Structurable algebras, TKK Lie algebras and their inner ideals

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30 april 2019

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Joint work with Tom De Medts

II International Workshop on Non-Associative Algebras in Porto

2 Inner ideals

Assumption

All algebras are finite-dimensional and defined over a field k of characteristic different from 2 and 3!

Definition

An inner ideal of a Lie algebra L is a subspace I such that $[I,[I,L]] \leq I$. If I is 1-dimensional, any non-zero element of I is called extremal.

Example

For $L = sl_2$ we have a basis $\{e, f, h\}$ with $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. Then e is extremal by $[e,[e,h]] = 0 = [e,[e,e]]$ and $[e,[e,f]] = [e,h] = -2e$.

3 TKK Lie algebras

- If $\mathcal{A}$ is a structurable algebra, then

$$K(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+$$

  is a 5-graded Lie algebra. Denote its i-th component by L_i.
- E.g.: $[x_+, y_-] = V_{x,y}$.
- $[L_i, [L_i, L_j]] \leq L_{2i+j}$.
- $\mathcal{S}_+$ will always be an inner ideal!
- If $\mathcal{S} = 0$, then $\mathcal{A}_+$ will always be an inner ideal.
- If $\mathcal{A}$ is central simple, any non-trivial inner ideal $I \leq K(\mathcal{A})$ satisfies $[I, I] = 0$.

4 Some automorphisms

- A Lie algebra automorphism maps inner ideals on to inner ideals.
- For any $a \in \mathcal{A}$ and $s \in \mathcal{S}$ we get $\mathrm{ad}(a_+ + s_+)^5 = 0$.
- If $\mathcal{A}$ is central simple

$$e_+(a,s) := \exp(\mathrm{ad}(a_+ + s_+)) = \sum_{i=0}^{4} \frac{1}{i!} \mathrm{ad}(a_+ + s_+)^i$$

  is a Lie algebra automorphism.
- Set $E_+(\mathcal{A}) = \{e_+(a,s) \mid s \in \mathcal{S}, a \in \mathcal{A}\}$ and similarly $E_-(\mathcal{A})$.
- $E(\mathcal{A})$ is the subgroup of $\mathrm{Aut}(L)$ generated by $E_+(\mathcal{A})$ and $E_-(\mathcal{A})$.

5 Inner ideals of Jordan algebras

Definition

A subspace I of a Jordan algebra J is an inner ideal if $U_I(J) \leq I$.

Example

Let $J = \mathrm{Jord}(Q, c)$ be the Jordan algebra corresponding to a non-degenerate quadratic form Q with basepoint c.
The non-trivial inner ideals are precisely the isotropic subspaces.

6 Some remarks

- If A is an associative algebra, $A^+ = (A, \circ)$ is a Jordan algebra with $a \circ b = (ab + ba)/2$.
- Motivation for terminology: $I \leq A^+$ is inner if $IAI \leq I$.
- Studied by McCrimmon in 1971 (*Inner ideals in quadratic Jordan algebras*).
- If J is division, there are no non-trivial inner ideals.
- A subspace $I \leq J$ is an inner ideal if and only if I_+ is an inner ideal in $K(J)$.

Structurable division algebras

Theorem (De Medts - M.)

Let J be a central simple Jordan division algebra.
Then any non-trivial inner ideal of $K(J)$ distinct from J_- equals $e_-(j)(J_+)$ for a unique $j \in J$.

Note: $sl_2 = K(k)$.

Theorem (De Medts - M.)

Let $\mathcal{A}$ be a central simple structurable division algebra with $\mathcal{S} \neq 0$.
Then any non-trivial inner ideal of $K(\mathcal{A})$ distinct from $\mathcal{S}_-$ equals $e_-(a, s)(\mathcal{S}_+)$ for unique $a \in \mathcal{A}$ and $s \in \mathcal{S}$.

8 Moufang sets

Definition

Let X be a set and $\{U_x \mid x \in X\}$ a collection of subgroups of $\mathrm{Sym}(X)$. The data $(X, \{U_x\}_{x\in X})$ is a Moufang set if:

- For each $x \in X$, U_x fixes x and acts sharply transitively on $X \backslash \{x\}$.
- For each $g \in G := \langle U_x \mid x \in X \rangle$ and each $y \in X$ we have $U_y^g = U_{y.g}$.

