On some properties preserved by the non-abelian tensor product of Hom-Lie algebras

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The main goal of this talk is to continue investigation done in

J. M. Casas, E. Khmaladze and N. Pacheco: A non-abelian tensor poduct of Hom-Lie algebras,Bull. Malaysian Math. Sci. Soc **40** (3) (2017), 1035–1054.

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and obtain further properties of the non-abelian tensor product of Hom-Lie algebras, concerning solvability, nilpotency and describe compatibility with the universal central extensions of perfect Hom-Lie algebras.

Definition

A Hom-Lie algebra (L, α_L) *is a* $\mathbb{K}$*-vector space endowed with a bilinear bracket operation* $[-,-] : L \times L \to L$ *and a linear map* $\alpha_L : L \to L$ *satisfying*

$$[x, y] = -[y, x], \quad \textbf{(skew-symmetry)}$$

$$[\alpha_L(x), [y, z]] + [\alpha_L(z), [x, y]] + [\alpha_L(y), [z, x]] = 0$$

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for all $x, y, z \in L$.

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*(**Hom-Jacobi identity**)*

for all $x, y, z \in L$.
A Hom-Lie algebras (L, α_L) is called multiplicative if α_L preserves the bracket, i. e. $\alpha_L[x, y] = [\alpha_L(x), \alpha_L(y)]$.

Definition

A Hom-Lie subalgebra (H, α_H) of (L, α_L) is a vector subspace H of L, which is closed for the bracket and invariant by α_L:

a) $[x, y] \in H$, for all $x, y \in H$,

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The center of a Hom-Lie algebra (L, α_L) is the vector subspace of L,

$$\mathbb{Z}(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}.$$

Definition

1) *Let (L, α_L) and (M, α_M) be Hom-Lie algebras. A* Hom-action *of (L, α_L) on (M, α_M) is a linear map $L \otimes M \to M$, $x \otimes m \mapsto {}^x m$, satisfying the following properties:*

   a) ${}^{[x,y]}\alpha_M(m) = {}^{\alpha_L(x)}({}^y m) - {}^{\alpha_L(y)}({}^x m)$,
   b) ${}^{\alpha_L(x)}[m, m'] = [{}^x m, \alpha_M(m')] + [\alpha_M(m), {}^x m']$,
   c) $\alpha_M({}^x m) = {}^{\alpha_L(x)}\alpha_M(m)$

   for all $x, y \in L$ and $m, m' \in M$.

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   for all $x, y \in L$ and $m, m' \in M$.

2) *Let (M, α_M) and (N, α_N) be Hom-Lie algebras with mutual Hom-actions on each other. The Hom-actions are said to be compatible if*

$$ {}^{({}^{m}n)}m' = [m', {}^{n}m] \quad \text{and} \quad {}^{({}^{n}m)}n' = [n', {}^{m}n] $$

   for all $m, m' \in M$ and $n, n' \in N$.

Example

Let (M, α_M) and (N, α_N) be two Hom-ideals of a Hom-Lie algebra (L, α_L). Then there are compatible Hom-actions of (M, α_M) and (N, α_N) on each other given by the structural bracket in L.

Definition

1. *Let (M, α_M) be a Hom-ideal of a Hom-Lie algebra (Q, α_Q). A series from (M, α_M) to (Q, α_Q) is a finite sequence of Hom-ideals $(M_i, \alpha_{Q|})$, $0 \leq i \leq k$, of (Q, α_Q) such that*

$$M = M_0 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_{k-1} \trianglelefteq M_k = Q.$$

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2. *A series from (M, α_M) to (Q, α_Q) is said to be*
   a) *central if all its factors are central, that is $[Q, M_i] \subseteq M_{i-1}$, equivalently $M_i/M_{i-1} \subseteq Z(Q/M_{i-1})$.*

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   a) *central if all its factors are central, that is $[Q, M_i] \subseteq M_{i-1}$, equivalently $M_i/M_{i-1} \subseteq Z(Q/M_{i-1})$.*
   b) *abelian if $[M_i, M_i] \subseteq M_{i-1}$, equivalently $[M_i/M_{i-1}, M_i/M_{i-1}] = 0$.*

Definition

a) *A Hom-Lie algebra* (Q, α_Q) *is said to be solvable if it has an abelian series. Let* k *be the minimal length of such series, then* k *is said to be the class of solvability of* (Q, α_Q)*.*

