TOPOLOGICAL MIRROR SYMMETRY FOR PARABOLIC HIGGS BUNDLES

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Abstract. We prove the topological mirror symmetry conjecture of Hausel–Thaddeus [20, 21] for the moduli space of strongly parabolic Higgs bundles of rank two and three, with full flags. Although the main theorem is proved only for rank at most three, most of the results are proved for any prime rank.

1. Introduction

The Hitchin system is an algebraic completely integrable system. Since it was introduced by Hitchin [22, 23] thirty years ago, it has been the subject of much interest, and it has turned out to have profound connections with several other areas of mathematics. Its basic ingredient is the moduli space $\mathcal{M}_d(G)$ of G-Higgs bundles (V,φ) of fixed topological type d on a closed Riemann surface X for a connected complex reductive group G. Here V is a holomorphic principal G-bundle on X and φ is a holomorphic 1-form with values in $\mathrm{Ad}(V)$. This moduli space is a holomorphic symplectic manifold carrying a hyper-Kähler metric. The integrable system is given by the Hitchin map $h\colon \mathcal{M}(G)\to\mathcal{A}$, where the Hitchin base $\mathcal{A}$ is an affine space whose dimension is half that of $\mathcal{M}(G)$ and the components of h are the coefficients of the characteristic polynomial of φ.

Mirror symmetry for the Hitchin system was introduced in the work of Hausel and Thaddeus [21] (announced in [20]). It involves the Hitchin systems for the pair of Langlands dual groups $G = \mathrm{SL}(n,\mathbb{C})$ and $G^L = \mathrm{PGL}(n,\mathbb{C})$. Hausel and Thaddeus show that the moduli spaces are mirror partners in the sense of Strominger–Yau–Zaslow (SYZ) [39]; since they consider the case when n and the degree $d = \deg(V)$ are coprime, this requires equipping the moduli spaces with suitable B-fields, or gerbes. Hausel and Thaddeus also show that, in the cases $n = 2, 3$, the moduli spaces satisfy topological mirror symmetry, which is an identity of suitably defined stringy E-polynomials (these encode stringy Hodge numbers and again involve the B-field). Moreover, they conjecture that this holds for any n and d with $(n, d) = 1$.

It was proved by Donagi–Pantev [13] that, more generally, SYZ mirror symmetry is satisfied by the moduli spaces (or stacks) of G-Higgs bundles for any complex reductive group G. On the other hand, a very recent preprint by Groechenig–Wyss–Ziegler [17] uses p-adic integration to prove topological mirror symmetry in the case $G = \mathrm{SL}(n,\mathbb{C})$ for any n and d with $(n, d) = 1$.

Parabolic Higgs bundles were introduced by Simpson [37] as the natural objects to consider for extending non-abelian Hodge theory to punctured Riemann surfaces. They are pairs consisting of a parabolic bundle V, i.e., a vector bundle with weighted flags in

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the fibers over fixed marked points in X, and a Higgs field φ with values in the parabolic endomorphisms of V.

The theory of parabolic Higgs bundles is in many way analogous to that of usual Higgs bundles. In particular, there is a parabolic version of the Hitchin system, which goes back to the work of Bottacin [11], Markman [30] and Nasatyr–Steer [33]. Parabolic Higgs bundles have subsequently been studied by many authors; we merely point to Boden–Yokogawa [10] and Logares–Martens [29] as convenient references for parabolic Higgs bundles and the parabolic Hitchin system. We emphasize that in this paper we consider exclusively *strongly* parabolic Higgs bundles, meaning that the residue of the Higgs field at the marked points is nilpotent. These provide the most immediate generalization of the Hitchin system in that their moduli spaces are symplectic leaves of more general (Poisson) moduli spaces of (non-strongly) parabolic Higgs bundles.

In the announcement [20] Hausel and Thaddeus also consider the parabolic case, and outline a proof that SYZ mirror symmetry holds for any n. Biswas–Dey [8] later proved this in the case when the moduli spaces of parabolic Higgs bundles are equipped with the natural gerbe, analogously to the non-parabolic case considered in [21]. In [20] Hausel and Thaddeus also state that topological mirror symmetry holds for parabolic Higgs bundles in the case $G = \mathrm{SL}(n, \mathbb{C})$ with $n = 2, 3$, and conjecture that it should be true for any n. The main result of the present paper is a proof of this conjecture for $n = 2, 3$ (Theorem 3.13 below). For simplicity we restrict ourselves to the case of full flags, though our calculations of E-polynomials can in fact be carried through in the general case.

Our proof follows the basic strategy of Hausel and Thaddeus. It rests on the observation that it suffices to prove that certain contributions on each side are identical in order to conclude that the full stringy E-polynomials coincide for the $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$ moduli spaces. On the $\mathrm{PGL}(n, \mathbb{C})$-moduli space, the relevant contribution to the stringy E-polynomial comes from the fixed loci in the moduli space by the natural action of non-trivial elements of the group Γ_n of n-torsion points of $\mathrm{Pic}^0(X)$. On the $\mathrm{SL}(n, \mathbb{C})$-moduli space the relevant contribution is the part of the E-polynomial which is *not invariant* under the action of Γ_n, also known as the *variant part*, and which is determined by the variant part of the E-polynomial of certain fixed point subvarieties under the natural $\mathbb{C}^*$-action.

