Representations of equivariant map superalgebras

Joint works with I. Bagci, M. Brito and L. Calixto

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May 03, 2019

II International workshop on non-associative algebras - Porto

Introduction

Definition of EMSAs

An equivariant map superalgebra $(\mathfrak{g} \otimes A)^\Gamma$ is constructed in the following way. Let:

- $\Bbbk$ be an algebraically closed field of characteristic zero,
- $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra over $\Bbbk$,
- A be an associative, commutative, finitely-generated $\Bbbk$-algebra,
- Γ be a finite abelian group acting on $\mathfrak{g}$ and on A by automorphisms and freely on A. (In particular, Γ acts diagonally on $\mathfrak{g} \otimes A$.)

The equivariant map superalgebra $(\mathfrak{g} \otimes A)^\Gamma$ is the subalgebra of fixed points under the diagonal action of Γ on $\mathfrak{g} \otimes A$.

Examples

An affine Lie superalgebra is a Lie superalgebra of the form

$$\hat{\mathfrak{g}} := \left(\mathfrak{g} \otimes \mathbb{k}[t^{\pm 1}]\right)^{\sigma} \oplus \mathbb{k}c \oplus \mathbb{k}d,$$

where σ is an automorphism of the loop superalgebra $\mathfrak{g} \otimes \mathbb{k}[t^{\pm 1}]$ (coming from a Dynkin-Kac diagram automorphism), c is a central element, and d is a derivation.

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A twisted loop algebra $(\mathfrak{g} \otimes \Bbbk[t^{\pm 1}])^{\sigma}$ is a subalgebra of $\hat{\mathfrak{g}}$. Every $\hat{\mathfrak{g}}$-module in which c, d act trivially is in fact a $(\mathfrak{g} \otimes \Bbbk[t^{\pm 1}])^{\sigma}$-module. Moreover, there exists no finite-dimensional $\hat{\mathfrak{g}}$-module in which c, d act nontrivially.

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A twisted current algebra $(\mathfrak{g} \otimes \Bbbk[t])^{\sigma}$ is a subalgebra of $\hat{\mathfrak{g}}$, closely related to the parabolic subalgebra $(\mathfrak{g} \otimes \Bbbk[t])^{\sigma} \oplus \Bbbk c \oplus \Bbbk d$. In fact, the standard triangular decomposition of $\hat{\mathfrak{g}}$ is

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+, \quad \hat{\mathfrak{h}} = \mathfrak{h} \oplus \Bbbk c \oplus \Bbbk d, \;\; \hat{\mathfrak{n}}^{\pm} = (\mathfrak{n}^{\pm})^{\sigma} \oplus \left(\mathfrak{g} \otimes t^{\pm 1}\Bbbk[t^{\pm 1}]\right)^{\sigma}.$$

Finite-dimensional simple Lie superalgebras

Classification of finite-dimensional simple Lie superalgebras

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$\mathfrak{g}$	$\mathfrak{r}$	Type
$A(m,n),\ \ m > n \geq 0$	$A_m \oplus A_n \oplus \mathbb{k}$	Basic, type I
$A(n,n),\ \ n \geq 1$	$A_n \oplus A_n$	Basic, type I
$B(m,n),\ \ m \geq 0,\ n \geq 1$	$B_m \oplus C_n$	Basic, type II
$C(n+1),\ \ n \geq 1$	$C_n \oplus \mathbb{k}$	Basic, type I
$D(m,n),\ \ m \geq 2,\ n \geq 1$	$D_m \oplus C_n$	Basic, type II
$D(2,1;\alpha),\ \ \alpha \neq 0,-1$	$A_1 \oplus A_1 \oplus A_1$	Basic, type II
$F(4)$	$A_1 \oplus B_3$	Basic, type II
$G(3)$	$A_1 \oplus G_2$	Basic, type II
$H(n),\ \ n \geq 4$	B_n or D_n	Cartan
$S(n),\ \ n \geq 3$	A_{n-1}	Cartan
$\tilde{S}(n),\ \ n = 2m,\ m \geq 2$	A_{n-1}	Cartan
$W(n),\ \ n \geq 2$	$A_{n-1} \oplus \mathbb{k}$	Cartan
$\mathfrak{p}(n),\ \ n \geq 2$	A_n	Strange
$\mathfrak{q}(n),\ \ n \geq 2$	A_n	Strange

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Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Every finite-dimensional irreducible $\mathfrak{g}$-module is a highest-weight module, that is, a cyclic module cyclically generated by a nonzero vector v satisfying

$$\mathfrak{n}^+ v = 0, \qquad hv = \lambda(h)v \quad \text{for all} \;\; h \in \mathfrak{h} \;\; \text{and some} \;\; \lambda \in \mathfrak{h}^*.$$

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We will denote these modules by $L(\lambda)$, and denote the set of $\lambda \in \mathfrak{h}^*$ such that $L(\lambda)$ is finite-dimensional and nonzero by X^+.