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b) *A Hom-Lie algebra (Q, α_Q) is said to be nilpotent if it has a central series. Let k be the minimal length of such series, then k is said to be the class of nilpotency of (Q, α_Q).*

Given a Hom-Lie algebra (Q, α_Q), consider the sequence of Hom-Lie subalgebras $(Q^{(i)}, \alpha_{Q|})$, $i \geq 0$, defined inductively by

$$Q^{(0)} = Q; Q^{(i+1)} = [Q^{(i)}, Q^{(i)}], i \geq 0$$

It is clear that

$$\cdots \subseteq Q^{(i)} \subseteq \cdots \subseteq Q^{(1)} \subseteq Q^{(0)}$$

In general, the terms $(Q^{(i)}, \alpha_{Q|})$ are Hom-Lie subalgebras but not Hom-ideals of (Q, α_Q). This fact is illustrated by the following example.

Example

Consider the four-dimensional Hom-Lie algebra (Q, α_Q) with basis $\{a_1, a_2, a_3, a_4\}$, bracket operation given by $[a_1, a_2] = -[a_2, a_1] = a_1$, $[a_1, a_3] = -[a_3, a_1] = a_2$ and the map $\alpha_Q = 0$. It is easy to see that $Q^{(2)} = \langle \{a_1\} \rangle$, which is not a Hom-ideal of (Q, α_Q).

Remark

In order to guarantee that the terms $(Q^{(i)}, \alpha_{Q|})$, $i \geq 1$, are Hom-ideals, the Hom-Lie algebra (Q, α_Q) should satisfy additional condition, such as:

a) *If the endomorphism α_Q is surjective.*
b) *Hom-Lie algebras satisfying the so-called α-identity condition, that is, Hom-Lie algebra (Q, α_Q) such that $[Q, \mathrm{Im}(\alpha_Q - \mathrm{id}_Q)] = 0$, equivalently $[x, y] = [\alpha_Q(x), y]$, for all $x, y \in Q$.*
c) *Hom-Lie algebras satisfying a weaker condition. Namely, $[[Q, Q], \mathrm{Im}(\alpha_Q - \mathrm{id}_Q)] = 0$, equivalently $[x, [y, z]] = [\alpha_Q(x), [y, z]]$, for all $x, y, z \in Q$. We call this condition weak α-identity condition.*

The lower central series of a Hom-Lie algebra (Q, α_Q) is the sequence of Hom-ideals $(Q^{[i]}, \alpha_{Q|})$, $i \geq 0$, defined inductively by

$$Q^{[0]} = Q \quad \text{and} \quad Q^{[i+1]} = [Q^{[i]}, Q], \quad i \geq 0.$$

It is readily checked that each $(Q^{[i]}, \alpha_{Q|})$ is indeed a Hom-ideal of (Q, α_Q)

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Theorem

a) *Let (Q, α_Q) be a Hom-Lie algebra such that α_Q is surjective or (Q, α_Q) satisfies the weak α-identity condition. Then (Q, α_Q) is solvable with class of solvability k if and only if $Q^{(k)} = 0$ and $Q^{(k-1)} \neq 0$.*

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b) *(Q, α_Q) is nilpotent Hom-Lie algebra with class of nipotency k if and only if $Q^{[k]} = 0$ and $Q^{[k-1]} \neq 0$.*

Definition

1. *The non-abelian tensor product of Hom-Lie algebras (M, α_M) and (N, α_N) with mutual compatible Hom-actions on each other, denoted by $(M \star N, \alpha_{M\star N})$, is the Hom-Lie algebra generated by the Hom-vector space $(M \otimes N, \alpha_{M\otimes N})$ given by the tensor product $M \otimes N$ of the underlying vector spaces and the linear map $\alpha_{M\otimes N} : M \otimes N \to M \otimes N$, $\alpha_{M\otimes N}(m \otimes n) = \alpha_M(m) \otimes \alpha_N(n)$, with the following defining relations:*
   - i) $[m, m'] \otimes \alpha_N(n) - \alpha_M(m) \otimes {}^{m'}n + \alpha_M(m') \otimes {}^{m}n,$
   - ii) $\alpha_M(m) \otimes [n, n'] - {}^{n'}m \otimes \alpha_N(n) + {}^{n}m \otimes \alpha_N(n'),$
   - iii) ${}^{n}m \otimes {}^{m}n,$
   - iv) ${}^{n}m \otimes {}^{m'}n' + {}^{n'}m' \otimes {}^{m}n,$
   - v) $[{}^{n}m, {}^{n'}m'] \otimes \alpha_N({}^{m''}n'') + [{}^{n'}m', {}^{n''}m''] \otimes \alpha_N({}^{m}n) + [{}^{n''}m'', {}^{n}m] \otimes \alpha_N({}^{m'}n'),$
   - vi) $[m \otimes n, m' \otimes n'] = -{}^{n}m \otimes {}^{m'}n'$
   - vii) $\alpha_{M\otimes N}(m \otimes n) = \alpha_M(m) \otimes \alpha_N(m)$

