Post-Lie algebra structures on classes of nilpotent Lie algebras

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Post-Lie algebra structures

Definition (Vallette 2007, Burde–Dekimpe–Vercammen 2010)

Let $(\mathfrak{g}, [\cdot,\cdot]), (\mathfrak{n}, \{\cdot,\cdot\})$ be two Lie algebras on a k-vector space V. A post-Lie algebra structure (PA-structure) on the pair $(\mathfrak{g}, \mathfrak{n})$ is a k-bilinear map $V \times V \to V, (x, y) \mapsto x \cdot y$ such that

$$x \cdot y - y \cdot x = [x, y] - \{x, y\}$$
$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$$
$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$$

for all $x, y, z \in V$.

All Lie algebras will be finite-dimensional and over $k = \mathbb{C}$.

Special cases

(i) If $\{\mathfrak{n}, \mathfrak{n}\} = 0$ ($\mathfrak{n}$ is an abelian Lie algebra), a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ is a pre-Lie algebra structure on $\mathfrak{g}$.

(ii) If $[\mathfrak{g}, \mathfrak{g}] = 0$ ($\mathfrak{g}$ is an abelian Lie algebra), a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ is an LR-structure on $\mathfrak{n}$.

(iii) If $\mathfrak{g} = \mathfrak{n}$ (i.e. $[x, y] = \{x, y\}$ for all $x, y \in V$), then by the first identity, $x \cdot y = y \cdot x$. So we call those post-Lie algebra structures commutative post-Lie algebra structures or CPA-structures on $\mathfrak{g}$.

Commutative post-Lie algebra structures

Definition (Burde–Dekimpe 2016)

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a k-Lie algebra. A commutative post-Lie algebra structure (CPA-structure) on $\mathfrak{g}$ is a k-bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (x, y) \mapsto x \cdot y$ such that

$$x \cdot y = y \cdot x$$
$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$$
$$x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z]$$

for all $x, y, z \in \mathfrak{g}$.

Special CPA-structures

"Some" CPA-structure on a Lie algebra $\mathfrak{g}$ satisfy

$$\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$$

or the stronger condition

$$\mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g}),$$

with $[\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{[g, h] : g, h \in \mathfrak{g}\}$ the commutator algebra of $\mathfrak{g}$

and $Z(\mathfrak{g}) = \{g \in \mathfrak{g} : [g, h] = 0 \quad \forall h \in \mathfrak{g}\}$ the center of $\mathfrak{g}$.

Poisson algebras

A Poisson algebra is a vector space $(A, \cdot, \{\cdot, \cdot\})$ with a bilinear, associative and commutative map $\cdot \colon A \times A \to A$ and a Lie bracket $\{\cdot, \cdot\}$ such that

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$$

for all $x, y, z \in A$.

Poisson-admissible algebras

An algebra $(V, \cdot)$ is called Poisson-admissible (Goze–Remm 2008), if $(V, \circ, [\cdot, \cdot])$, given by

$$[x, y] := x \cdot y - y \cdot x,$$
$$x \circ y := \frac{1}{2}(x \cdot y + y \cdot x)$$

is a Poisson algebra.

Proposition (Goze–Remm, 2008)

There is a bijective correspondence between Poisson-admissible algebras and Poisson algebras.

CPA-structures and associative algebras

Lemma

Let $\mathfrak{g}$ be a Lie algebra and $x \cdot y$ a CPA-structure on $\mathfrak{g}$. Then:

$$\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0 \Leftrightarrow (\mathfrak{g}, \cdot) \text{ is an associative algebra.}$$

Proof: By commutativity and a property of PA-structures, we have

$$\begin{aligned}(x \cdot y) \cdot z - x \cdot (y \cdot z) &= z \cdot (x \cdot y) - x \cdot (y \cdot z)\\ &= x \cdot (z \cdot y) + [z, x] \cdot y - x \cdot (y \cdot z)\\ &= x \cdot (y \cdot z) - x \cdot (y \cdot z) + [z, x] \cdot y\\ &= y \cdot [z, x],\end{aligned}$$

meaning $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0 \Leftrightarrow (\mathfrak{g}, \cdot)$ is an associative algebra.

