The classification of nilpotent bicommutative algebras

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joint work with Ivan Kaygorodov and Pilar Páez-Guillán

30 April 2019

Content

Bicommutative algebras

Definition

Bicommutative algebras:

$$(xy)z = (xz)y \text{ -- right commutative} \quad (1)$$

$$x(yz) = y(xz) \text{ -- left commutative} \quad (2)$$

Previous works

1. First appearance of one-sided:
   Cayley A., *The theory of analytical forms called trees*, Phil. Mag., **13** (1857), 19–30; Collected Math. Papers, University Press, Cambridge, **3** (1890), 242–246.
2. $\mathrm{Bicom}^2 \subseteq \mathrm{CommAssoc}$. Simple bicommutative is field:
   Dzhumadildaev A. and Tulenbaev K., *Bicommutative algebras*, Russian Mathematical Surveys, **58** (2003), 6, 1196–1197.
3. Dzhumadildaev A., Ismailov N. and Tulenbaev K., *Free bicommutative algebras*, Serdica Mathematical Journal, **37** (2011), 1, 25–44.
4. $\mathrm{Bicom}^{(+)} \subseteq \mathrm{Comm}$:
   Dzhumadildaev A. and Ismailov N., *Polynomial identities of bicommutative algebras, Lie and Jordan elements*, Communications in Algebra, **46** (2018), 12, 5241–5251.

Previous works

1. $\text{Bicom}^{(-)} \subseteq \text{MetabelianLie}$:
   Burde D., Dekimpe K. and Deschamps S., *LR-algebras*, New developments in Lie theory and geometry, 125—140, Contemp. Math., **491**, Amer. Math. Soc., Providence, RI, 2009.
2. Drensky V. and Zhakhayev B., *Noetherianity and Specht problem for varieties of bicommutative algebras*, Journal of Algebra, **499** (2018), 1, 570–582.

Examples

1. commutative associative algebras
2. $\mathcal{B}$ — bicommutative algebra, then $\mathcal{B}^{\mathrm{op}}$ — bicommutative algebra.

$$a \circ b = ba$$

4-dimensional nilpotent non-pure bicommutative algebras

Definition

$(xy)z = 0$ and $x(yz) = 0$ — non-pure bicommutative algebra.

Dimension 4: non-anticommutative can be found in:

1. Demir I., Misra K. and Stitzinger E., *On classification of four-dimensional nilpotent Leibniz algebras*, Communications in Algebra, **45** (2017), 3, 1012–1018.

Only one nilpotent and anticommutative.

Our goal

Problem

Classify all 4-dimensional nilpotent pure bicommutative algebras over $\mathbb{C}$.

Nilpotent bicommutative algebras of small dimensions

- Nilpotent algebras of dimension 1 are trivial.
- There is only one non-trivial nilpotent algebra of dimension 2 and it is bicommutative:

$$\mathcal{B}^{2*}_{01} \quad : \quad e_1 e_1 = e_2.$$

- There is 5 non-trivial nilpotent algebras of dimension 3, which is bicommutative:

$$\begin{array}{lllll}
\mathcal{B}^{3*}_{02} & : & e_1 e_1 = e_3, & e_2 e_2 = e_3; & \\
\mathcal{B}^{3*}_{03} & : & e_1 e_2 = e_3, & e_2 e_1 = -e_3; & \\
\mathcal{B}^{3*}_{04}(\alpha)_{\alpha\neq 0} & : & e_1 e_1 = \alpha e_3, & e_2 e_1 = e_3, & e_2 e_2 = e_3; \\
\mathcal{B}^{3*}_{04}(0) & : & e_1 e_2 = e_3; & & \\
\mathcal{B}^{3}_{01} & : & e_1 e_1 = e_2, & e_2 e_1 = e_3; & \\
\mathcal{B}^{3}_{02}(\alpha) & : & e_1 e_1 = e_2, & e_1 e_2 = e_3, & e_2 e_1 = \alpha e_3.
\end{array}$$

The last item follows from

1. Calderón Martín A., Fernández Ouaridi A., Kaygorodov I., *The classification of bilinear maps with radical of codimension 2*, arXiv:1806.07009

Procedure

$\mathbf{A}'$ — (nilpotent) bicommutative algebra of dimension $n - s$.
To classify all n-dimensional central extensions of $\mathbf{A}'$:

1. Determine $\mathrm{H}^2(\mathbf{A}', \mathbb{C})$, $\mathrm{Ann}(\mathbf{A}')$ and $\mathrm{Aut}(\mathbf{A}')$.
2. Determine the set of $\mathrm{Aut}(\mathbf{A}')$-orbits on $T_s(\mathbf{A}')$.
3. For each orbit, construct the bicommutative algebra associated with a representative of it.