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   - iii) ${}^{n}m \otimes {}^{m}n$,
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   - v) $[{}^{n}m, {}^{n'}m'] \otimes \alpha_N({}^{m''}n'') + [{}^{n'}m', {}^{n''}m''] \otimes \alpha_N({}^{m}n) + [{}^{n''}m'', {}^{n}m] \otimes \alpha_N({}^{m'}n')$,
   - vi) $[m \otimes n, m' \otimes n'] = -{}^{n}m \otimes {}^{m'}n'$
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Proposition

Let (M, α_M) and (N, α_N) be Hom-Lie algebras with compatible Hom-actions on each other.

a) *There are homomorphisms of Hom-Lie algebras*

$$\psi_M : (M \star N, \alpha_{M\star N}) \to (M, \alpha_M), \quad \psi_M(m \star n) = -{}^n m,$$
$$\psi_N : (M \star N, \alpha_{M\star N}) \to (N, \alpha_N), \quad \psi_N(m \star n) = {}^m n.$$

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b) *There is a Hom-action of (M, α_M) (resp. (N, α_N)) on the Hom-Lie tensor product $(M \star N, \alpha_{M\star N})$ given by*

$$^{m'}(m \star n) = [m', m] \star \alpha_N(n) + \alpha_M(m) \star {}^{m'}n$$
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$$\left(resp. \quad {}^{n'}(m \star n) = {}^{n'} m \star \alpha_N(n) + \alpha_M(m) \star [n', n]\right)$$

c) $\mathrm{Ker}(\psi_M) \subseteq Z((M \star N, \alpha_{M\star N})$ *(resp.* $\mathrm{Ker}(\psi_N) \subseteq Z((M \star N, \alpha_{M\star N}))$*.*

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c) $\mathrm{Ker}(\psi_M) \subseteq Z((M \star N, \alpha_{M\star N})$ *(resp.* $\mathrm{Ker}(\psi_N) \subseteq Z((M \star N, \alpha_{M\star N}))$*.*

d) *The induced Hom-action of* $\mathrm{Im}(\psi_M)$ *(resp.* $\mathrm{Im}(\psi_N)$*) on* $\mathrm{Ker}(\psi_M)$ *(resp.* $\mathrm{Ker}(\psi_N)$*) is trivial.*

Proposition

If $f:(M,\alpha_M)\to(M',\alpha_{M'})$ and $g:(N,\alpha_N)\to(N',\alpha_{N'})$ are homomorphisms of Hom-Lie algebras together with compatible Hom-actions of (M,α_M) (resp. $(M',\alpha_{M'})$) and (N,α_N) (resp. $(N',\alpha_{N'})$) on each other such that f, g preserve these Hom-actions, that is

$$f({}^{n}m)={}^{g(n)}f(m),\quad g({}^{m}n)={}^{f(m)}g(n),\qquad m\in M,\ n\in N.$$

Then there is a unique homomorphism of Hom-Lie algebras

$$f\star g:(M\star N,\alpha_{M\star N})\to(M'\star N',\alpha_{M'\star N'}),\quad \textit{given by}$$

$$(f\star g)(m\star n)=f(m)\star g(n).$$

Furthermore, if f, g are onto then so is $f\star g$.

