On Whitehead's quadratic functor for supermodules

Pilar Páez-Guillán

(Joint work with T. Fakhr Taha and M. Ladra)

International Workshop on Non-associative Algebras in Porto
29th April - 3rd May.

Universidade de Santiago de Compostela

FEDER
MTM2016-79661-P (MINECO)

1. Whitehead quadratic functor for supermodules

2. Whitehead quadratic functor for crossed modules

Index

1. Whitehead quadratic functor for supermodules

2. Whitehead quadratic functor for crossed modules

Whitehead quadratic functor for Λ-modules

D. Simson, A. Tyc.
Connected sequences of stable derived functors and their applications.
Dissertationes Math. **111** (1974).

Whitehead quadratic functor for Λ-modules

Let M be a Λ-module. The Whitehead Λ-module $\Gamma(M)$ is defined as the free module Λ^M generated by the elements

$$e_x, \; x \in M,$$

and subject to the relations

$$e_{\lambda x} = \lambda^2 e_x,$$
$$e_{\lambda x+y} + \lambda e_x + \lambda e_y = \lambda e_{x+y} + e_{\lambda x} + e_y,$$
$$e_{x+y+z} + e_x + e_y + e_z = e_{x+y} + e_{x+z} + e_{y+z}.$$

Whitehead quadratic functor for Λ-modules

Let M be a Λ-module. The Whitehead Λ-module $\Gamma(M)$ is defined as the free module Λ^M generated by the elements

$$e_x,\ x \in M,$$

and subject to the relations

$$\begin{aligned} e_{\lambda x} &= \lambda^2 e_x, \\ e_{\lambda x+y} + \lambda e_x + \lambda e_y &= \lambda e_{x+y} + e_{\lambda x} + e_y, \\ e_{x+y+z} + e_x + e_y + e_z &= e_{x+y} + e_{x+z} + e_{y+z}. \end{aligned}$$

$\varphi\colon M \longrightarrow N$ quadratic:

- $\varphi(\lambda x) = \lambda^2 \varphi(x)$;
- $b_\varphi\colon M \times M \longrightarrow N$ given by $b_\varphi(x, y) = \varphi(x + y) - \varphi(x) - \varphi(y)$ is bilinear.

Whitehead quadratic functor for Λ-modules

Let M be a Λ-module. The Whitehead Λ-module $\Gamma(M)$ is defined as the free module Λ^M generated by the elements

$$e_x, \; x \in M,$$

and subject to the relations

$$\begin{aligned} e_{\lambda x} &= \lambda^2 e_x, \\ e_{\lambda x+y} + \lambda e_x + \lambda e_y &= \lambda e_{x+y} + e_{\lambda x} + e_y, \\ e_{x+y+z} + e_x + e_y + e_z &= e_{x+y} + e_{x+z} + e_{y+z}. \end{aligned}$$

$\varphi\colon M \longrightarrow N$ quadratic:

- $\varphi(\lambda x) = \lambda^2 \varphi(x)$;
- $b_\varphi\colon M \times M \longrightarrow N$ given by $b_\varphi(x, y) = \varphi(x+y) - \varphi(x) - \varphi(y)$ is bilinear.

$$\begin{aligned} \gamma\colon M &\longrightarrow \Gamma(M) \\ x &\longmapsto \overline{e_x} \end{aligned}$$

Whitehead quadratic functor for Λ-modules

Let M be a Λ-module. The Whitehead Λ-module $\Gamma(M)$ is defined as the free module Λ^M generated by the elements

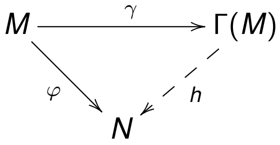
$$e_x, \ x \in M,$$

and subject to the relations

$$\begin{aligned} e_{\lambda x} &= \lambda^2 e_x, \\ e_{\lambda x+y} + \lambda e_x + \lambda e_y &= \lambda e_{x+y} + e_{\lambda x} + e_y, \\ e_{x+y+z} + e_x + e_y + e_z &= e_{x+y} + e_{x+z} + e_{y+z}. \end{aligned}$$

$\varphi\colon M \longrightarrow N$ quadratic:

- $\varphi(\lambda x) = \lambda^2 \varphi(x)$;
- $b_\varphi\colon M \times M \longrightarrow N$ given by $b_\varphi(x,y) = \varphi(x+y) - \varphi(x) - \varphi(y)$ is bilinear.