If $\mathcal{A}$ is central simple division, the set of non-trivial inner ideals of $K(\mathcal{A})$ is a Moufang set.
Note: $U_{\mathcal{S}_-} = E_-(\mathcal{A})$.

Skew-dimension one structurable algebras

- Recall the skew-dimension one structurable algebra with as elements
$$\begin{pmatrix} a & i \\ j & b \end{pmatrix},$$
with $a, b \in k$ and $i, j \in J$.
- We denote it by $M(J)$.
- J is a cubic Jordan **division** algebra.
- Example of J: an Albert division algebra.
- Other example: k with cubic form $N(a) = a^3$.

10 Reduction

Lemma (De Medts - M.)

Any non-trivial inner ideal of $K(M(J))$ is the image of an inner ideal containing $\mathcal{S}_+$ under an element of $E(\mathcal{A})$.

- $\mathcal{S}_+$ is 1-dimensional.
- What are the inner ideals containing $\mathcal{S}_+$?
- The inner ideals of a structurable algebra $\mathcal{A}$ are the subspaces I satisfying $U_I(\mathcal{A}) \leq I$.
- In this case all non-trivial inner ideals of $M(J)$ are 1-dimensional and form a Moufang set.

A tale of two types of inner ideals

Lemma (De Medts - M.)

The only non-trivial inner ideals properly containing $\mathcal{S}_+$ are $\mathcal{S}_+ \oplus \langle a_+ \rangle$, with $\langle a \rangle$ an inner ideal.

- Each non-trivial inner ideal is 1 or 2-dimensional.
- If its dimension is 2, all subspaces are inner ideals.
- In particular: the Moufang set in $M(J)$ embeds in the lattice of inner ideals of $K(M(J))$.

Generalized hexagon

- A generalized hexagon is a point-line geometry which does not contain triangles, quadrangles or pentagons and such that any two distinct points, any two distinct lines and any point and line lie in a hexagon.
- In particular: two points are at distance at most 3.

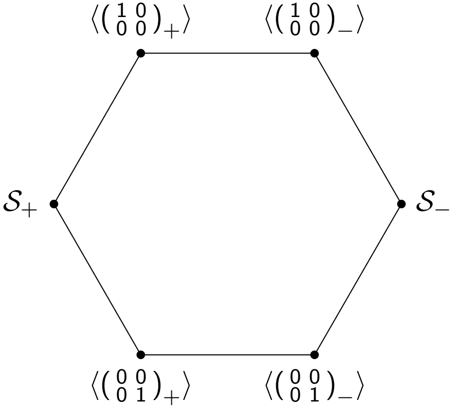
Theorem (De Medts - M.)

The point-line geometry with as points the 1-dimensional inner ideals, as lines all non-trivial inner ideals of dimension > 1 and inclusion as incidence is a generalized hexagon.

- $\mathcal{S}_+$ and $\mathcal{S}_-$ are at distance 3.

13 A hexagon

The generic hexagon in $K(M(J))$ is:



14 Some remarks

- In *Groups with Steinberg Relations and Coordinatization of Polygonal Geometries* ('77) Faulkner constructs a Lie algebra starting from a cubic Jordan division algebra. He then constructs a generalized hexagon with as points and lines some specific inner ideals.
- Similar ideas can also be used to determine inner ideals of some other structurable algebras and TKK Lie algebras that will correspond to projective planes.

15 Extremal geometries

Joint with Hans Cuypers (TU Eindhoven)

- Relying on results of Stavrova:

Theorem (Cuypers - M.)

If $L \neq sl_2$ is a non-degenerate simple Lie algebra generated by its extremal elements, then $L = K(\mathcal{A})$ for some skew-dimension one structurable algebra $\mathcal{A}$.

- Cohen and Ivanyos: construction of point-line geometries called extremal geometries with as points extremal elements.
- Currently: trying to see the lines of these geometries as minimal inner ideals containing extremal elements and extend the results of Cohen and Ivanyos.

16 Open questions

- What happens in characteristic 2 (and 3)? Extremal elements in characteristic 2 are well-defined, what is "good" definition for inner ideals of Lie algebras?
- Inner ideals of a Jordan algebra correspond to inner ideals of its Lie algebra, what happens when structurable algebra is not Jordan?
- Which class of structurable algebras corresponds to rank 2 geometries?

Thanks for your attention!