Proposition

Let

$$0 \to (M, \alpha_M) \xrightarrow{i_1} (L, \alpha_L) \xrightarrow{\sigma_1} (P, \alpha_P) \to 0$$
$$0 \to (N, \alpha_N) \xrightarrow{i_2} (K, \alpha_K) \xrightarrow{\sigma_2} (Q, \alpha_Q) \to 0$$

be two short exact sequences of Hom-Lie algebras. Suppose that α_L and α_K are surjective endomorphisms, (L, α_L) and (K, α_K), as well as (P, α_P) and (Q, α_Q) act compatibly on each other, and all σ_1, σ_2, α_L and α_K preserve the corresponding Hom-actions.

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$$(M \star K, \alpha_{M\star K}) \rtimes (L \star N, \alpha_{L\star N}) \xrightarrow{\eta} (L \star K, \alpha_{L\star K}) \xrightarrow{\sigma_1 \star \sigma_2} (P \star Q, \alpha_{P\star Q}) \to 0$$

is exact,

Proposition

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is exact,where the homomorphism η is the map given by
$\eta((m \star k), (l \star n)) = (i_1(m) \star k) + (\alpha_L(l) \star \alpha_K(i_2(n)))$.

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b) ${}^{\alpha_B(b)}({}^{c}a) = {}^{\alpha_c(c)}({}^{b}a)$, *for all* $a \in A, b \in B, c \in C$.

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a) *(A, α_A) and (B, α_B) Hom-act compatibly on each other, and (A, α_A) and (C, α_C) Hom-act compatibly on each other.*

b) ${}^{\alpha_B(b)}({}^{c}a) = {}^{\alpha_c(c)}({}^{b}a)$*, for all $a \in A, b \in B, c \in C$.*

c) *The canonical maps $({}^{C}A, \alpha_{A|}) \star (\alpha_B(B), \alpha_{B|}) \longrightarrow (A, \alpha_A) \star (\alpha_B(B), \alpha_{B|})$ and $({}^{B}A, \alpha_{A|}) \star (\alpha_C(C), \alpha_{C|}) \longrightarrow (A, \alpha_A) \star (\alpha_C(C), \alpha_{C|})$ are trivial, and the induced Hom-actions of ${}^{B}A$ on C and of ${}^{C}A$ on B are also trivial.*

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a) (A, α_A) *and* (B, α_B) *Hom-act compatibly on each other, and* (A, α_A) *and* (C, α_C) *Hom-act compatibly on each other.*

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c) *The canonical maps* $({}^{C}A, \alpha_{A|}) \star (\alpha_B(B), \alpha_{B|}) \longrightarrow (A, \alpha_A) \star (\alpha_B(B), \alpha_{B|})$ *and* $({}^{B}A, \alpha_{A|}) \star (\alpha_C(C), \alpha_{C|}) \longrightarrow (A, \alpha_A) \star (\alpha_C(C), \alpha_{C|})$ *are trivial, and the induced Hom-actions of* ${}^{B}A$ *on* C *and of* ${}^{C}A$ *on* B *are also trivial.*

Then

$$(A, \alpha_A) \star ((B, \alpha_B) \times (C, \alpha_C)) \cong ((A, \alpha_A) \star (B, \alpha_B)) \times ((A, \alpha_A) \star (C, \alpha_C)).$$

Proposition

Let (A, α_A), (B, α_B) *and* (C, α_C) *be Hom-Lie algebras, such that:*

a) (A, α_A) *and* (B, α_B) *Hom-act compatibly on each other, and* (A, α_A) *and* (C, α_C) *Hom-act compatibly on each other.*

b) ${}^{\alpha_B(b)}({}^{c}a) = {}^{\alpha_c(c)}({}^{b}a)$, *for all* $a \in A, b \in B, c \in C$.

c) *The canonical maps* $({}^{C}A, \alpha_{A|}) \star (\alpha_B(B), \alpha_{B|}) \longrightarrow (A, \alpha_A) \star (\alpha_B(B), \alpha_{B|})$ *and* $({}^{B}A, \alpha_{A|}) \star (\alpha_C(C), \alpha_{C|}) \longrightarrow (A, \alpha_A) \star (\alpha_C(C), \alpha_{C|})$ *are trivial, and the induced Hom-actions of* ${}^{B}A$ *on* C *and of* ${}^{C}A$ *on* B *are also trivial.*

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- The Hom-Lie pairing $(K \times H, \alpha_{K\times H}) \to (G, \alpha_G) \star (H, \alpha_H)$, $(k, h) \to k \star h$, induces a homomorphism $\iota_1 : (K \star H, \alpha_{K\star H}) \to (G \star H, \alpha_{G\star H})$, $\iota_1 = inc \star \mathrm{id}$.