CPA-structures and Poisson(-admissible) algebras

Similarly, one can prove:

Lemma

Let $\mathfrak{g}$ be a Lie algebra and $x \cdot y$ a CPA-structure on $\mathfrak{g}$. Then:

$$\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0 \Leftrightarrow (\mathfrak{g}, \cdot) \text{ is a Poisson-admissible algebra}$$

Lemma

Let $\mathfrak{g}$ be a Lie algebra and $x \cdot y$ a CPA-structure on $\mathfrak{g}$. Then:

$$\mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g}) \Leftrightarrow (\mathfrak{g}, \cdot) \text{ is a Poisson-algebra}$$

LR-structures and associative/Poisson-admissible algebras

For LR-structures, one can prove a similar statement:

Lemma

Let $\mathfrak{g}$ be a Lie algebra and $x \cdot y$ an LR-structure on $\mathfrak{g}$. Then:

$$\begin{aligned} \mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g}) &\Leftrightarrow (\mathfrak{g}, \cdot) \text{ is an associative algebra} \\ &\Leftrightarrow (\mathfrak{g}, \cdot) \text{ is a Poisson-admissible algebra} \end{aligned}$$

Corollary

If $(\mathfrak{g}, \cdot)$ is an associative LR-structure, then $\mathfrak{g}$ is 2-step nilpotent. Moreover, $(\mathfrak{g}, \cdot)$ is complete (i.e., all right-multiplication operators are nilpotent).

CPA-structures

This suggests to study CPA-structures satisfying $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ and $\mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g})$.

We want to present some classification results in this direction.

Theorem (Burde–Dekimpe–Moens 2019)

All CPA-structures on the free-nilpotent Lie algebras $\mathfrak{g} = F_{3,3}, F_{2,c}, 3 \leq c \leq 10$ (and most likely also for most other $F_{n,c}$) satisfy $\mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g})$, thus are Poisson-algebras.

Nilpotent Lie algebras

Given a Lie algebra $\mathfrak{g}$, we call

$$\mathfrak{g}^0 := \mathfrak{g}, \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}^0], \mathfrak{g}^2 := [\mathfrak{g}, \mathfrak{g}^1], \ldots, \mathfrak{g}^k := [\mathfrak{g}, \mathfrak{g}^{k-1}]$$

its lower central series.

We say that $\mathfrak{g}$ is nilpotent, if there is a k with $\mathfrak{g}^k = 0$. The smallest such k is called the nilpotency class $c(\mathfrak{g})$ of $\mathfrak{g}$.

Nilpotent Lie algebras

Examples:

- $c(\mathfrak{g}) = 1$ (i.e. $[\mathfrak{g}, \mathfrak{g}] = 0$): $\mathfrak{g}$ is called abelian.
- $c(\mathfrak{g}) = 2$: $\mathfrak{g}$ is called two-step nilpotent.
- $c(\mathfrak{g}) = \dim(\mathfrak{g}) - 1$: $\mathfrak{g}$ is called filiform.
- The Lie algebra $\mathfrak{g}$ of strictly upper triangular matrices of size $n \times n$ is nilpotent of class $c(\mathfrak{g}) = n - 1$.

We want to understand/classify (commutative) post-Lie algebra structures (CPA-structures) on nilpotent Lie algebras.

2-step nilpotent Lie algebras

A Lie algebra $(\mathfrak{g}, \cdot)$ is called 2-step nilpotent, if $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$.

Example

The Heisenberg algebra of dimension $2k + 1$ has a basis $\{p_1, \ldots, p_k, q_1, \ldots, q_k, z\}$ and Lie brackets $[p_i, q_i] = z, i = 1, \ldots, 2k + 1$.

2-step nilpotent Lie algebras

Theorem (Burde–E.–Moens)

Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra.

Then there is a bijective correspondence between associative CPA-structures on $\mathfrak{g}$ and LR-structures on $\mathfrak{g}$ with $\mathfrak{g} \cdot Z(\mathfrak{g}) = 0$ (and also to certain pre-Lie algebra structures on $\mathfrak{g}$).

2-step nilpotent Lie algebras

Proposition

Let $\mathfrak{g}$ be a Heisenberg algebra of dimension $n \geq 5$.

Then all LR-structures on $\mathfrak{g}$ satisfy $\mathfrak{g} \cdot Z(\mathfrak{g}) = 0$ and all CPA-structures on $\mathfrak{g}$ are associative.

Corollary

Let $\mathfrak{g}$ be a Heisenberg algebra of dimension $n \geq 5$.

Then there is a bijective correspondence between LR-structures on $\mathfrak{g}$ and CPA-structures on $\mathfrak{g}$.