The description of the fixed loci of elements of Γ_n is broadly parallel to that of [21] and essentially rests on the work of Narasimhan–Ramanan [32]. The result is that the fixed point loci are described in terms of Prym varieties of unramified covers of X modulo the action of the Galois group. However, in the parabolic situation, it turns out that this action can be absorbed in the parabolic data, and this simplifies the arguments somewhat compared to the non-parabolic situation.

On the other hand, the fixed points of the $\mathbb{C}^*$-action are so-called Hodge bundles. These are Higgs bundles whose underlying vector bundle has a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_l$ with respect to which the Higgs field φ has weight one. For rank $n = 2, 3$, it is known that only fixed loci consisting of Hodge bundles whose summands are all line bundles contribute to the variant E-polynomial, but the corresponding result for higher prime rank — that only $\mathbb{C}^*$-fixed loci of type $(1, 1, \ldots, 1)$ contribute to the variant E-polynomial — is not known to be true. This is the only missing step for generalising our proof to any prime rank n, since our calculations are done for every such n. This is completely analogous to the non-parabolic case as treated in [21], since also there the only missing step for proving topological mirror symmetry for any prime rank was exactly the same.

It turns out that the B-field does not play a very prominent role in the parabolic situation. Indeed, for SYZ mirror symmetry to hold in the strict sense, i.e., without a B-field, it is required that there is a Lagrangian section of the fibration, providing the natural base points of the abelian varieties which are the fibers of the integrable system. In degree zero this is provided by the Hitchin section (see Biswas–Arés-Gastesi–Govindarajan [7] for the parabolic version). Moreover, there is an isomorphism between moduli spaces of parabolic Higgs bundles for any two different degrees (requiring an adjustment of the weights), as long as at least one of the flags are full. This provides the Lagrangian section in any other degree. One thus also expects the usual (i.e. without the B-field correction) stringy E-polynomials to agree, and this is indeed what we show to be the case. On the other hand, one could of course consider the B-field twisted stringy E-polynomials, and as expected our calculations indicate that the end result is the same.

Here is an outline of the contents of the paper. Section 2 reviews basic facts about parabolic Higgs bundles and their moduli. We also recall how in the parabolic setting moduli spaces for different degrees d are isomorphic (with a change in parabolic weights). In Section 3 we recall the SYZ mirror symmetry result for the parabolic Hitchin system, review the stringy E-polynomials, describe the topological mirror symmetry conjecture of Hausel–Thaddeus, and state and prove our result. Section 4 is devoted to the calculation of the contribution to the variant part of the E-polynomial of the $\mathrm{SL}(n,\mathbb{C})$-moduli space arising only from the $\mathbb{C}^*$-fixed point loci of type $(1,1,\dots,1)$. In Section 5 we recall some classical results on Prym varieties of unramified covers. These are used in Section 6, where the contribution from the fixed point loci of non-trivial elements of Γ_n to the stringy E-polynomial of $\mathrm{PGL}(n,\mathbb{C})$-moduli space is calculated.

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2. Parabolic Higgs bundles and their moduli

In this section we recall basic facts about parabolic Higgs bundles and their moduli spaces.

2.1. **Parabolic vector bundles.** Denote by X a smooth projective curve over $\mathbb{C}$, and mark it with distinct points labeled by the divisor

$$D = p_1 + \cdots + p_{|D|},$$

with $p_i \neq p_j$ for $i \neq j$ and where $|D| = \deg D$. Let g be the genus of X and assume $g \geqslant 2$. This data will be fixed throughout.

Parabolic vector bundles on X associated to D, are vector bundles together with extra structure over each point of D.

Definition 2.1. A holomorphic *parabolic vector bundle of rank n on X, associated to the divisor D*, is a holomorphic vector bundle V of rank n over X, endowed with a *parabolic structure along D*. By this is meant a collection of weighted flags of the fibers of V over each point $p \in D$:

$$\begin{aligned} V_p = V_{p,1} \supsetneq V_{p,2} \supsetneq \cdots \supsetneq V_{p,s_p} \supsetneq V_{p,s_p+1} = \{0\}, \\ 0 \leqslant \alpha_1(p) < \cdots < \alpha_{s_p}(p) < 1, \end{aligned} \tag{2.1}$$