Irreps vs. Kacs

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When $L(\lambda)$ is projective, the weight λ is called typical (otherwise, atypical). In the typical case, $L(\lambda)$ coincides with the generalized Kac module.

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Assume $\mathfrak{g} \not\cong \mathfrak{q}(n), \tilde{S}(n)$, and define $\mathfrak{r} = \begin{cases} \mathfrak{g}_0, & \text{if } \mathfrak{g} \text{ is of Cartan type,} \\ \mathfrak{g}_{\bar{0}}, & \text{otherwise.} \end{cases}$

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Every triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ induces a triangular decomposition $\mathfrak{r} = \mathfrak{n}_{\mathfrak{r}}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\mathfrak{r}}^+$. Denote by $R_{\mathfrak{r}}^+$ the associated set of positive roots, by $\Delta_{\mathfrak{r}}$ the associated set of simple roots, and by $\{x_\beta^\pm, h_\alpha \mid \beta \in R_{\mathfrak{r}}^+, \alpha \in \Delta_{\mathfrak{r}}\}$ a Chevalley basis of $\mathfrak{r}'$.

Generalized Kac modules

Given $\lambda \in X^+$, the generalized Kac module associated to λ is defined to be the cyclic $\mathfrak{g}$-module $K(\lambda)$ given as a quotient of $\mathbf{U}(\mathfrak{g})$ by the left ideal generated by

$$\mathfrak{n}^+, \qquad h - \lambda(h) \quad \text{for all } h \in \mathfrak{h}, \qquad (x_\alpha^-)^{\lambda(h_\alpha)+1} \quad \text{for all } \alpha \in \Delta_{\mathfrak{r}}.$$

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In fact, Kac only studied Kac modules for basic classical Lie algebras and certain triangular decompositions known as *distinguished*, that is, a triangular decomposition such that there is only one $\alpha \in \Delta_\mathfrak{r}$ such that $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq \{0\}$.

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The important feature of the distinguished case is that character formulas for "distinguished" Kac modules are easier to compute.

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- If $\mathfrak{g}$ is of Cartan type and $(\mathfrak{n}^- \cap \mathfrak{g}_{\bar{0}}) \subseteq \mathfrak{r}$, then $K(\lambda)$ is finite dimensional for all $\lambda \in X^+$.
- If $\mathfrak{g}$ is of Cartan type, $(\mathfrak{n}^- \cap \mathfrak{g}_{\bar{0}}) \not\subseteq \mathfrak{r}$ and $\mathfrak{r} + \mathfrak{n}^+$ is a subalgebra of $\mathfrak{g}$, then $K(\lambda)$ is infinite dimensional for all $\lambda \in X^+$.

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In an ongoing work (M. Brito, L. Calixto, M.), we proved that: if $\alpha \in \Delta$ is odd, Δ' is the set $(\Delta \setminus \{\alpha\}) \cup \{-\alpha\}$ of simple roots, $\mathfrak{b}'$ is the Borel subalgebra of $\mathfrak{g}$ associated to Δ', and $\lambda \in X_{\mathfrak{b}}^+$ is such that $\lambda(h_\alpha) \neq 0$, then there is an isomorphism of $\mathfrak{g}$-modules $K_{\mathfrak{b}}(\lambda) \cong K_{\mathfrak{b}'}(\lambda - \alpha)$.

Finite-dimensional irreducible modules for EMSAs

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In the Lie superalgebra case, it has been an effort of:

- A. Savage: basic classical cases,
- L. Calixto, A. Moura and A. Savage: queer case,
- I. Bagci: Cartan cases,
- L. Calixto, M.: periplectic case.

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- Basic classical type II case, periplectic case, $H(n)$, $S(n)$, $\tilde{S}(n)$: take $n_1 = \cdots = n_\ell = 1$. Then $\bigotimes_{k=1}^{\ell} \mathrm{ev}_{\mathsf{m}_k}^* L(\lambda_k)$ are all the finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules.

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- Basic classical type I and $W(n)$: maybe $n_1, \dots, n_\ell > 1$, but $\bigotimes_{k=1}^{\ell} \mathrm{ev}_{\mathsf{m}_k^{n_k}}^* V_k$ are all the finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules.

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- Queer case: take $n_1 = \cdots = n_\ell = 1$. However $\mathrm{ev}_{\mathsf{m}_i}^* V_i \otimes \mathrm{ev}_{\mathsf{m}_j}^* V_j$ may not be irreducible, but a direct sum of two copies of an irreducible module. Denote by $\mathrm{ev}_{\mathsf{m}_i}^* V_i \widehat{\otimes} \mathrm{ev}_{\mathsf{m}_j}^* V_j$ one of these irreducible copies. Then $\widehat{\bigotimes}_{k=1}^{\ell} \mathrm{ev}_{\mathsf{m}_k}^* V_k$ are all the finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules.

