Leibniz A-algebras

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Introduction

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Every non-abelian nilpotent Lie algebra admits a non-trivial solution of the constant Yang-Mills equations.

Moreover, if a subalgebra of L admits a non-trivial solution of the Yang-Mills equations, then so does L.

It is, therefore, useful to know if a given non-nilpotent Lie algebra has a non-abelian nilpotent subalgebra.

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Residually finite varieties

Let F be a finite field of characteristic $p > 3$. All algebras of the variety B of Lie algebras over F are residually finite if and only if B is generated by one finite Lie A-algebra. (Premet and Semenov, 1988)

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Definition

The Leibniz kernel of L is the set Leib(L)=span$\{x^2 : x \in L\}$. This is a two-sided ideal of L which is the smallest ideal such that L/Leib(L) is a Lie algebra. Also [L,Leib(L)] $= 0$.

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$$L^1 = L, L^{k+1} = [L^k, L] \text{ and } L^{(0)} = L, L^{(k+1)} = [L^{(k)}, L^{(k)}] \text{ for all } k = 1, 2, 3, \ldots$$

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Then L is nilpotent (resp. solvable) if $L^n = 0$ (resp.$L^{(n)} = 0$) for some $n \in \mathbb{N}$. If $L^{(n)} = 0$ but $L^{(n-1)} \neq 0$ we say that L has derived length n. The nilradical, $N(L)$, (resp. radical, $R(L)$) is the largest nilpotent (resp. solvable) ideal of L.

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Example

Let L be the two-dimensional cyclic Leibniz algebra with basis a, a^2 and product $[a^2, a] = a^2$.Then this is a solvable Leibniz algebra which is not nilpotent, but which is an A-algebra.

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Lemma

Let A be an abelian ideal of a Leibniz algebra L and suppose that $x^2 \in A$. Then $L_x^n(A) \subseteq R_x^{n-1}(A)$ for all $n \geqslant 1$.

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Lemma

Let L be a Leibniz A-algebra and let N be its nilradical. Then

(i) *N is the unique maximal abelian ideal of L;*

(ii) *if B and C are abelian ideals of L, we have $[B, C] = 0$.*

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Lemma

If L is a Leibniz A-algebra over any field and B is an ideal of L, then L/B is a Leibniz A-algebra.

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Let B, C be ideals of the Leibniz algebra L.

(i) *If L/B, L/C are A-algebras, then $L/(B \cap C)$ is an A-algebra.*

(ii) *If $L = B \oplus C$, where B, C are A-algebras, then L is an A-algebra.*

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The nilpotent residual, $\gamma_\infty(L)$, of L be the smallest ideal of L such that $L/\gamma_\infty(L)$ is nilpotent. Clearly this is the intersection of the terms of the lower central series for L. Then the lower nilpotent series for L is the sequence of ideals $N_i(L)$ of L defined by $N_0(L) = L$, $N_{i+1}(L) = \gamma_\infty(N_i(L))$ for $i \geqslant 0$.

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Lemma

Let L be a Leibniz A-algebra. Then the lower nilpotent series coincides with the derived series.

The non-solvable case

The non-solvable case

Theorem

Let L be a Leibniz A-algebra over a field F. If F has characteristic $\neq 2, 3$ and cohomological dimension $\leqslant 1$ (this means that the Brauer group of any algebraic extension of the underlying field is trivial), then

(i) $L^2 \cap Z(L) = 0$; *and*

(ii) *L has a Levi decomposition and every Levi subalgebra is representable as a direct sum of simple ideals, each one of which splits over some finite extension of the ground field into a direct sum of ideals isomorphic to $sl(2)$.*

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Lemma

Let L be a Leibniz algebra over a field of characteristic different from 2 such that $L/Z(L)$ is a simple three-dimensional Lie algebra. Then $L = L^2 \dot{+} Z(L)$.

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Lemma

Let L be a Leibniz algebra over a field of characteristic different from 2 such that $L/Z(L)$ is a simple three-dimensional Lie algebra. Then $L = L^2 \dot{+} Z(L)$.

