On Leibniz cohomology

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Plan of the talk

1. Motivation
2. Leibniz algebras and bimodules
3. Cohomology
4. Spectral sequences
5. Cohomology vanishing for semisimple Leibniz algebras

Motivation: Hochschild-Serre spectral sequence in Lie algebra cohomology

$$0 \to \mathfrak{k} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{k} \to 0$$

induces a filtration

$$\mathcal{F}^p C^n(\mathfrak{g}, M) = \{c \mid c(x_1, \ldots, x_n) = 0 \text{ if } n-p \text{ elements } x_i \text{ are in } \mathfrak{k}\}$$

and a spectral sequence with

$$E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{k}, H^q(\mathfrak{k}, M)).$$

Theorem (Whitehead)

$\mathfrak{g}$ finite-dimensional semisimple over a field k with $\mathrm{char}(k) = 0$. M finite-dimensional $\mathfrak{g}$-module with $M^{\mathfrak{g}} = 0$. Then $H^n(\mathfrak{g}, M) = 0$ for all $n \geq 0$.

Theorem (Hochschild-Serre)

$\mathfrak{g}$ finite-dimensional over a field k with $\mathrm{char}(k) = 0$ and Levy decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathrm{Rad}(\mathfrak{g})$. M finite-dimensional $\mathfrak{g}$-module. Then

$$H^n(\mathfrak{g}, M) = \bigoplus_{p+q=n} H^p(\mathfrak{s}, k) \otimes H^q(\mathrm{Rad}(\mathfrak{g}), M)^{\mathfrak{g}}$$

for all $n \geq 0$.

Definition

A (left) Leibniz algebra is a vector space $\mathfrak{h}$ *with a bracket such that for all* $x, y, z \in \mathfrak{h}$

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$

$\mathrm{Leib}(\mathfrak{h})$ is the *ideal of squares* with corresponding quotient Lie algebra $\mathfrak{h}_{\mathrm{Lie}} := \mathfrak{h}/\mathrm{Leib}(\mathfrak{h})$.

A finite-dimensional Leibniz algebra $\mathfrak{h}$ is called *semisimple* if the maximal solvable ideal $\mathrm{Rad}(\mathfrak{h})$ is included in $\mathrm{Leib}(\mathfrak{h})$.

Definition

A Leibniz $\mathfrak{h}$-bimodule is a vector space M with left and right operations of $\mathfrak{h}$ such that for all $x, y \in \mathfrak{h}$ and all $m \in M$

1. $x \cdot (y \cdot m) = [x, y] \cdot m + y \cdot (x \cdot m)$ *(LLM),*
2. $x \cdot (m \cdot y) = (x \cdot m) \cdot y + m \cdot [x, y]$ *(LML),*
3. $m \cdot [x, y] = (m \cdot x) \cdot y + x \cdot (m \cdot y)$ *(MLL).*

M, k-vector space satisfying (LLM), becomes a bimodule in two ways:
$m \cdot x = -x \cdot m$ gives a bimodule called *symmetric bimodule*.
$m \cdot x = 0$ gives a bimodule called *antisymmetric bimodule*.

With $M_0 := \mathrm{span}_k(x \cdot m + m \cdot x \,|\, x \in \mathfrak{h}, m \in M)$, get for any bimodule M an exact sequence

$$0 \to M_0 \to M \to M/M_0 =: M^s \to 0.$$

$M^{\mathfrak{h}} := \{m \in M \,|\, \forall x \in \mathfrak{h} : m \cdot x = 0\}$ right invariants

Lemma (1.1)

$M^{\mathfrak{h}} = 0$ implies M symmetric. Furthermore $\mathrm{Leib}(\mathfrak{h}) \subset \mathrm{Ann}^{\mathrm{bi}}(M)$.

Corollary (1.3)

$M \neq \{0\}$ irreducible bimodule: $M^{\mathfrak{h}} = 0 \Leftrightarrow M$ symmetric and non-trivial.

Cohomology: $CL^n(\mathfrak{h}, M) = \mathrm{Hom}_k(\mathfrak{h}^{\otimes n}, M) \ni c$

$$dc(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot c(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) + \\ + (-1)^{n+1} c(x_1, \ldots, x_n) \cdot x_{n+1} + \\ + \sum_{1 \leq i < j \leq n+1} (-1)^i c(x_1, \ldots, \widehat{x_i}, \ldots, [x_i, x_j], \ldots, x_{n+1}).$$

Lemma (1.5)

M antisymmetric $\mathfrak{h}$-bimodule:

$$HL^0(\mathfrak{h}, M) = M$$

$$HL^n(\mathfrak{h}, M) \cong HL^{n-1}(\mathfrak{h}, \mathrm{Hom}(\mathfrak{h}, M)^s),$$

induced by $\varphi(c)(x_1, \ldots, x_{n-1})(x) := c(x_1, \ldots, x_{n-1}, x)$ *for* $n \geq 1$.