Extensions for EMSAs

(L. Calixto, M.) Let $\mathcal{V}(\pi) = \widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}^*_{\mathsf{m}_i^{n_i}} V_i$ and $\mathcal{V}(\pi') = \widehat{\bigotimes}_{i=1}^{\ell} \mathrm{ev}^*_{\mathsf{m}_i^{n_i}} V_i'$.
For each $i \in \{1, \ldots, \ell\}$, let d_i denote $\delta_{1,n_i} \dim_{A/\mathsf{m}_i} \mathsf{m}_i/\mathsf{m}_i^2$.

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(a) If V_i is not isomorphic to V'_i for two or more indices i, then

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(b) If V_i is isomorphic to V'_i for all but one index i, then

$$\mathrm{Ext}^1_{(\mathfrak{g}\otimes A)^\Gamma}(\mathcal{V}(\pi), \mathcal{V}(\pi'))^{\oplus 2^{\kappa(\pi)+\kappa(\pi')}} \cong \mathrm{Ext}^1_{\mathfrak{g}\otimes A/\mathsf{m}_i^{n_i}}(V_i, V'_i) \oplus \mathrm{Hom}_{\mathfrak{g}}\left(\mathfrak{g} \otimes V_i, V'_i\right)^{\oplus d_i}.$$

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Block decompositions

As a consequence, we obtained a block decomposition of the category of finite-dimensional $(\mathfrak{g} \otimes A)^\Gamma$-modules. (In the nonsuper case, this block decomposition had been obtained by E. Neher and A. Savage.)

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An open question remains: What are the finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathrm{m}^n$-modules ($n > 1$)? And what are their extensions?

Local Weyl modules

Similar to the relation between Kac and irreducible $\mathfrak{g}$-modules, there exist also universal highest-weight $\mathfrak{g} \otimes A$-modules, which are called local Weyl modules, and which map onto the irreducible $\mathfrak{g} \otimes A$-modules.

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Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. For each $\psi \in \mathfrak{h} \otimes A^*$ such that $\psi|_{\mathfrak{h}} \in X^+$, define the local Weyl module $W(\psi)$ to be the cyclic $\mathbf{U}(\mathfrak{g} \otimes A)$-module given as the quotient of $\mathbf{U}(\mathfrak{g} \otimes A)$ by the left ideal generated by

$$\mathfrak{n}^+ \otimes A, \quad H - \psi(H) \text{ for all } H \in \mathfrak{h} \otimes A, \quad (x_\alpha^-)^{\psi(h_\alpha)+1} \text{ for all } \alpha \in \Delta_{\mathfrak{r}}.$$

Finite-dimensionality of local Weyl modules

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In the superalgebra case, I. Bagci, L. Calixto and M. have proved that:

if $(\mathfrak{n}^- \cap \mathfrak{g}_{\bar{0}}) \subseteq \mathfrak{r}$ and $-\theta_{\mathfrak{g}} \in R_{\mathfrak{r}}$, then $W(\psi)$ is finite dimensional.

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In particular, if $\mathfrak{g}$ is of type II, then every local Weyl module is finite dimensional.

In the other hand, if $\mathfrak{g}$ is of Cartan type and the Borel subalgebra is such that $(\mathfrak{n}^- \cap \mathfrak{g}_{\bar{0}}) \not\subseteq \mathfrak{r}$ and $\mathfrak{r} + \mathfrak{n}^+$ is a subalgebra of $\mathfrak{g}$, then every local Weyl module is infinite dimensional.

Bases and characters of local Weyl modules

Of course, character and bases for finite-dimensional irreducible $(\mathfrak{g} \otimes A)^\Gamma$-modules will be known as soon as character and bases for finite-dimensional irreducible $\mathfrak{g} \otimes A/\mathrm{m}^n$-modules are known.

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We'd like to find character formulas and bases for local Weyl modules.

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For Lie superalgebras*, E. Feigin, I. Makedonskyi and D. Kus have worked out the case $\mathfrak{osp}(1,2)$.

Bases and characters of local graded Weyl modules

M. Brito, L. Calixto and M. are working on the case $\mathfrak{g} = \mathfrak{sl}(1,2)$ with a choice of a good (non-distinguished) Borel subalgebra, $A = \mathbb{k}[t]$, and $\psi(\mathfrak{h} \otimes t\mathbb{k}[t]) = 0$. In this case, $W(\psi)$ is called local graded Weyl module and ψ depends only on $\psi|_{\mathfrak{h}} \in X^+$.

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In particular, we obtained that:

- $W(\lambda) \cong_{\Bbbk} K(\kappa_1, 1) \otimes \cdots \otimes K(\kappa_{\lambda_2}, 1)$,
- $\dim W(\lambda) = 4^{\lambda_2}$,
- $\mathrm{ch} W(\lambda) = e^{-\lambda_2}(e+1)^{2\lambda_2}$,
- $\mathrm{sch} W(\lambda) = e^{-\lambda_2}(e-1)^{2\lambda_2}$,

where $\lambda \in X^+$ and $\lambda_2 = \lambda \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right)$.

Thanks :)