Let P be a Lie algebra.

$$\Gamma(P^{ab}) \xrightarrow{\psi} P \otimes P \xrightarrow{\pi} P \wedge P \longrightarrow 0.$$

Let P be a Lie algebra.

$$\Gamma(P^{ab}) \xrightarrow{\psi} P \otimes P \xrightarrow{\pi} P \wedge P \longrightarrow 0.$$

Lie superalgebras?

Let P be a Lie algebra.

$$\Gamma(P^{ab}) \xrightarrow{\psi} P \otimes P \xrightarrow{\pi} P \wedge P \longrightarrow 0.$$

Lie superalgebras?

We need to construct $\Gamma(M)$ for a supermodule $M = M_{\bar{0}} \oplus M_{\bar{1}}$.

Let P be a Lie algebra.

$$\Gamma(P^{ab}) \xrightarrow{\psi} P \otimes P \xrightarrow{\pi} P \wedge P \longrightarrow 0.$$

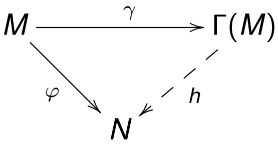
Lie superalgebras?

We need to construct $\Gamma(M)$ for a supermodule $M = M_{\bar{0}} \oplus M_{\bar{1}}$.

Problem: $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is not a $\mathbb{Z}/2\mathbb{Z}$-graded set, and we cannot construct the free supermodule generated by e_x for $x \in M$.

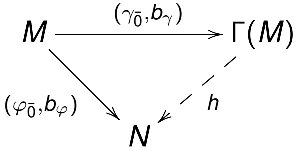
Whitehead quadratic functor for Λ-supermodules

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Λ-supermodule.



Whitehead quadratic functor for Λ-supermodules

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Λ-supermodule.

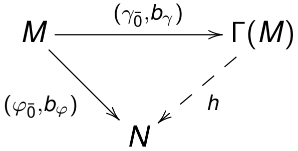


The pair $\varphi = (\varphi_{\bar{0}}, b_\varphi)$ is a quadratic mapping between M and N if:

- $\varphi_{\bar{0}} : M_{\bar{0}} \longrightarrow N_{\bar{0}}$ is quadratic (modules sense).

Whitehead quadratic functor for Λ-supermodules

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Λ-supermodule.



The pair $\varphi = (\varphi_{\bar{0}}, b_\varphi)$ is a quadratic mapping between M and N if:

- $\varphi_{\bar{0}} : M_{\bar{0}} \longrightarrow N_{\bar{0}}$ is quadratic (modules sense).
- $b_\varphi : M \times M \longrightarrow N$ is a bilinear mapping satisfying
    - $b_\varphi(x, y) = (-1)^{|x||y|} b_\varphi(y, x)$;
    - $b_\varphi(x_{\bar{1}}, x_{\bar{1}}) = 0$;
    - $b_\varphi(x_{\bar{0}}, y_{\bar{0}}) = \varphi_{\bar{0}}(x_{\bar{0}} + y_{\bar{0}}) - \varphi_{\bar{0}}(x_{\bar{0}}) - \varphi_{\bar{0}}(y_{\bar{0}})$.

E. Neher.
Quadratic Jordan superpairs covered by grids.
J. of Algebra **269** (2003) (1) 28–53.

An alternative construction

J. Helmstetter and A. Micali.
Quadratic mappings and Clifford algebras.
Birkhäuser Verlag, Basel, 2008.

An alternative construction

Let M be a Λ-module. We define $\Gamma'(M)$ as the sum

$$\Lambda^M \oplus (M \otimes M)$$

subject to the relations

$$(e_{\lambda x}, 0) = (\lambda^2 e_x, 0),$$
$$(e_{x+y} - e_x - e_y, 0) = (0, x \otimes y).$$