2-step nilpotent Lie algebras

Let $\mathfrak{g}$ be a Heisenberg algebra of dimension $n \geq 5$.
Then the correspondence is given as follows:

$$\begin{aligned} \{\text{LR-structures on } \mathfrak{g}\} &\longleftrightarrow \{\text{CPA-structures on } \mathfrak{g}\}, \\ x \cdot y &\longmapsto x \cdot y + \frac{1}{2}[x, y], \\ x \circ y - \frac{1}{2}[x, y] &\longleftarrow x \circ y. \end{aligned}$$

As CPA-structures on stem ($Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$) nilpotent Lie algebras are always complete (Burde–Dekimpe–Moens 2019), we obtain that all LR-structures on Heisenberg algebras of dimension ≥ 5 are complete.

2-step nilpotent Lie algebras

Classification of CPA-structures on the 3-dimensional Heisenberg algebra (Burde–E.–Moens)

With respect to the basis $\{p, q, z\}, [p, q] = z$, all CPA-structures on the 3-dimensional Heisenberg algebra are given by the three types

(i) $p \cdot q = q \cdot p = \frac{1}{2}z, q \cdot q = \alpha p + \beta z;$

(ii) $p \cdot p = \alpha p + \beta q + \gamma z,$
$p \cdot q = q \cdot p = -\alpha^2/\beta p - \alpha q + \delta z,$
$q \cdot q = \alpha^3/\beta^2 p + \alpha^2/\beta q + (2\alpha\beta\delta - \alpha^2\gamma)/\beta^2 z, \beta \neq 0;$

(iii) $p \cdot p = \alpha z, p \cdot q = q \cdot p = \beta z, q \cdot q = \gamma p + \delta z; \alpha\gamma = 0;$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ arbitrary.

Filiform Lie algebras

An n-dimensional Lie algebra $\mathfrak{g}$ is called filiform, if $\mathfrak{g}$ is $(n-1)$-step nilpotent.

Theorem (Burde–E.)

Let $\mathfrak{g}$ be a filiform Lie algebra. Then

(i) all CPA-structures on $\mathfrak{g}$ satisfy $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ (are associative) if and only if $\mathfrak{g}$ is non-metabelian and

(ii) all CPA-structures on $\mathfrak{g}$ satisfy $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.

Filiform Lie algebras

Every filiform Lie algebra $\mathfrak{g}$ has an adapted basis (Vergne, 1970), that is, a vector space basis $\{e_1, e_2, \ldots, e_n\}$ such that

$$\begin{aligned} [e_1, e_i] &= e_{i+1}, & i &= 2, ..., n-1 \\ [e_i, e_j] &\in \operatorname{span}(e_{i+j}, ..., e_n), & i, j &\geq 2, i+j \leq n \\ [e_{i+1}, e_{n-i}] &= (-1)^i \alpha e_n, & 1 &\leq i < n-1 \end{aligned}$$

with a scalar α, which is always 0 if n is odd.

Define $I_k := \operatorname{span}(e_k, \ldots, e_n)$. The lower central series of $\mathfrak{g}$ is given by

$$\mathfrak{g} = I_1 \geq I_3 \geq I_4 \geq \ldots \geq I_n \geq I_{n+1} = 0.$$

CPA-structures on filiform Lie algebras

Lemma

Let $(\mathfrak{g}, [\cdot,\cdot], \cdot)$ be a CPA-structure on a filiform Lie algebra $\mathfrak{g}$ and $(e_1, \ldots, e_n)$ be an adapted basis of $\mathfrak{g}$ with $\mathfrak{g} \cdot \mathfrak{g} \subseteq I_3$ and $\mathfrak{g} \cdot I_2 \subseteq I_4$. Suppose that for some $\ell \geq 0$ we have

$$e_1 \cdot e_j \in I_{j+\ell+2} \text{ for all } 3 \leq j \leq n,$$
$$e_i \cdot e_j \in I_{i+j+\ell} \text{ for all } (i,j) \neq (1,1),(1,2),(2,1),(2,2).$$

Then the same is true for $\ell + 1$, i.e., we have

$$e_1 \cdot e_j \in I_{j+\ell+3} \text{ for all } 3 \leq j \leq n,$$
$$e_i \cdot e_j \in I_{i+j+\ell+1} \text{ for all } (i,j) \neq (1,1),(1,2),(2,1),(2,2).$$

Filiform Lie algebras

Theorem (Burde–E.)