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We say that L is monolithic with monolith W if W is the unique minimal ideal of L.

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Now there is a finite extension K of F over which $(L/Z(L))_K$ splits as a direct sum of ideals $S_1/Z(L) \oplus \ldots \oplus S_n/Z(L)$ isomorphic to $sl(2)$, by the corresponding result for Lie algebras.

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Now there is a finite extension K of F over which $(L/Z(L))_K$ splits as a direct sum of ideals $S_1/Z(L) \oplus \ldots \oplus S_n/Z(L)$ isomorphic to $sl(2)$, by the corresponding result for Lie algebras.Using our Lemma about splitting over $Z(L)$ we then get the claimed result.

Solvable Leibniz A-algebras

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Theorem

Let L be a solvable Leibniz A-algebra. Then L splits over each term in its derived series. Moreover, the Cartan subalgebras of $L^{(i)}/L^{(i+2)}$ are precisely the subalgebras that are complementary to $L^{(i+1)}/L^{(i+2)}$ for $i \geqslant 0$.

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Corollary

Let L be a solvable Leibniz A-algebra of derived length $n+1$. Then

(i) $L = A_n \dot{+} A_{n-1} \dot{+} \dots \dot{+} A_0$ *where A_i is an abelian subalgebra of L for each $0 \leqslant i \leqslant n$; and*

(ii) $L^{(i)} = A_n \dot{+} A_{n-1} \dot{+} \dots \dot{+} A_i$ *for each $0 \leqslant i \leqslant n$*

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Proof.

Let L be a minimal counter-example. Then, again, a straightforward argument shows that L is monolithic with monolith $Z(L)$ and has a unique maximal ideal M that is abelian. It follows that $L = M \dot{+} Fx$ for some $x \in L$. Let $L = L_0 \dot{+} L_1$ be the Fitting decomposition of L relative to R_x. Then $L_1 = \cap_{i=1}^{\infty} R_x^i(L) \subseteq M$, and $[L_1, L_0] \subseteq L_1$, so L_1 is a right ideal of L. Now

$$[x, [L, x]] \subseteq [[x, L], x] + [x^2, L] \subseteq [L, x] + [x^2, M + Fx] \subseteq [L, x]$$

since $x^2 \in \text{Leib}(L) \subseteq M$, so $[x^2, M] = 0$. Then an induction argument shows that $[x, R_x^k(L)] \subseteq R_x^k(L)$. It follows that $[L, L_1] = [x, L_1] \subseteq L_1$ and L_1 is an ideal of L.
If $L_1 \neq 0$ then $Z(L) \subseteq L_1 \cap L_0 = 0$, a contradiction. Hence $L_1 = 0$ and R_x is nilpotent.
But then $L = M + Fx$ is nilpotent and hence abelian, and the result follows. □

Solvable Leibniz A-algebras

Lemma

Let L be a solvable Leibniz A-algebra of derived length $\leqslant n+1$, and suppose that $L = B \dot{+} C$ where $B = L^{(n)}$ and C is a subalgebra of L. If D is an ideal of L then $D = (B \cap D) \dot{+} (C \cap D)$.

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Theorem

Let L be a solvable Leibniz A-algebra of derived length $n+1$ with nilradical N, and let K be an ideal of L and A a minimal ideal of L. Then, with the same notation as in the Corollary above,

(i) $K = (K \cap A_n) \dot{+} (K \cap A_{n-1}) \dot{+} \dots \dot{+} (K \cap A_0)$;

(ii) $N = A_n \oplus (N \cap A_{n-1}) \oplus \dots \oplus (N \cap A_0)$;

(iii) $Z(L^{(i)}) = N \cap A_i$ *for each* $0 \leqslant i \leqslant n$; *and*

(iv) $A \subseteq N \cap A_i$ *for some* $0 \leqslant i \leqslant n$.

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In particular, over a field of characteristic zero every solvable Leibniz A-algebra is metabelian.