An alternative construction

Let M be a Λ-module. We define $\Gamma'(M)$ as the sum

$$\Lambda^M \oplus (M \otimes M)$$

subject to the relations

$$\begin{aligned} (e_{\lambda x}, 0) &= (\lambda^2 e_x, 0), \\ (e_{x+y} - e_x - e_y, 0) &= (0, x \otimes y). \end{aligned}$$

We have a quadratic map γ'

$$\begin{aligned} \gamma' \colon M &\longrightarrow \Gamma'(M) \\ x &\longmapsto \overline{(e_x, 0)} \end{aligned}$$

An alternative construction

Let M be a Λ-module. We define $\Gamma'(M)$ as the sum

$$\Lambda^M \oplus (M \otimes M)$$

subject to the relations

$$\begin{aligned} (e_{\lambda x}, 0) &= (\lambda^2 e_x, 0), \\ (e_{x+y} - e_x - e_y, 0) &= (0, x \otimes y). \end{aligned}$$

We have a quadratic map γ'

$$\begin{array}{ccc} M & \xrightarrow{\gamma'} & \Gamma'(M) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle h} & \\ & N & \end{array}$$

$$h(\overline{(e_x, y \otimes z)}) = \varphi(x) + \varphi(y+z) - \varphi(y) - \varphi(z).$$

An alternative construction

Let M be a Λ-module. We define $\Gamma'(M)$ as the sum

$$\Lambda^M \oplus (M \otimes M)$$

subject to the relations

$$(e_{\lambda x}, 0) = (\lambda^2 e_x, 0),$$
$$(e_{x+y} - e_x - e_y, 0) = (0, x \otimes y).$$

We have a quadratic map γ'

$$\begin{array}{ccc} M & \xrightarrow{\gamma'} & \Gamma'(M) \\ & {\scriptstyle\varphi}\searrow & \swarrow {\scriptstyle h} \\ & N & \end{array}$$

$$\Gamma(M) \cong \Gamma'(M).$$

$$h(\overline{(e_x, y \otimes z)}) = \varphi(x) + \varphi(y+z) - \varphi(y) - \varphi(z).$$

An alternative construction

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Λ-supermodule. We define $\Gamma(M)$ as the sum

$$\Lambda^{M_{\bar{0}}} \oplus (M \otimes M)$$

subject to the relations

$$\begin{aligned}(e_{\lambda x_{\bar{0}}}, 0) &= (\lambda^2 e_{x_{\bar{0}}}, 0),\\ (e_{x_{\bar{0}}+y_{\bar{0}}} - e_{x_{\bar{0}}} - e_{y_{\bar{0}}}, 0) &= (0, x_{\bar{0}} \otimes y_{\bar{0}}),\end{aligned}$$

We have a quadratic map γ

$$\begin{aligned}\gamma \colon M_{\bar{0}} &\longrightarrow \Gamma(M)_{\bar{0}} = \Lambda^{M_{\bar{0}}} \oplus (M_{\bar{0}} \otimes M_{\bar{0}}) \oplus (M_{\bar{1}} \otimes M_{\bar{1}})\\ x_{\bar{0}} &\longmapsto \overline{(e_{x_{\bar{0}}}, 0)}\end{aligned}$$

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Λ-supermodule. We define $\Gamma(M)$ as the sum

$$\Lambda^{M_{\bar{0}}} \oplus (M \otimes M)$$

subject to the relations

$$\begin{aligned}
(e_{\lambda x_{\bar{0}}}, 0) &= (\lambda^2 e_{x_{\bar{0}}}, 0),\\
(e_{x_{\bar{0}}+y_{\bar{0}}} - e_{x_{\bar{0}}} - e_{y_{\bar{0}}}, 0) &= (0, x_{\bar{0}} \otimes y_{\bar{0}}),\\
(0, x \otimes y) &= (-1)^{|x||y|}(0, y \otimes x),\\
(0, x_{\bar{1}} \otimes x_{\bar{1}}) &= 0.
\end{aligned}$$

We have a quadratic map Γ and a bilinear map b_γ

$$\begin{aligned}
\gamma \colon M_{\bar{0}} &\longrightarrow \Gamma(M)_{\bar{0}},\\
x_{\bar{0}} &\longmapsto \overline{(e_{x_{\bar{0}}}, 0)}
\end{aligned}
\qquad
\begin{aligned}
b_\gamma \colon M \times M &\longrightarrow \Gamma(M)\\
(x, y) &\longmapsto \overline{(0, x \otimes y)}
\end{aligned}$$