Let $\mathfrak{g}$ be a filiform Lie algebra. Then

(i) all CPA-structures on $\mathfrak{g}$ satisfy $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ (are associative) if and only if $\mathfrak{g}$ is non-metabelian and

(ii) all CPA-structures on $\mathfrak{g}$ satisfy $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.

CPA-structures on filiform Lie algebras

On non-metabelian filiform Lie algebras $\mathfrak{g}$ of dimension ≥ 4, there is always a non-associative CPA-structure:

Let $\{e_1, e_2, \ldots, e_n\}$ be an adapted basis of $\mathfrak{g}$. Let $[e_2, e_3] = \alpha_5 e_5 + \alpha_6 e_6 + \ldots + \alpha_n e_n$. Then

$$\begin{aligned} e_1 \cdot e_i &= [e_1, e_i], 1 \leq i \leq n, \\ e_2 \cdot e_j &= [e_2, e_j], 3 \leq j \leq n, \\ e_2 \cdot e_2 &= 2(\alpha_5 e_4 + \alpha_6 e_5 + \ldots + \alpha_n e_{n-1}) \end{aligned}$$

is a CPA-structure on $\mathfrak{g}$ which is not associative.

CPA-structures on filiform Lie algebras

With this theorem, one can derive all possible CPA-structures on certain important classes of filiform Lie algebras (e.g. L_n, Q_n, R_n, W_n).

CPA-structures on filiform Lie algebras

Definition

The Lie algebra L_n of dimension n has a basis $\{e_1, \ldots, e_n\}$ and

$$[e_1, e_i] = e_{i+1} \qquad \text{for} \qquad 2 \leq i \leq n-1.$$

Proposition (Burde–E.)

All CPA-structures on $L_n, n \geq 5$ are given by ...

Type 1:

$$\begin{aligned} e_1 \cdot e_1 &= \alpha_2 e_2 + \ldots + \alpha_n e_n, \\ e_1 \cdot e_2 = e_2 \cdot e_1 &= \beta e_{n-1} + \gamma e_n, \\ e_1 \cdot e_3 = e_3 \cdot e_1 &= \beta e_n, \\ e_2 \cdot e_2 &= \delta e_n, \end{aligned}$$

$\alpha_i, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha_2 \delta + \beta = 0$.

CPA-structures on filiform Lie algebras

Definition

The Lie algebra L_n of dimension n has a basis $\{e_1, \ldots, e_n\}$ and

$$[e_1, e_i] = e_{i+1} \qquad \text{for} \qquad 2 \leq i \leq n-1.$$

Proposition (Burde–E.)

Type 2:

$$\begin{aligned} e_1 \cdot e_1 &= \alpha_2 e_2 + \ldots + \alpha_n e_n, \\ e_1 \cdot e_2 = e_2 \cdot e_1 &= e_3 + \beta e_{n-1} + \gamma e_n, \\ e_1 \cdot e_3 = e_3 \cdot e_1 &= e_4 + \beta e_n, \\ e_1 \cdot e_k = e_k \cdot e_1 &= e_{k+1}, 4 \leq k \leq n-1, \\ e_2 \cdot e_2 &= \delta e_n, \end{aligned}$$

$\alpha_i, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha_2 \delta - \beta = 0$.

CPA-structures on filiform Lie algebras

Definition

The Lie algebra Q_n of dimension n even has basis $\{e_1, \ldots, e_n\}$ and

$$[e_1, e_i] = e_{i+1} \qquad \text{for} \qquad 2 \le i \le n-1,$$
$$[e_i, e_{n-i+1}] = (-1)^{i+1} e_n \qquad \text{for} \qquad 2 \le i \le n/2.$$

Proposition (Burde–E.)

All CPA-structures on $Q_n, n \ge 6$ are given by

$$e_1 \cdot e_1 = \alpha e_{n-1} + \beta e_n,$$
$$e_1 \cdot e_2 = e_2 \cdot e_1 = -\alpha e_{n-1} + \gamma e_n,$$
$$e_2 \cdot e_2 = \alpha e_{n-1} + \delta e_n,$$

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

CPA-structures on filiform Lie algebras

Definition

The Lie algebra R_n of dimension n has a basis $\{e_1, \ldots, e_n\}$ and

$$[e_1, e_i] = e_{i+1} \quad \text{for} \quad 2 \leq i \leq n-1,$$
$$[e_2, e_i] = e_{i+2} \quad \text{for} \quad 3 \leq i \leq n-2.$$

Proposition (Burde–E.)

All CPA-structures on R_n, $n \geq 6$ are given by...