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- Let be $K_1 = \mathrm{Im}(\iota_1)$, then $(K_1, \alpha_{G \star H|})$ is a Hom-ideal of $(G \star H, \alpha_{G \star H})$.

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- Analogously, let be $K_2 = \text{Im}(\iota_2 : (G \star K, \alpha_{G\star K}) \to (G \star H, \alpha_{G\star H}))$, where $\iota_2 = \text{id} \star inc$.

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- The Hom-Lie pairing $(K \times H, \alpha_{K\times H}) \to (G, \alpha_G) \star (H, \alpha_H)$, $(k, h) \to k \star h$, induces a homomorphism $\iota_1 : (K \star H, \alpha_{K\star H}) \to (G \star H, \alpha_{G\star H})$, $\iota_1 = inc \star \mathrm{id}$.
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Proposition

$$\left(\frac{G}{K}, \overline{\alpha}_G\right) \star \left(\frac{H}{K}, \overline{\alpha}_H\right) \cong \left(\frac{G \star H}{K_1 + K_2}, \overline{\alpha}_{G\star H}\right)$$

Here $\overline{\alpha}$ denotes the endomorphism induced by α on the quotient.

Definition

We say that a Hom-Lie algebra (L, α_L) is k-Engel if $R_x^k = 0$ for all $x \in L$, where R_x^k is the k-adjoint representation of (L, α_L), defined for $x \in L$.

$$R_x^k : L \to L, \quad y \mapsto [\dots[[y, \underbrace{x], x], \dots, x]}_{k}$$

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$$R_x^k : L \to L, \quad y \mapsto [\dots[[y, \underbrace{x], x], \dots, x]}_{k}$$

Theorem

Let (M, α_M) and (N, α_N) be Hom-Lie algebras with compatible actions on each other.

a) *If $({}^N M, \alpha_{M|})$ is nilpotent, then so are $(M \star N, \alpha_{M\star N})$ and $({}^M N, \alpha_{N|})$. Furthermore, if $({}^N M, \alpha_{M|})$ is nilpotent of class k, then*

$$k \leq \text{ncl}\,(M \star N, \alpha_{M\star M}) \leq k+1 \quad \textit{and} \quad \text{ncl}\,({}^M N, \alpha_{{}^M N}) \leq k+1.$$

Theorem

b) *Let (M, α_M) and (N, α_N) be Hom-Lie algebras such that α_M and α_N are surjective endomorphisms or both (M, α_M) and (N, α_N) satisfy the weak α-identity condition. If $\left({}^N M, \alpha_{M|}\right)$ is solvable, then so are $(M \star N, \alpha_{M\star N})$ and $\left({}^M N, \alpha_{N|}\right)$. Furthermore, if $\left({}^N M, \alpha_{M|}\right)$ is solvable of class k, then*

$$k \leq \operatorname{scl}(M \star N, \alpha_{M\star N}) \leq k+1 \quad \textit{and} \quad \operatorname{scl}\left({}^M N, \alpha_{N|}\right) \leq k+1.$$

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$$k \leq \operatorname{scl}(M \star N, \alpha_{M\star N}) \leq k+1 \quad \textit{and} \quad \operatorname{scl}\left({}^M N, \alpha_{N|}\right) \leq k+1.$$

c) *If $\left({}^N M, \alpha_{M|}\right)$ is k-Engel, then $(M \star N, \alpha_{M\star N})$ and $\left({}^M N, \alpha_{N|}\right)$ are $(k+1)$-Engel.*

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Proposition

Let (M, α_M) and (N, α_N) be perfect Hom-Lie algebras, with compatible Hom-action on each other, such that ${}^N M = M$ and ${}^M N = N$. Then $(M \star N, \alpha_{M \star N})$ is a perfect Hom-Lie algebra as well.

