$$
$$[L_{x,y}, \varphi_z] = -(-1)^{|x||y|+\epsilon|x|}\varphi_{(yxz)}\,,\ [\varphi_x, \varphi_y] = 0\,.$$

TKK for ϵ-sJTS in detail

With this definition $Lie(V)$ is a 3-graded transitive Lie superalgebra which is simple if and only if V is simple.

$Lie(V)$ is canonically endowed with a grade-reversing ϵ-involution:

$$\sigma(x) = \phi_x\,,\ \sigma(L_{x,y}) = -(-1)^{|x||y|+\epsilon|x|} L_{y,x}\,,\ \sigma(\phi_x) = (-1)^{\epsilon|x|} x\,.$$

and the triple product can be recovered by

$$[[x, \sigma(y)], z] = [[x, \phi_y], z] = [L_{x,y}, z] = (xyz)\,.$$

This construction holds for linearly-compact spaces in general, allowing us to deal with infinite-dimensional linearly-compact spaces.

sJTS with Lie superalgebra $F(4)$

- $C\ell(U)$ denote the Clifford algebra of U which acts on $\mathbb{S}_5$ by $\circ$.
- $(,)_{\mathbb{S}}$ denote an invariant bilinear form on $\mathbb{S}_5$, i.e. $(u \circ s, t)_{\mathbb{S}} = (s, u \circ t)_{\mathbb{S}}$;
- Ψ_B denote the bilinear map from $\mathbb{S}_5 \otimes \mathbb{S}_5 \to \mathfrak{so}(U)$ satisfying $\eta(\Psi_B(s,t)u, v) = (u \wedge v \circ s, t)_{\mathbb{S}}$;
- Ψ_U denote the bilinear map from $\mathbb{S}_5 \otimes \mathbb{S}_5 \to U$ satisfying $\eta(\Psi_U(s,t), u) = (u \circ s, t)_{\mathbb{S}}$.

$$(xyz) = \begin{pmatrix} -2(a_x a_y a_z + (a_x s_z - a_z s_x, (\mathrm{vol}_p \circ s_y))_B) + \\ + \sqrt{2}(s_z, (r_W(u_y)) \circ (s_x))_B \,, \\ -(u_x, r_W(u_y))u_z - (u_z, r_W(u_y))u_x + (u_x, u_z) r_W(u_y) + \\ + \frac{4}{3} a_y \Psi_U(s_x, s_z) + \\ + \frac{2\sqrt{2}}{3}(\Psi_U(s_x, u_z \circ (\mathrm{vol}_p \circ s_y)) - \Psi_U(s_z, u_x \circ (\mathrm{vol}_p \circ s_y)))\,, \\ - a_y(a_x s_z + a_z s_x) - \frac{1}{\sqrt{2}}(a_z u_x + a_x u_z) \circ (\mathrm{vol}_p \circ s_y) - \\ - \frac{1}{2}((u_x(r_W(u_y))) \circ s_z + (u_z(r_W(u_y))) \circ s_x) - \\ - \frac{4}{3}\Psi_B(s_x, (\mathrm{vol}_p \circ s_y)) \cdot s_z + \frac{2}{3}(s_x, (\mathrm{vol}_p \circ s_y))_B s_z \end{pmatrix}$$

ϵ-sJTSwith Lie superalgebra $H(2m|n+2)$

Example

Let $\mathfrak{g} = H(2m|n+2)$ denote the Lie superalgebra of vector fields of $\mathbb{C}^{2m|n+2}$ which annihilate a symplectic-orthogonal form.
We identify $\mathfrak{g}$ with $P(2m,n) \cong (\Lambda(2m,n+2),\{,\})$ by sending

$$\Lambda(2m,n+2) \ni f \to H_f = \sum_{i=1}^{m}\left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{2m+1-i}} - \frac{\partial f}{\partial x_{2m+1-i}}\frac{\partial}{\partial x_i}\right) - -(-1)^{|f|}\sum_j \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_{n+1-j}}$$

with the product given by the Poisson bracket

$$\{f,g\} = \sum_{k=1}^{m}\left(\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_{2m+1-k}} - \frac{\partial f}{\partial x_{2m+1-k}}\frac{\partial g}{\partial x_k}\right) - -(-1)^{|f|}\sum_{k=1}^{n+2}\frac{\partial f}{\partial \xi_k}\frac{\partial g}{\partial \xi_{n+1-k}}$$

Let $\mathfrak{g}$ be endowed with the 3-grading $(0,\ldots,0|1,0,\ldots,0,-1)$. Then all elements of the form $\xi_0 f$, $\forall f \in \Lambda(2m,n)$ are in $\mathfrak{g}_{-1}$.

ϵ-sJTScon Lie superalgebra $H(2m|n+2)$

Example

Let $P(2m, n)$ denote the Poisson superalgebra, i.e. $\Lambda(2m, n)$ with the Poisson bracket, and let $\mathfrak{g} = H(2m, n+2)$. The 3-grading of $\mathfrak{g}$ is given by

$$\begin{aligned} \mathfrak{g}_{-1} &\cong &< \xi_0 > \otimes P(2m, n) \\ \mathfrak{g}_0 &\cong &< 1, \xi_0\xi_{n+1} > \otimes P(2m, n) \\ \mathfrak{g}_1 &\cong &< \xi_{n+1} > \otimes P(2m, n) \end{aligned}$$

The TKK for ϵ-sJTS associates to $\mathfrak{g}$ the Poisson superalgebra $P(2m, n)$ with triple product:

$$(fgh)_\phi = \{f, \phi(g)\}h + (-1)^{\alpha(f,g,h)}\{h, \phi(g)\}f - (-1)^{|f||g|}\{\phi(g), f\}h$$

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