Hom-associatively deformed Weyl algebras

International workshop on non-associative algebras in Porto, 2019

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Introduction

Hom-associative algebras – algebras with the associativity condition twisted by a *hom*omorphism – arose with hom-Lie algebras, introduced by Hartwig, Larsson, and Silvestrov [HLS06].

Non-commutative polynomial rings – or *Ore extensions* – were introduced by Ø. Ore Ore33, and generalized to non-associative such by Nystedt, Öinert, and Richter [NÖR18].

Naive idea – why not *hom-associative Ore extensions*?

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Preliminaries: hom-associative algebras

Definition (Hom-associative algebra)

A hom-associative algebra *over an associative, commutative, and unital ring R, is a triple $(M, \cdot, \alpha)$ consisting of an R-module M, a binary operation $\cdot\colon M \times M \to M$ linear over R in both arguments, and an R-linear map $\alpha\colon M \to M$, satisfying, for all $a, b, c \in M$,*

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c).$$

Definition (Weakly unital hom-associative algebra)

Let A be a hom-associative algebra. If for all $a \in A$,

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for some $e \in A$, we say that A is weakly unital *with* weak unit *e.*

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Proposition ([BRS18])

Any multiplicative hom-associative algebra can be embedded into a multiplicative, weakly unital hom-associative algebra.

Proposition ([Yau09])

Let A be a unital, associative algebra with unit 1_A, α an algebra endomorphism on A, and define $*\colon A \times A \to A$ for all $a, b \in A$ by

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$$\big[\alpha(a),[b,c]\big]+\big[\alpha(c),[a,b]\big]+\big[\alpha(b),[c,a]\big]=0.$$

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Definition (Left R-additivity)

If R is a non-associativ, non-unital ring, a map $\beta\colon R \to R$ is left R-additive *if for all $r, s, t \in R$, $r \cdot \beta(s+t) = r \cdot \big(\beta(s) + \beta(t)\big)$.*

Given a non-associative, non-unital ring R with left R-additive maps $\delta\colon R \to R$ and $\sigma\colon R \to R$, by a *non-associative, non-unital Ore extension* of R, $R[x;\sigma,\delta]$, we mean $\left\{\sum_{i\in\mathbb{N}} a_i x^i\right\}$, finitely many $a_i \in R$ non-zero, endowed with the addition

$$\sum_{i\in\mathbb{N}} a_i x^i + \sum_{i\in\mathbb{N}} b_i x^i := \sum_{i\in\mathbb{N}} (a_i + b_i) x^i, \quad a_i, b_i \in R,$$

two polynomials being equal iff their coefficients are, $\forall a, b \in R$,

$$ax^m \cdot bx^n := \sum_{i\in\mathbb{N}} \left(a \cdot \pi_i^m(b)\right) x^{i+n}.$$

Here π_i^m is the sum of all $\binom{m}{i}$ compositions of i copies of σ and $m-i$ copies of δ. For example $\pi_0^0 = \mathrm{id}_R$ and $\pi_1^2 = \sigma \circ \delta + \delta \circ \sigma$. Imposing distributivity of the multiplication over addition makes $R[x;\sigma,\delta]$ a ring.

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For instance,

$$ax^0 \cdot bx^0 = \sum_{i\in\mathbb{N}} \big(a \cdot \pi_i^0(b)\big)x^{i+0} = (a\cdot b)x^0, \text{ so } R \cong Rx^0,$$
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Assume $\alpha\colon R \to R$ is the twisting map of a non-unital, hom-associative ring R, and extend the map homogeneously to $R[X;\sigma,\delta]$. Assume further that α commutes with δ and σ, and that σ is an endomorphism and δ a σ-derivation. Then $R[X;\sigma,\delta]$ is hom-associative.

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Proposition ([BRS18])

Let $R[x;\sigma,\delta]$ be a non-unital, associative Ore extension of a non-unital, associative ring R, and $\alpha\colon R\to R$ a ring endomorphism that commutes with σ and δ, a σ-derivation. Then $\big(R[x;\sigma,\delta],*,\alpha\big)$ is a multiplicative, non-unital, hom-associative Ore extension with α extended homogeneously to $R[x;\sigma,\delta]$.

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The hom-associative Weyl algebras

The first (associative) Weyl algebra A_1 over a field K of characteristic zero is $K\langle x, y\rangle$ modulo $[x, y] := x \cdot y - y \cdot x = 1_{A_1}$, $A_1 \cong K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$.