Example

Let (M, α_M) be the perfect three-dimensional Hom-Lie algebra with basis $\{e_1, e_2, e_3\}$, $\alpha_M = 0$, and bracket operation given by

$$[e_1, e_2] = -[e_2, e_1] = e_3, \qquad [e_1, e_3] = -[e_3, e_1] = e_2,$$
$$[e_2, e_3] = -[e_3, e_2] = e_1.$$

Let (N, α_N) be the perfect four-dimensional Hom-Lie algebra with basis $\{a_1, a_2, a_3, a_4\}$, $\alpha_N = 0$, and bracket operation given by

$$[a_1, a_2] = -[a_2, a_1] = a_3, [a_1, a_3] = -[a_3, a_1] = a_2,$$
$$[a_1, a_4] = -[a_4, a_1] = a_4, [a_2, a_3] = -[a_3, a_2] = a_1.$$

Example

The Hom-action of M over N given by

$$ {}^{e_1}a_2 = -{}^{e_2}a_1 = a_3, {}^{e_1}a_3 = -{}^{e_3}a_1 = a_2, \qquad {}^{e_1}a_4 = -{}^{e_4}a_1 = a_4, $$
$$ {}^{e_2}e_3 = -{}^{e_3}a_2 = a_1. $$

and the Hom-action of N over M given by

$$ {}^{a_1}e_2 = -{}^{a_2}e_1 = e_3, \quad {}^{a_1}e_3 = -{}^{a_3}e_1 = e_2, \quad {}^{a_2}e_3 = -{}^{a_3}e_2 = e_1. $$

are compatible on each other. Moreover ${}^{N}M = M$ and ${}^{M}N = N$. Keep in mind that the unwritten brackets and actions are trivial.

Definition

A central extension of a Hom-Lie algebra (L, α_L) is an epimorphism of Hom-Lie algebras $(K, \alpha_K) \stackrel{\pi}{\twoheadrightarrow} (M, \alpha_M)$ such that $[\mathrm{Ker}(\pi), K] = 0$, i. e. $\mathrm{Ker}(\pi) \subseteq \mathbb{Z}(K)$. It is called a universal central extension if in addition, for every central extension $(K', \alpha_{K'}) \stackrel{\pi'}{\twoheadrightarrow} (M, \alpha_M)$ there exists a unique homomorphism of Hom-Lie algebras $h : (K, \alpha_K) \to (K', \alpha_{K'})$ such that $\pi' \circ h = \pi$.

It is known by

J. M. Casas, M. A. Insua and N. Pacheco: On universal central extensions of Hom-Lie algebras,Hacet. J. Math. Stat. 2016; 44 (2): 277–288.

that a Hom-Lie algebra (M, α_M) admits a universal central extension if and only if (M, α_M) is perfect, given by

$$(H_2^{\alpha}(M), \alpha_{\mathfrak{uce}(M)|}) \rightarrowtail (\mathfrak{uce}(M), \alpha_{\mathfrak{uce}(M)}) \overset{u}{\twoheadrightarrow} (M, \alpha_M)$$

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where $\mathfrak{uce}(M)$ is the quotient vector space $M \wedge M/\mathrm{Im}(d_3)$, d_3 denotes the boundary of the Hom-Lie homology complex and $H_2^{\alpha}(M)$ denotes the second homology with trivial coefficients of (M, α_M).

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gives another description of the universal central extension via the non-abelian tensor product. Namely, given a perfect Hom-Lie algebra (M, α_M), then

$$(M \star M, \alpha_{M \star M}) \overset{\psi_M}{\twoheadrightarrow} (M, \alpha_M), \quad \psi_M(m \star m') = [m, m'],$$

is the universal central extension of (M, α_M).

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$$(M \star M, \alpha_{M\star M}) \overset{\psi_M}{\twoheadrightarrow} (M, \alpha_M), \quad \psi_M(m \star m') = [m, m'],$$

is the universal central extension of (M, α_M).
Since the universal central extension is unique up to isomorphisms, we have

$$(\mathfrak{uce}(M), \alpha_{\mathfrak{uce}(M)}) \cong (M \star M, \alpha_{M\star M})$$

Proposition

For a perfect Hom-Lie algebra (G, α_G) with surjective endomorphism α_G, there is an exact sequence of Hom-Lie algebras

$$(H, \alpha_H) \stackrel{\eta}{\longrightarrow} (\mathfrak{uce}(G) \star \mathfrak{uce}(G), \alpha) \stackrel{\psi_G * \psi_G}{\longrightarrow} (G \star G, \alpha_{G \star G}) \longrightarrow 0 \ ,$$

where $H = \left(H_2^{\alpha}(G) \star \mathfrak{uce}(G)\right) \rtimes \left(\mathfrak{uce}(G) \star H_2^{\alpha}(G)\right)$ *and* $\alpha = \alpha_{\mathfrak{uce}(G)} \star \alpha_{\mathfrak{uce}(G)}$.