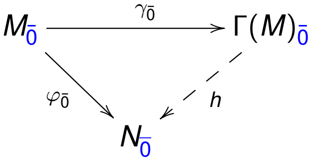
An alternative construction

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Λ-supermodule. We define $\Gamma(M)$ as the sum

$$\Lambda^{M_{\bar{0}}} \oplus (M \otimes M)$$

subject to the relations

$$\begin{aligned}
(e_{\lambda x_{\bar{0}}}, 0) &= (\lambda^2 e_{x_{\bar{0}}}, 0), \\
(e_{x_{\bar{0}}+y_{\bar{0}}} - e_{x_{\bar{0}}} - e_{y_{\bar{0}}}, 0) &= (0, x_{\bar{0}} \otimes y_{\bar{0}}), \\
(0, x \otimes y) &= (-1)^{|x||y|}(0, y \otimes x), \\
(0, x_{\bar{1}} \otimes x_{\bar{1}}) &= 0.
\end{aligned}$$



$$h(\overline{(e_{x_{\bar{0}}}, x \otimes y)}) = \varphi(x_{\bar{0}}) + b_\varphi(x, y)$$

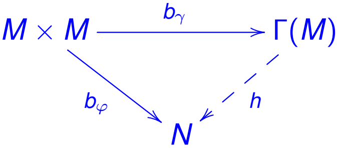
An alternative construction

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Λ-supermodule. We define $\Gamma(M)$ as the sum

$$\Lambda^{M_{\bar{0}}} \oplus (M \otimes M)$$

subject to the relations

$$\begin{aligned}
(e_{\lambda x_{\bar{0}}}, 0) &= (\lambda^2 e_{x_{\bar{0}}}, 0),\\
(e_{x_{\bar{0}}+y_{\bar{0}}} - e_{x_{\bar{0}}} - e_{y_{\bar{0}}}, 0) &= (0, x_{\bar{0}} \otimes y_{\bar{0}}),\\
(0, x \otimes y) &= (-1)^{|x||y|}(0, y \otimes x),\\
(0, x_{\bar{1}} \otimes x_{\bar{1}}) &= 0.
\end{aligned}$$



$$h(\overline{(e_{x_{\bar{0}}}, x \otimes y)}) = \varphi(x_{\bar{0}}) + b_\varphi(x, y)$$

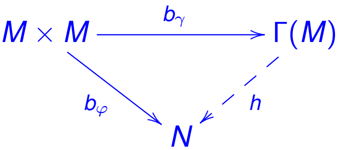
An alternative construction

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a Λ-supermodule. We define $\Gamma(M)$ as the sum

$$\Lambda^{M_{\bar{0}}} \oplus (M \otimes M)$$

subject to the relations

$$\begin{aligned}
(e_{\lambda x_{\bar{0}}}, 0) &= (\lambda^2 e_{x_{\bar{0}}}, 0), \\
(e_{x_{\bar{0}}+y_{\bar{0}}} - e_{x_{\bar{0}}} - e_{y_{\bar{0}}}, 0) &= (0, x_{\bar{0}} \otimes y_{\bar{0}}), \\
(0, x \otimes y) &= (-1)^{|x||y|}(0, y \otimes x), \\
(0, x_{\bar{1}} \otimes x_{\bar{1}}) &= 0.
\end{aligned}$$



$$h(\overline{(e_{x_{\bar{0}}}, x \otimes y)}) = \varphi(x_{\bar{0}}) + b_\varphi(x, y)$$

Functoriality in **SMod**:

$$\begin{array}{ccc} M & \xrightarrow{(\gamma^M, b^M_\gamma)} & \Gamma(M) \\ \varphi \downarrow & & \downarrow h = \Gamma(\varphi) \\ N & \xrightarrow{(\gamma^N, b^N_\gamma)} & \Gamma(N) \end{array}$$