Type 1:

$$e_1 \cdot e_1 = \alpha_3 e_3 + \ldots + \alpha_n e_n,$$
$$e_1 \cdot e_2 = e_2 \cdot e_1 = \alpha_3 e_4 + \ldots + \alpha_{n-2} e_{n-1} + \beta e_n,$$
$$e_2 \cdot e_2 = \alpha_3 e_5 + \ldots + \alpha_{n-3} e_{n-1} + \gamma e_n,$$

$\alpha_i, \beta, \gamma \in \mathbb{C}$.

CPA-structures on filiform Lie algebras

Definition

The Lie algebra R_n of dimension n has a basis $\{e_1, \ldots, e_n\}$ and

$$[e_1, e_i] = e_{i+1} \quad \text{for} \quad 2 \leq i \leq n-1,$$
$$[e_2, e_i] = e_{i+2} \quad \text{for} \quad 3 \leq i \leq n-2.$$

Proposition (Burde–E.)

Type 2:

$$e_1 \circ e_i = e_i \circ e_1 = e_1 \cdot e_i + [e_1, e_i], \qquad 1 \leq i \leq n,$$
$$e_2 \circ e_2 = 2e_4 + e_2 \cdot e_2,$$
$$e_2 \circ e_i = e_i \circ e_2 = [e_2, e_i], \qquad 3 \leq i \leq n,$$

where $e_i \cdot e_j$ is a CPA-structure of type 1.

CPA-structures on filiform Lie algebras

Definition

The Lie algebra W_n (Witt algebra) of dimension n has a basis $\{e_1, \ldots, e_n\}$ and Lie brackets

$$[e_i, e_j] = (j - i)e_{i+j} \quad \text{for} \quad i + j \leq n.$$

Proposition (Burde–E.)

All CPA-structures on the Witt algebra W_n, $n \geq 7$ are given by

$$e_1 \cdot e_1 = \binom{n-2}{n-4} \alpha e_{n-2} + \beta e_{n-1} + \gamma e_n,$$

$$e_1 \cdot e_2 = e_2 \cdot e_1 = \alpha e_{n-1} + \delta e_n,$$

$$e_2 \cdot e_2 = \varepsilon e_n,$$

$\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{C}$.

Lie algebras of strictly upper triangular matrices

Theorem (Burde–E.)

Let $\mathfrak{g}$ be the ($(n-1)$-step nilpotent) Lie algebra of strictly upper triangular matrices of size $n \times n$, $n \geq 5$.Then all CPA-structures on $\mathfrak{g}$ are associative and satisfy $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathfrak{g}^{n-3}$.

Proof: Inductively: Let $\mathfrak{h}$ the Lie algebra of strictly upper triangular $(n-1) \times (n-1)$-matrices.

For the ideals $\mathfrak{a} = \operatorname{span}(E_{1,i} | 2 \leq i \leq n) + \mathfrak{g}^{n-3}$, $\mathfrak{b} = \operatorname{span}(E_{i,n} | 1 \leq i \leq n-1) + \mathfrak{g}^{n-3}$, we have $\mathfrak{g}/\mathfrak{a} \cong \mathfrak{h}/Z(\mathfrak{h})$, $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{h}/Z(\mathfrak{h})$.

Induction gives us $(\mathfrak{g}/\mathfrak{a}) \cdot (\mathfrak{g}/\mathfrak{a}) \subseteq Z(\mathfrak{g}/\mathfrak{a})$ and $(\mathfrak{g}/\mathfrak{a}) \cdot [(\mathfrak{g}/\mathfrak{a}), (\mathfrak{g}/\mathfrak{a})] = 0$.

This implies $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathfrak{g}^{n-4}$ and $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}^{n-3}$; the CPA-identities and the structure of $\operatorname{Der}(\mathfrak{g})$ further imply $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g} \cdot \mathfrak{g}^{n-4} = 0$.

Lie algebras of strictly upper triangular matrices

Theorem (Burde–E.)

Let $\mathfrak{g}$ be the Lie algebra of strictly upper triangular matrices of size $n \times n$, $n \geq 5$. Then all CPA-structures on $\mathfrak{g}$ are associative and satisfy $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathfrak{g}^{n-3}$.

Corollary

Let $\mathfrak{g}$ be the Lie algebra of non-strictly upper triangular matrices of size $n \geq 5$. Then all CPA-structures on $\mathfrak{g}$ are associative and satisfy $\mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g}) + Z([\mathfrak{g}, \mathfrak{g}])$.

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