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where $H = \left(H_2^{\alpha}(G) \star \mathfrak{uce}(G)\right) \rtimes \left(\mathfrak{uce}(G) \star H_2^{\alpha}(G)\right)$ *and* $\alpha = \alpha_{\mathfrak{uce}(G)} \star \alpha_{\mathfrak{uce}(G)}$.

Proof.

The result is a consequence of a Proposition presented before,for the short exact sequence,

$$0 \longrightarrow (H_2^{\alpha}(G), \alpha_{\mathfrak{uce}(G)|}) \longrightarrow (\mathfrak{uce}(G), \alpha_{\mathfrak{uce}(G)}) \xrightarrow{\psi_G} (G, \alpha_G) \longrightarrow 0\ .$$

Corollary

For a perfect Hom-Lie algebra (G, α_G) *with surjective endomorphism* α_G, *there is a central extension*

$$0 \longrightarrow \eta(H) \longrightarrow (\mathfrak{uce}(G) \star \mathfrak{uce}(G), \alpha) \stackrel{\psi_G * \psi_G}{\longrightarrow} (G \star G, \alpha_{G\star G}) \longrightarrow 0\ . \tag{1}$$

Theorem

For any perfect Hom-Lie algebra (G, α_G) with surjective endomorphism α_G, there is a surjective homomorphism of Hom-Lie algebras

$$\omega : (\mathfrak{uce}(G \star G), \alpha_{\mathfrak{uce}(G \star G)}) \twoheadrightarrow (\mathfrak{uce}(G) \star \mathfrak{uce}(G), \alpha) \ .$$

Theorem

For any perfect Hom-Lie algebra (G, α_G) with surjective endomorphism α_G, there is a surjective homomorphism of Hom-Lie algebras

$$\omega : (\mathfrak{uce}(G \star G), \alpha_{\mathfrak{uce}(G \star G)}) \twoheadrightarrow (\mathfrak{uce}(G) \star \mathfrak{uce}(G), \alpha) \ .$$

Proof.

$(G \star G, \alpha_{G \star G})$ is perfect, hence it admits the following universal central extension

$$(H_2^{\alpha}(G \star G), \alpha_{\mathfrak{uce}(G \star G)|}) \rightarrowtail (\mathfrak{uce}(G \star G), \alpha_{\mathfrak{uce}(G \star G)}) \overset{\psi_{G * G}}{\twoheadrightarrow} (G \star G, \alpha_{G \star G}).$$

□

Proof.

Now, the result follows from the following commutative diagram applied to the universal central extension constructed above:

$$\begin{array}{ccccc} (H_2^{\alpha}(G \star G), \alpha_{\mathfrak{uce}(G\star G)|}) & \longrightarrow & (\mathfrak{uce}(G \star G), \alpha_{\mathfrak{uce}(G\star G)}) & \xrightarrow{\psi_{G*G}} & (G \star G, \alpha_{G\star G}) \\ \downarrow{\scriptstyle \omega_|} & & \downarrow{\scriptstyle \omega} & & \\ \eta(H) & \longrightarrow & (\mathfrak{uce}(G) \star \mathfrak{uce}(G), \alpha) & \xrightarrow{\psi_G * \psi_G} & (G \star G, \alpha_{G\star G}) \end{array}$$

□

Corollary

Let (G, α_G) *be a perfect Hom-Lie algebra with surjective endomorphism* α_G *and* $H_2^{\alpha}(G \star G) = \mathrm{Im}(\eta)$*, then*

$$(\mathfrak{uce}(G \star G), \alpha_{\mathfrak{uce}(G \star G)}) \cong (\mathfrak{uce}(G) \star \mathfrak{uce}(G), \alpha).$$

Details of this talk can be found in:

J. M. Casas, E. Khmaladze and N. Pacheco: On some properties preserved by the non-abelian tensor product of Hom-Lie algebras, arXiv:1902.06538 (2019)

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Thank you for your attention!