Functoriality in **SMod**:

$$\begin{array}{ccc} M & \xrightarrow{(\gamma^M, b^M_\gamma)} & \Gamma(M) \\ \varphi \downarrow & & \downarrow h = \Gamma(\varphi) \\ N & \xrightarrow{(\gamma^N, b^N_\gamma)} & \Gamma(N) \end{array}$$

$$\Gamma(M \oplus N) \cong \Gamma(M) \oplus \Gamma(N) \oplus (M \otimes N).$$

Properties

Functoriality in **SMod**:

$$\begin{array}{ccc} M & \xrightarrow{(\gamma^M, b^M_\gamma)} & \Gamma(M) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle h=\Gamma(\varphi)} \\ N & \xrightarrow{(\gamma^N, b^N_\gamma)} & \Gamma(N) \end{array}$$

$$\Gamma(M \oplus N) \cong \Gamma(M) \oplus \Gamma(N) \oplus (M \otimes N).$$

If M is a free supermodule, $\Gamma(M)$ is also free.

Properties

Functoriality in **SMod**:

$$\begin{array}{ccc} M & \xrightarrow{(\gamma^M, b_\gamma^M)} & \Gamma(M) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle h=\Gamma(\varphi)} \\ N & \xrightarrow{(\gamma^N, b_\gamma^N)} & \Gamma(N) \end{array}$$

$$\Gamma(M \oplus N) \cong \Gamma(M) \oplus \Gamma(N) \oplus (M \otimes N).$$

If M is a free supermodule, $\Gamma(M)$ is also free.

If 2 has an inverse in Λ, then $S^2(M) \cong \Gamma(M)$.

Properties

Let P be a Lie superalgebra.

$$\Gamma(P^{ab}) \xrightarrow{\psi} P \otimes P \xrightarrow{\pi} P \wedge P \longrightarrow 0$$

being $\psi(\overline{(\gamma(x_{\bar{0}}), x \otimes y)}) = x_{\bar{0}} \otimes x_{\bar{0}} + x \otimes y + (-1)^{|x||y|} y \otimes x$.

Index

1. Whitehead quadratic functor for supermodules

2. Whitehead quadratic functor for crossed modules

$$? \longrightarrow (M,P,\partial)\otimes(M,P,\partial) \xrightarrow{\pi} (M,P,\partial)\wedge(M,P,\partial) \longrightarrow 0.$$

Crossed modules of Lie superalgebras

Definition

A crossed module of Lie superalgebras is a triple (M, P, ∂), where M and P are Lie superalgebras and

$$\partial \colon M \longrightarrow P$$

is a homomorphism, together with an action of P on M such that

- $\partial({}^{p}m) = [p, \partial(m)]$;
- ${}^{\partial(m)}m' = [m, m']$.

Crossed modules of Lie superalgebras

Definition

A crossed module of Lie superalgebras is a triple (M, P, ∂), where M and P are Lie superalgebras and

$$\partial\colon M \longrightarrow P$$

is a homomorphism, together with an action of P on M such that

- $\partial({}^{p}m) = [p, \partial(m)]$;
- ${}^{\partial(m)}m' = [m, m']$.

Definition

An abelian crossed module of Lie superalgebras is a triple (A, B, ∂) where A and B are abelian Lie superalgebras (i.e. supermodules) and $\partial\colon A \longrightarrow B$ is a homomorphism.

Morphisms of crossed modules

Definition

A morphism $(\xi, \varphi)\colon (M, P, \partial) \longrightarrow (N, Q, \sigma)$ of crossed modules is a pair of homomorphisms of Lie superalgebras

$$\xi\colon M \longrightarrow N, \qquad \varphi\colon P \longrightarrow Q$$

such that $\xi({}^{p}m) = {}^{\varphi(p)}\xi(m)$ and the following square is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\partial} & P \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ N & \xrightarrow{\sigma} & Q. \end{array}$$

Morphisms of crossed modules

Definition

A morphism $(\xi, \varphi)\colon (M, P, \partial) \longrightarrow (N, Q, \sigma)$ of abelian crossed modules is a pair of homomorphisms of Lie superalgebras

$$\xi\colon M \longrightarrow N, \qquad \varphi\colon P \longrightarrow Q$$

such that ~~$\xi({}^{p}m) = {}^{\varphi(p)}\xi(m)$~~ and the following square is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\partial} & P \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ N & \xrightarrow{\sigma} & Q. \end{array}$$

Whitehead functor for abelian crossed modules

T. Pirashvili.
Ganea term for CCG-homology of crossed modules.
Extracta Math. **15** (2000) (1) 231–235.

H. Ravanbod, A. R. Salemkar.
The non-abelian tensor and exterior products of crossed modules of Lie algebras.
J. Lie Theory **28** (2018) (1) 169–185.