Procedure from:

1. Hegazi A., Abdelwahab H., Calderón Martín A., The classification of n-dimensional non-Lie Malcev algebras with $(n-4)$-dimensional annihilator, Linear Algebra and its Applications, 505 (2016), 32–56.

Main result

Theorem (Kaygorodov, Páez-Guillán, V., 2019)

Let $\mathcal{B}$ be a nonzero 4-dimensional nilpotent pure bicommutative algebra over $\mathbb{C}$. Then, $\mathcal{B}$ is isomorphic to one of the algebras listed in Table.

Table

$\mathcal{B}$	Multiplication table					
$\mathcal{B}^4_{01}$	$e_1e_1 = e_2$	$e_2e_1 = e_3$				
$\mathcal{B}^4_{02}(\alpha)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = \alpha e_3$			
$\mathcal{B}^4_{03}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$			
$\mathcal{B}^4_{04}(\alpha)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = \alpha e_4$	$e_3e_3 = e_4$		
$\mathcal{B}^4_{05}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_3e_3 = e_4$	
$\mathcal{B}^4_{06}(\alpha \neq 0)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	$e_2e_1 = \alpha e_4$		
$\mathcal{B}^4_{07}$	$e_1e_1 = e_2$	$e_2e_1 = e_4$	$e_3e_3 = e_4$			
$\mathcal{B}^4_{08}$	$e_1e_1 = e_2$	$e_1e_3 = e_4$	$e_2e_1 = e_4$			
$\mathcal{B}^4_{09}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_3e_1 = e_4$			
$\mathcal{B}^4_{10}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_3e_2 = e_4$		
$\mathcal{B}^4_{11}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_3e_2 = e_4$			
$\mathcal{B}^4_{12}$	$e_1e_2 = e_3$	$e_1e_1 = e_4$	$e_2e_1 = e_4$	$e_3e_2 = e_4$		
$\mathcal{B}^4_{13}$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_4$			
$\mathcal{B}^4_{14}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_2e_2 = e_4$		
$\mathcal{B}^4_{15}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$			
$\mathcal{B}^4_{16}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_4$			
$\mathcal{B}^4_{17}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$				
$\mathcal{B}^4_{18}$	$e_1e_2 = e_3$	$e_1e_1 = e_4$	$e_3e_2 = e_4$			
$\mathcal{B}^4_{19}$	$e_1e_2 = e_3$	$e_3e_2 = e_4$				
$\mathcal{B}^4_{20}$	$e_1e_1 = e_2$	$e_2e_1 = e_3$	$e_1e_2 = e_4$	$e_3e_1 = e_4$		
$\mathcal{B}^4_{21}$	$e_1e_1 = e_2$	$e_2e_1 = e_3$	$e_3e_1 = e_4$			
$\mathcal{B}^4_{22}$	$e_1e_1 = e_2,$	$e_1e_2 = e_3,$	$e_1e_3 = e_4,$	$e_2e_1 = e_4$		
$\mathcal{B}^4_{23}$	$e_1e_1 = e_2,$	$e_1e_2 = e_3,$	$e_1e_3 = e_4,$	$e_2e_1 = e_3 + e_4,$	$e_2e_2 = e_4,$	$e_3e_1 = e_4$
$\mathcal{B}^4_{24}(\alpha)$	$e_1e_1 = e_2,$	$e_1e_2 = e_3,$	$e_1e_3 = e_4,$	$e_2e_1 = \alpha e_3,$	$e_2e_2 = \alpha e_4,$	$e_3e_1 = \alpha e_4$

Geometric classification

If we fix a basis $e_1, \ldots, e_n$ of $\mathbb{V}$, then any $\mu \in Hom(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is determined by n^3 structure constants $c_{ij}^k \in \mathbb{C}$ such that

$$\mu(e_i \otimes e_j) = \sum_{k=1}^{n} c_{ij}^k e_k$$

Definition

A subset of $Hom(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is called *Zariski-closed* if it can be defined by a set of polynomial equations in the variables c_{ij}^k $(1 \leq i, j, k \leq n)$.