Conjecture (Dixmier [Dix68])
All endomorphisms on A_1 are automorphisms.

The hom-associative Weyl algebras A_1^k are $(A_1, *, \alpha_k)$ where $\alpha_k(1_{A_1}) = 1_{A_1}$, $\alpha_k(x) := x$ and $\alpha_k(y) = y + k$ for $k \in K$ $(A_1 = A_1^0)$,

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Proposition ([BR19])

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Corollary ([BR19])

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Corollary ([BR19])

δ is a derivation on A_1^k for k nonzero if and only if $\delta = [cy + p(x), \cdot] = e^{-k\frac{\partial}{\partial y}}[cy + p(x), \cdot]_*$ for some $c \in K$ and $p(x) \in K[x]$.

Proposition ([BR19])

Any homomorphism $f\colon A_1^k \to A_1^l$ for $k, l \neq 0$ is an isomorphism of the form $f(x) = \frac{l}{k}x + c$, $f(y) = \frac{k}{l}y + p(x)$ for some $c \in K$ and $p(x) \in K[x]$.

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Formal hom-associative deformations

Definition (One-parameter formal hom-associative deformation)

A one-parameter formal hom-associative deformation *of a hom-associative algebra over R, $(M, \cdot_0, \alpha_0)$ is a hom-associative algebra over $R[[t]]$, $(M[[t]], \cdot_t, \alpha_t)$, where*

$$\cdot_t = \sum_{i \in \mathbb{N}} \cdot_i t^i, \quad \alpha_t = \sum_{i \in \mathbb{N}} \alpha_i t^i.$$

Proposition ([BR19])

A_1^k is a one-parameter formal hom-associative deformation of A_1.

Remark

A_1 is formally rigid as an associative *algebra.*

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$\left(A_1^k, [\cdot,\cdot]_*, \alpha_k\right)$ *is a one-parameter formal hom-Lie deformation of* $\left(A_1, [\cdot,\cdot], \mathrm{id}_{A_1}\right)$.

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Hom-noetherian Ore extensions

A family $\mathcal{F}$ of subsets of a set S satisfies the *ascending chain condition* if there is no properly ascending infinite chain $S_1 \subset S_2 \subset \ldots$ of subsets from $\mathcal{F}$. Furthermore is an element in $\mathcal{F}$ called a *maximal element* of $\mathcal{F}$ provided there is no subset of $\mathcal{F}$ that properly contains that element.

Proposition ([BR18])

Let R be a non-unital, hom-associative ring. Then the following conditions are equivalent:

(N1) R satisfies the ascending chain condition on its right (left) hom-ideals.

(N2) Any nonempty family of right (left) hom-ideals of R has a maximal element.

(N3) Any right (left) hom-ideal of R is finitely generated.

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Let R be a non-unital, hom-associative ring. Then the following conditions are equivalent:

(N1) R satisfies the ascending chain condition on its right (left) hom-ideals.

(N2) Any nonempty family of right (left) hom-ideals of R has a maximal element.

(N3) Any right (left) hom-ideal of R is finitely generated.

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Definition (Hom-noetherian ring)

A non-unital, hom-associative ring R is called right (left) hom-noetherian *if it satisfies the three equivalent conditions on its right (left) hom-ideals. If R satisfies the conditions on both its right and its left hom-ideals, it is called* hom-noetherian.

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Let $\alpha: R \to R$ be the twisting map of a unital, hom-associative ring R, and extend the map homogeneously to $R[x; \sigma, \delta]$. Assume further that α commutes with δ and σ, and that σ is an automorphism and δ a σ-derivation on R. If R is right (left) hom-noetherian, then so is $R[x; \sigma, \delta]$.

Corollary ([BR18])

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Example (Octonionic Weyl algebra [BR18])

Denote by $\mathbb{O}$ *the octonions;* $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$. *We define the* octonionic Weyl algebra *as* $\mathbb{O}[y][x; \mathrm{id}_{\mathbb{O}[y]}, \delta]$, *where* $\delta := \frac{\mathrm{d}}{\mathrm{d}y}$*; hence* $x \cdot y - y \cdot x = 1_{\mathbb{O}}$.

Proposition ([BR18])

The octonionic Weyl algebra is noetherian.

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