Whitehead functor for abelian crossed modules

Let (A, B, ∂) be an abelian crossed module. We construct $B \underline{\otimes} A$ as the tensor product $B \otimes A$ subject to the relations

$$\partial(a) \otimes a' = (-1)^{|a||a'|} \partial(a') \otimes a;$$
$$\partial(a_{\bar{1}}) \otimes a_{\bar{1}} = 0.$$

Whitehead functor for abelian crossed modules

Let (A, B, ∂) be an abelian crossed module. We construct $B \underline{\otimes} A$ as the tensor product $B \otimes A$ subject to the relations

$$\partial(a) \otimes a' = (-1)^{|a||a'|}\partial(a') \otimes a;$$
$$\partial(a_{\bar{1}}) \otimes a_{\bar{1}} = 0.$$

$$f \colon A \otimes A \longrightarrow (B \underline{\otimes} A) \oplus \Gamma(A)$$
$$a \otimes a' \longmapsto (\overline{\partial(a) \otimes a'}, -\overline{(0, a \otimes a')}),$$

Whitehead functor for abelian crossed modules

Let (A, B, ∂) be an abelian crossed module. We construct $B \underline{\otimes} A$ as the tensor product $B \otimes A$ subject to the relations

$$\partial(a) \otimes a' = (-1)^{|a||a'|}\partial(a') \otimes a;$$
$$\partial(a_{\bar{1}}) \otimes a_{\bar{1}} = 0.$$

$$f\colon A \otimes A \longrightarrow (B \underline{\otimes} A) \oplus \Gamma(A)$$
$$a \otimes a' \longmapsto (\overline{\partial(a) \otimes a'}, -\overline{(0, a \otimes a')}),$$

$$\partial_\Gamma\colon \text{coker } f \longrightarrow \Gamma(B)$$
$$\overline{((\overline{b \otimes a}, \overline{(\gamma(a_{\bar{0}}), \alpha \otimes \alpha')}))} \longmapsto \overline{(\gamma(\partial(a_{\bar{0}})), b \otimes \partial(a) + \partial(\alpha) \otimes \partial(\alpha'))}$$

Whitehead functor for abelian crossed modules

Let (A, B, ∂) be an abelian crossed module. We construct $B\underline{\otimes}A$ as the tensor product $B \otimes A$ subject to the relations

$$\partial(a) \otimes a' = (-1)^{|a||a'|}\partial(a') \otimes a;$$
$$\partial(a_{\bar{1}}) \otimes a_{\bar{1}} = 0.$$

$$f\colon A \otimes A \longrightarrow (B\underline{\otimes}A) \oplus \Gamma(A)$$
$$a \otimes a' \longmapsto (\overline{\partial(a) \otimes a'}, -\overline{(0, a \otimes a')}),$$

$$\partial_\Gamma : \overline{\Gamma}(A, B, \partial) \longrightarrow \Gamma(B)$$
$$(\overline{(\overline{b \otimes a}, \overline{(\gamma(a_{\bar{0}}), \alpha \otimes \alpha')})}) \longmapsto \overline{(\gamma(\partial(a_{\bar{0}})), b \otimes \partial(a) + \partial(\alpha) \otimes \partial(\alpha'))}$$