Let T be such a set of polynomial identities. It holds that every algebra structure on $\mathbb{V}$ satisfying polynomial identities from T forms a Zariski-closed subset of the variety $Hom(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$; it is denoted by $\mathbb{L}(T)$.

We will denote by $O(\mu)$ the orbit of $\mu \in \mathbb{L}(T)$ under the action of $GL(\mathbb{V})$, and by $\overline{O(\mu)}$ the Zariski closure of $O(\mu)$.

Definitions

Definition

$\mu, \lambda \in \mathbb{L}(T)$ represent $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that $\mathcal{A}$ *degenerates* to $\mathcal{B}$, and write $\mathcal{A} \to \mathcal{B}$, if $\lambda \in \overline{O(\mu)}$.

Definition

We call $\mathcal{A}$ *rigid* in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$.

How to find degeneration or non-degeneration

If there exist $a_i^j(t) \in \mathbb{C}$ $(1 \le i, j \le n,\ t \in \mathbb{C}^*)$ such that $E_i^t = \sum_{j=1}^{n} a_i^j(t) e_j$ $(1 \le i \le n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$, and the structure constants of μ in the basis $E_1^t, \dots, E_n^t$ are such polynomials $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then $\mathcal{A} \to \mathcal{B}$.

$E_1^t, \dots, E_n^t$ — *parametrized basis* for $\mathcal{A} \to \mathcal{B}$.

Theorem

If $\mathcal{A} \to \mathcal{B}$ then

1. $dim(Der(\mathcal{A})) < dim(Der(\mathcal{B}))$
2. $dim(\mathcal{A}^2) \ge dim(\mathcal{B}^2)$
3. $dim(Ann(\mathcal{A})) \le dim(Ann(\mathcal{B}))$

Geometric classification

Theorem (Kaygorodov, Páez-Guillán, V., 2019)

The variety of 4-dimensional nilpotent bicommutative algebras has two irreducible components defined by the rigid algebra $\mathcal{B}^4_{10}$ and the infinite family of algebras $\mathcal{B}^4_{24}(\alpha)$.

1. Kaygorodov I., Páez-Guillán P., Voronin V., *The algebraic and geometric classification of nilpotent bicommutative algebras*, 2019, arXiv:1903.08997

Thanks for your attention!
Obrigado pela atenção!

Definitions

Definition

We define the $\mathbb{C}$-linear space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ as the set of all bilinear maps $\theta\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ such that

$$\theta(xy, z) = \theta(xz, y),$$

$$\theta(x, yz) = \theta(y, xz).$$

These maps will be called *cocycles*.

Definition

Consider a linear map f from $\mathbf{A}$ to $\mathbb{V}$, and set $\delta f\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ with $\delta f(x, y) = f(xy)$. Then, δf is a cocycle, and we define $\mathrm{B}^2(\mathbf{A}, \mathbb{V}) = \{\theta = \delta f \ : f \in \mathit{Hom}(\mathbf{A}, \mathbb{V})\}$, a linear subspace of $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$; its elements are called *coboundaries*.

Definition

The *second cohomology space* $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$ is defined to be the quotient space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V}) \big/ \mathrm{B}^2(\mathbf{A}, \mathbb{V})$.

Definitions

Definition

The *Grassmannian* $G_k(\mathbb{V})$ is the set of all k-dimensional linear subspaces of $\mathbb{V}$. Define $G_s\left(\mathrm{H}^2(\mathbf{A},\mathbb{C})\right)$ as the Grassmannian of subspaces of dimension s in $\mathrm{H}^2(\mathbf{A},\mathbb{C})$.

Definition

$$T_s(\mathbf{A}) = \left\{ W = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in G_s\left(\mathrm{H}^2(\mathbf{A},\mathbb{C})\right) : \bigcap_{i=1}^{s} \mathrm{Ann}(\theta_i) \cap \mathrm{Ann}(\mathbf{A}) = 0 \right\}$$