This is not the case, however, if the ground field has characteristic $p > 0$.

Completely solvable Leibniz A-algebras

General case

Completely solvable case

Completely solvable Leibniz A-algebras

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Completely solvable Leibniz A-algebras (continued)

Lemma

Let L be a completely solvable Leibniz A-algebra, and let U be a maximal nilpotent subalgebra of L. Then $U = (U \cap L^{(1)}) \oplus (U \cap C)$ where C is a Cartan subalgebra of L.

Completely solvable Leibniz A-algebras (continued)

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Lemma

Let L be a metabelian Leibniz algebra, and let U be a maximal nilpotent subalgebra of L. Then $U \cap L^{(1)}$ is an abelian ideal of L and $L^{(1)} = (U \cap L^{(1)}) \oplus K$ where K is an ideal of L and $[U, K] = K$.

Monolithic solvable Leibniz A-algebras

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Definition

The Frattini subalgebra, $F(L)$ of L is the intersection of the maximal subalgebras of L; the Frattini ideal, $\varphi(L)$, of L is the biggest ideal of L inside $F(L)$. We call L φ-free if $\varphi(L) = 0$.

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Theorem

Let L be a monolithic solvable Leibniz A-algebra of derived length $n+1$ with monolith W. Then, with the same notation as in the Corollary above,

(i) *W is abelian;*

(ii) *$Z(L) = 0$ and either $[L, W] = W$ or $[W, L] = W$;*

(iii) *$N = A_n = L^{(n)}$;*

(iv) *$N = C_L(W)$; and*

(v) *L is φ-free if and only if $W = N$.*

Monolithic solvable Leibniz A-algebras

Lemma

Let $L = L^2 \dot{+} B$ be a metabelian Leibniz algebra, where B is a subalgebra of L, and suppose that $[L^2, b] = L^2$ for all $b \in B$. Then L is a completely solvable A-algebra.

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Theorem

Let L be a monolithic Leibniz algebra. Then L is a completely solvable A-algebra if and only if $L = L^2 \dot{+} B$ is metabelian, where B is a subalgebra of L and $[L^2, b] = L^2$ for all $b \in B$ (or, equivalently, R_b acts invertibly on L^2).

Cyclic Leibniz algebras

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Definition

Cyclic Leibniz algebras, L, are generated by a single element. In this case L has a basis $a, a^2, \ldots, a^n (n > 1)$ and product $[a^n, a] = \alpha_2 a^2 + \ldots + \alpha_n a^n$.

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Let T be the matrix for R_a with respect to the above basis. Then T is the companion matrix for $p(x) = x^n - \alpha_n x^{n-1} - \ldots - \alpha_2 x = p_1(x)^{n_1} \ldots p_r(x)^{n_r}$, where the p_j are the distinct irreducible factors of $p(x)$.

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Theorem

L is a cyclic Leibniz A-algebra if and only if $\alpha_2 \neq 0$, and then $L = L^2 \dot{+} F(a^n - \alpha_n a^{n-1} - \cdots - \alpha_2 a)$ and we can take $p_1(x)^{n_1} = x$.

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Theorem

The cyclic Leibniz A-algebra L is monolithic if and only if $p(x)$ has exactly two irreducible factors (one of which is x).

Cyclic Leibniz algebras

Corollary

The cyclic Leibniz A-algebra L is monolithic and φ-free if and only if $p(x) = xp_2(x)$

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Corollary

If the underlying field is algebraically closed, then the cyclic Leibniz A-algebra L is monolithic and φ-free if and only if it is two dimensional with $[a^2, a] = a^2$.

Solvable Leibniz A-algebras over an algebraically closed field

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Over a field of characteristic zero the derived length of a solvable Leibniz A-algebra is at most 2, but over a field of characteristic p it can have any finite length. However, over an algebraically closed field we have the following result.

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Theorem

Let L be a solvable Leibniz A-algebra over an algebraically closed field F. Then the derived length of L is at most 3.