Let (A, B, ∂) be an abelian crossed module. We construct $B\underline{\otimes}A$ as the tensor product $B \otimes A$ subject to the relations

$$\partial(a) \otimes a' = (-1)^{|a||a'|}\partial(a') \otimes a;$$
$$\partial(a_{\bar 1}) \otimes a_{\bar 1} = 0.$$

$$f\colon A \otimes A \longrightarrow (B\underline{\otimes}A) \oplus \Gamma(A)$$
$$a \otimes a' \longmapsto (\overline{\partial(a) \otimes a'}, -\overline{(0, a \otimes a')}),$$

$$\partial_\Gamma\colon \bar{\Gamma}(A, B, \partial) \longrightarrow \Gamma(B)$$
$$(\overline{(\overline{b \otimes a}, \overline{(\gamma(a_{\bar 0}), \alpha \otimes \alpha')})}) \longmapsto \overline{(\gamma(\partial(a_{\bar 0})), b \otimes \partial(a) + \partial(\alpha) \otimes \partial(\alpha'))}$$

$$\Gamma(A, B, \partial) = (\bar{\Gamma}(A, B, \partial), \Gamma(B), \partial_\Gamma)$$

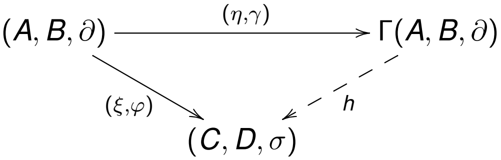
.

Properties

If ∂ is onto,

$$\Gamma((M,P,\partial)^{ab}) \longrightarrow (M,P,\partial)\otimes(M,P,\partial) \xrightarrow{\pi} (M,P,\partial)\wedge(M,P,\partial) \longrightarrow 0.$$

Universality of $\Gamma(A, B, \partial)$



Universality of $\Gamma(A, B, \partial)$

$$\begin{array}{ccc} (A, B, \partial) & \xrightarrow{(\eta,\gamma)} & \Gamma(A, B, \partial) \\ & {\scriptstyle (\xi,\varphi)} \searrow \quad \swarrow {\scriptstyle h} & \\ & (C, D, \sigma) & \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\partial} & B \\ {\scriptstyle \xi} \downarrow & & \downarrow {\scriptstyle \varphi} \\ C & \xrightarrow{\sigma} & D. \end{array}$$

Universality of $\Gamma(A, B, \partial)$

$$\begin{array}{ccc} (A, B, \partial) & \xrightarrow{(\eta,\gamma)} & \Gamma(A, B, \partial) \\ & {\scriptstyle (\xi,\varphi)} \searrow \quad \swarrow {\scriptstyle h} & \\ & (C, D, \sigma) & \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\partial} & B \\ {\scriptstyle (\xi_{\bar{0}}, b_\xi) = \xi} \downarrow & & \downarrow {\scriptstyle \varphi = (\varphi_{\bar{0}}, b_\varphi)} \\ C & \xrightarrow{\sigma} & D. \end{array}$$

Universality of $\Gamma(A, B, \partial)$

$$
\begin{array}{ccc}
(A, B, \partial) & \xrightarrow{(\eta,\gamma)} & \Gamma(A, B, \partial) \\
 {\scriptstyle (\xi,\varphi)} \searrow & & \swarrow {\scriptstyle h} \\
 & (C, D, \sigma) &
\end{array}
$$

$$
\begin{array}{ccc}
A & \xrightarrow{\partial} & B \\
{\scriptstyle (\xi_{\bar{0}}, b_\xi) = \xi} \downarrow & & \downarrow {\scriptstyle \varphi = (\varphi_{\bar{0}}, b_\varphi)} \\
C & \xrightarrow{\sigma} & D.
\end{array}
$$

$$
\begin{array}{ccc}
A_{\bar{0}} & \xrightarrow{\partial} & B_{\bar{0}} \\
{\scriptstyle \xi_{\bar{0}}} \downarrow & & \downarrow {\scriptstyle \varphi_{\bar{0}}} \\
C_{\bar{0}} & \xrightarrow{\sigma} & D_{\bar{0}}.
\end{array}
\qquad\qquad
\begin{array}{ccc}
B \times A & \xrightarrow{\mathrm{id} \times \partial} & B \times B \\
{\scriptstyle b_\xi} \downarrow & & \downarrow {\scriptstyle b_\varphi} \\
C & \xrightarrow{\sigma} & D.
\end{array}
$$

Universality of $\Gamma(A, B, \partial)$

$b_\xi : B \times A \longrightarrow C$ is bilinear and

- $b_\xi(\partial(a), a') = (-1)^{|a||a'|} b_\xi(\partial(a'), a)$;
- $b_\xi(\partial(a_{\bar{1}}), a_{\bar{1}}) = 0$;
- $b_\xi(\partial(a_{\bar{0}}), a'_{\bar{0}}) = \xi_{\bar{0}}(a_{\bar{0}} + a'_{\bar{0}}) - \xi_{\bar{0}}(a_{\bar{0}}) - \xi_{\bar{0}}(a'_{\bar{0}})$.