Spectral sequences I: comparison spectral sequence between Lie- and Leibniz cohomology for a Lie algebra (Pirashvili)

Theorem (Pirashvili)

$\mathfrak{g}$ *Lie algebra, M symmetric $\mathfrak{g}$-bimodule.*

$$H^*(\mathfrak{g}, M) = 0 \;\Rightarrow\; HL^*(\mathfrak{g}, M^s) = 0.$$

Spectral sequences II: Given a short exact sequence

$$0 \to \mathfrak{j} \to \mathfrak{h} \stackrel{\pi}{\to} \mathfrak{q} \to 0,$$

and a $\mathfrak{q}$-bimodule M, there is a *relative complex*

$$CL^{\bullet}(\mathfrak{h}|\mathfrak{q}, M) := \mathrm{coker}(\pi^*).$$

Lemma (3.1)

There is a long exact sequence

$$\ldots \to HL^n(\mathfrak{q}, M) \to HL^n(\mathfrak{h}, M) \to HL^{n-1}(\mathfrak{h}|\mathfrak{q}, M) \to \ldots$$

$$\mathcal{F}^p CL^n(\mathfrak{h}, M) = \{c \mid c(x_1, \ldots, x_n) = 0 \text{ if } \exists i \geq n - p + 1 : x_i \in \mathfrak{j}\}$$

leads to

Theorem (Beaudouin)

$$E_2^{p,q} = HL^{q-1}(\mathfrak{h}, HL^p(\mathfrak{q}, M))$$

for $q \geq 1$ and zero otherwise.

$$\mathcal{F}^p CL^n(\mathfrak{h}, M) = \{c \mid c(x_1, \ldots, x_n) = 0 \text{ if } \exists i \leq p : x_i \in \mathfrak{j}\}$$

leads for $\mathfrak{j} \subset Z_L(\mathfrak{h})$ to

Theorem (Pirashvili)

$$E_2^{p,q} = \begin{cases} HL^p(\mathfrak{q}, \mathrm{Hom}_k(\mathfrak{j}, HL^0(\mathfrak{h}, M))^s) & \text{for } \quad q = 0 \\ HL^p(\mathfrak{q}, (\mathfrak{j}^*)^s) \otimes HL^q(\mathfrak{h}, M) & \text{for } \quad q \geq 1 \end{cases}$$

Theorem (FW,4.1)

$\mathfrak{h}$ Leibniz such that $\mathfrak{h}_{\mathrm{Lie}}$ finite-dimensional semisimple over k with $\mathrm{char}(k) = 0$. M finite-dimensional bimodule such that $M^{\mathfrak{h}} = 0$. Then $HL^n(\mathfrak{h}, M) = 0$ for all $n \geq 0$.

Proof: $M^{\mathfrak{h}} = 0$ implies M symmetric and $\mathrm{Leib}(\mathfrak{h}) \subset \mathrm{Ann}^{\mathrm{bi}}(M)$. Thus M is a symmetric $\mathfrak{h}_{\mathrm{Lie}}$-bimodule with $M^{\mathfrak{h}_{\mathrm{Lie}}} = 0$. Whitehead implies $H^n(\mathfrak{h}_{\mathrm{Lie}}, M) = 0$ for all $n \geq 0$. Comparison Lie/Leibniz cohomology implies $HL^n(\mathfrak{h}_{\mathrm{Lie}}, M) = 0$ for all $n \geq 0$. Beaudouin implies $E_2^{p,q} = 0$. Thus the relative cohomology is zero and the long exact sequence (3.1) leads to the conclusion $HL^n(\mathfrak{h}, M) = 0$ for all $n \geq 0$. □

Theorem (FW,4.2)

$\mathfrak{h}$ finite-dimensional semisimple Leibniz over k with $\mathrm{char}(k) = 0$. M finite-dimensional bimodule. Then $HL^n(\mathfrak{h}, M) = 0$ for all $n \geq 2$ and $HL^1(\mathfrak{h}, M) = 0$ if M is symmetric.

Proof: First step: M symmetric.
M has a composition series $\Rightarrow$ it suffices to show the claim for M irreducible. If M is non-trivial, we have by Corollary 1.3 that $M^{\mathfrak{h}} = 0$ and thus by Theorem 4.1 that $HL^n(\mathfrak{h}, M) = 0$ for all $n \geq 0$.

Otherwise, if $M = k$ is trivial, have $HL^n(\mathfrak{h}, k) = HL^{n-1}(\mathfrak{h}, (\mathfrak{h}^*)^s)$ and $(\mathfrak{h}^*)^s$ is symmetric non-trivial.
$HL^0(\mathfrak{h}, (\mathfrak{h}^*)^s) = H^1(\mathfrak{h}_{\mathrm{Lie}}, k) = 0$, because $\mathfrak{h}_{\mathrm{Lie}}$ is perfect.

Second step: M antisymmetric.
Conclude by Lemma 1.5 and the first step.

Third step: M arbitrary.
The short exact sequence

$$0 \to M_0 \to M \to M^s \to 0$$

and the previous steps permit to conclude that $HL^n(\mathfrak{h}, M) = 0$ for all $n \geq 2$. $\square$

Corollary

$HL^2(\mathfrak{h}, \mathfrak{h}) = 0$ for $\mathfrak{h}$ finite-dimensional semisimple over k with $\mathrm{char}(k) = 0$.

Cohomology decomposition induced by the Levy decomposition

Thanks to Barnes, every finite-dimensional Leibniz algebra $\mathfrak{h}$ has a Levy decomposition $\mathfrak{h} = \mathfrak{s} \ltimes \mathrm{Rad}(\mathfrak{h})$.

Theorem (FW)

Let $\mathfrak{h}$ be a finite-dimensional Leibniz algebra over k with $\mathrm{char}(k) = 0$. Let M be a symmetric $\mathfrak{h}$-bimodule on which $\mathrm{Rad}(\mathfrak{h})$ acts trivially. Then

$$HL^n(\mathfrak{h}, M) = \begin{cases} M^{\mathfrak{s}} & \text{if} \quad n = 0 \\ HL^{n-1}(\mathfrak{h}, \mathrm{Hom}_k(\mathrm{Rad}(\mathfrak{h}), M^{\mathfrak{s}})^s) & \text{if} \quad n \geq 1 \end{cases}$$

For example, have

$$HL^n(\mathfrak{h}, k) = HL^{n-1}(\mathfrak{h}, (\mathrm{Rad}(\mathfrak{h})^*)^s).$$