Solvable Leibniz A-algebras over an algebraically closed field

Proof.

Suppose that L is a minimal counter-example, so the derived length of L is four.

Solvable Leibniz A-algebras over an algebraically closed field

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Solvable Leibniz A-algebras over an algebraically closed field

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Suppose that L is a minimal counter-example, so the derived length of L is four. Let A be a minimal ideal of L contained in Leib(L), and put $N = L^{(2)}$. We have that $L^{(3)} = A$. Put $\bar{L} = L/\text{Leib}(L)$ and for each $x \in L$ write $\bar{x} = x + \text{Leib}(L)$. Then A is an irreducible right $\bar{L}$-module, and hence an irreducible right U-module, where U is the universal enveloping algebra of $\bar{L}$.

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Let $n_1, n_2 \in N$. Then $\bar{n}_1^p, \bar{n}_2^p \in Z$, so $\alpha_1 \bar{n}_1^p + \alpha_2 \bar{n}_2^p \in \ker(\psi)$, for some $\alpha_1, \alpha_2 \in F$, since dim $\psi(Z) \leqslant 1$, by Schur's Lemma.

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Suppose that L is a minimal counter-example, so the derived length of L is four. Let A be a minimal ideal of L contained in $\text{Leib}(L)$, and put $N = L^{(2)}$. We have that $L^{(3)} = A$. Put $\bar{L} = L/\text{Leib}(L)$ and for each $x \in L$ write $\bar{x} = x + \text{Leib}(L)$. Then A is an irreducible right $\bar{L}$-module, and hence an irreducible right U-module, where U is the universal enveloping algebra of $\bar{L}$. Let ψ be the corresponding representation of U and let $\bar{x} \in \bar{L}$, $n \in N$. Then $[[\bar{x}, \bar{n}], \bar{n}] = \bar{0}$, whence $[\bar{x}, \bar{n}^p] = 0$ and so $\bar{n}^p \in Z = Z(U)$.

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□

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Now we can include $L^{(3)}$ in a chief series for L. So let $0 = A_0 \subset A_1 \subset \ldots \subset A_r = L^{(3)}$ be a chain of ideals of L each maximal in the next.

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Questions

Questions

Residually finite varieties

Is the following true: "Let F be a finite field of characteristic $p > 3$. All algebras of the variety B of Leibniz algebras over F are residually finite if and only if B is generated by one finite Leibniz A-algebra"?

Questions

Residually finite varieties

Is the following true: “Let F be a finite field of characteristic $p > 3$. All algebras of the variety B of Leibniz algebras over F are residually finite if and only if B is generated by one finite Leibniz A-algebra"?

Question

Are there analogues of the following results for Lie A-algebras?

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Question

Are there analogues of the following results for Lie A-algebras?

Theorem

Let L be a monolithic solvable Lie A-algebra of dimension greater than one over an algebraically closed field F, with monolith W. Then either

- $L = L^2 \dot{+} Fb$ *where* L^2 *is abelian and* $L^2(\operatorname{ad} b - \lambda 1)^k = 0$ *for some* $k > 0$ *and some* $0 \neq \lambda \in F$, *and* $\dim W = 1$; *or*
- F *has characteristic* $p > 0$, $\dim W = p$ *and* $L = L^{(2)} \dot{+} B$ *where* $L^{(2)}$ *is abelian,* $B = Fb + Fn$, $[n, b] = n$, $L^{(2)}(\operatorname{ad} n - \lambda 1)^k = 0$ *and* $L^{(2)}((\operatorname{ad} b)^p - \operatorname{ad} b - \mu^p 1)^k = 0$ *for some* $k > 0$ *and some* $0 \neq \lambda, \mu \in F$.

Questions

Theorem

Let L be a φ-free completely solvable Lie A-algebra over an algebraically closed field F. Then

$$L = \sum_{i=1}^{m} Fa_i + \sum_{i=1}^{n} Fb_i \text{ where } [a_i, b_j] = \lambda_{ij} a_i$$

for all $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$, other products being zero.