Universality of $\Gamma(A, B, \partial)$

$b_\xi : B \times A \longrightarrow C$ is bilinear and

- $b_\xi(\partial(a), a') = (-1)^{|a||a'|} b_\xi(\partial(a'), a)$;
- $b_\xi(\partial(a_{\bar{1}}), a_{\bar{1}}) = 0$;
- $b_\xi(\partial(a_{\bar{0}}), a'_{\bar{0}}) = \xi_{\bar{0}}(a_{\bar{0}} + a'_{\bar{0}}) - \xi_{\bar{0}}(a_{\bar{0}}) - \xi_{\bar{0}}(a'_{\bar{0}})$.

Quadratic mapping between abelian crossed modules

Properties

If (A, B, ∂) is free, $\Gamma(A, B, \partial)$ is not free, in general.

P. Carrasco, A. M. Cegarra, A. R.-Grandjeán.
(Co)homology of crossed modules.
J. Pure Appl. Algebra **168** (2002) (2-3) 147–176.

Properties

If (A, B, ∂) is free, $\Gamma(A, B, \partial)$ is not free, in general.

$X = X_{\bar{0}} \cup X_{\bar{1}}$.

Properties

If (A, B, ∂) is free, $\Gamma(A, B, \partial)$ is not free, in general.

$X = X_{\bar{0}} \cup X_{\bar{1}}$.

$(\oplus_X \Lambda, (\oplus_X \Lambda) \oplus (\oplus_X \Lambda), \iota_1)$. Free abelian crossed module on X.

Properties

If (A, B, ∂) is free, $\Gamma(A, B, \partial)$ is not free, in general.

$X = X_{\bar{0}} \cup X_{\bar{1}}$.

$(\oplus_X \Lambda, (\oplus_X \Lambda) \oplus (\oplus_X \Lambda), \iota_1)$. Free abelian crossed module on X.

$$X = \{*\}, \qquad \Lambda = \mathbb{Z}.$$

Properties

If (A, B, ∂) is free, $\Gamma(A, B, \partial)$ is not free, in general.

$X = X_{\bar{0}} \cup X_{\bar{1}}$.

$(\oplus_X \Lambda, (\oplus_X \Lambda) \oplus (\oplus_X \Lambda), \iota_1)$. Free abelian crossed module on X.

$$X = \{*\}, \qquad \Lambda = \mathbb{Z}.$$

$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \iota_1)$

Properties

If (A, B, ∂) is free, $\Gamma(A, B, \partial)$ is not free, in general.

$X = X_{\bar{0}} \cup X_{\bar{1}}$.

$(\oplus_X \Lambda, (\oplus_X \Lambda) \oplus (\oplus_X \Lambda), \iota_1)$. Free abelian crossed module on X.

$$X = \{*\}, \qquad \Lambda = \mathbb{Z}.$$

$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \iota_1)$

$\Gamma(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \iota_1)$,

Properties

If (A, B, ∂) is free, $\Gamma(A, B, \partial)$ is not free, in general.

$X = X_{\bar{0}} \cup X_{\bar{1}}$.

$(\oplus_X \Lambda, (\oplus_X \Lambda) \oplus (\oplus_X \Lambda), \iota_1)$. Free abelian crossed module on X.

$$X = \{*\}, \qquad \Lambda = \mathbb{Z}.$$

$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \iota_1)$

$\Gamma(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \iota_1)$,

$$\Gamma(\mathbb{Z} \oplus \mathbb{Z}) = \Gamma(\mathbb{Z}) \oplus \Gamma(\mathbb{Z}) \oplus \mathbb{Z} \otimes \mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

THANK YOU