where s_p is an integer between 1 and n. The real number $\alpha_i(p) \in [0,1)$ is *the weight of the subspace* $V_{p,i}$. The *multiplicity of the weight* $\alpha_i(p)$ is the number $m_i(p) = \dim(V_{p,i}/V_{p,i+1})$, thus $\sum_i m_i(p) = n$. The data given only by the flags over D (i.e., without the weights) is called the *quasi-parabolic structure* of V. The parabolic structure is obtained from a quasi-parabolic structure by specifying the weights. The *type of the quasi-parabolic structure* is $\boldsymbol{m}$, where $\boldsymbol{m} = (m_1(p), \dots, m_{s_p}(p))_{p\in D}$, is the collection of all multiplicities over all points of D. The *type of the parabolic structure* is $(\boldsymbol{m}, \boldsymbol{\alpha})$, with $\boldsymbol{\alpha} = (\alpha_1(p), \dots, a_{s_p}(p))_{p\in D}$ being the collection of all weights. The *type* of the parabolic vector bundle is $(n, d, \boldsymbol{m}, \boldsymbol{\alpha})$, where $d = \deg(V)$ is its degree. Finally, a flag over a point $p \in D$ is *full* if $s_p = n$ or, equivalently, $m_i(p) = 1$ for all i.

We shall denote a parabolic vector bundle by just V whenever the parabolic structure is clear from the context.

Remark 2.2. Given a parabolic vector bundle V, with parabolic structure of type $(\boldsymbol{m}, \boldsymbol{\alpha})$, and a line bundle L the tensor product $V \otimes L$ acquires a parabolic structure, of the same type $(\boldsymbol{m}, \boldsymbol{\alpha})$, in the obvious way, i.e., by taking the flags on $V \otimes L$ along D induced by the ones of V, with the same weights. Except when explicitly mentioned to the contrary, this will be the parabolic structure we shall consider on $V \otimes L$. In fact, it corresponds to the general tensor product of parabolic bundles (see Yokogawa [42]) in the particular case where L has trivial parabolic structure.

Next we come to morphisms of parabolic bundles. These will be the vector bundle homomorphisms which preserve the parabolic structures; however these can be preserved in a week or a strong sense.

Definition 2.3. Let V and W be parabolic vector bundles whose parabolic structures are of type $(\boldsymbol{m}, \boldsymbol{\alpha})$ and $(\boldsymbol{l}, \boldsymbol{\beta})$ respectively, and let $\phi : V \to W$ be a holomorphic map. The map ϕ is called *parabolic* if we have, for all $p \in D$,

$$\alpha_i(p) > \beta_j(p) \Longrightarrow \phi(V_{p,i}) \subseteq W_{p,j+1}.$$

Denote by $\mathrm{ParHom}(V, W)$ the bundle of parabolic homomorphisms from V to W and, if $W = V$, write $\mathrm{ParEnd}(V)$ instead. The map ϕ is said *strongly parabolic* if

$$\alpha_i(p) \geqslant \beta_j(p) \Longrightarrow \phi(V_{p,i}) \subseteq W_{p,j+1},$$

for all $p \in D$. Denote by $\mathrm{SParHom}(V, W)$ the bundle of strongly parabolic homomorphisms from V to W and, if $W = V$, write $\mathrm{SParEnd}(V)$ instead.

2.2. **Parabolic Higgs bundles.** We shall need to consider parabolic Higgs bundles with various structure groups G. Indeed, there is a theory of parabolic G-Higgs bundles (see, for example, Biquard–García-Prada–Mundet [6] for a general notion for real reductive G) but, since we shall only need the groups $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$, we can make the following ad hoc definitions.

Let $K = \Omega^1_X$ be the canonical bundle on X and write $K(D) = K \otimes \mathcal{O}_X(D)$.

Definition 2.4. A *strongly parabolic* $\mathrm{GL}(n, \mathbb{C})$*-Higgs bundle* is a pair (V, φ), where V is a parabolic bundle of rank n and the *Higgs field* $\varphi : V \to V \otimes K(D)$ is a strongly parabolic homomorphism, i.e., φ is a holomorphic section of $\mathrm{SParEnd}(V) \otimes K(D)$, where $V \otimes K(D)$ has the parabolic structure defined by V (cf. Remark 2.2). The *type* of a parabolic $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle (V, φ) is the type of the parabolic vector bundle V; cf. Definition 2.1.

Thus, in a strongly parabolic Higgs bundle (V, φ), the Higgs field φ is a meromorphic endomorphism valued one-form with at most simple poles along $p \in D$ and whose residue at p is nilpotent with respect to the flag. In other words, if the parabolic structure on V is given by (2.1) then,

$$\varphi(V_{p,i}) \subseteq V_{p,i+1} \otimes K(D)_p.$$

Remark 2.5. If we require φ to be just parabolic, rather than strongly parabolic, we get the notion of parabolic Higgs bundle (for the structure groups considered). We shall, however, never use this notion in the present paper. Thus we shall frequently omit the adverb "strongly", but the reader should keep in mind that the Higgs field φ is always required to be strongly parabolic.

If V is a parabolic bundle of rank n, consider the determinant line bundle $\Lambda^n V$. Though this has a natural parabolic structure, in the following definition we ignore it and consider just the underlying line bundle.

Definition 2.6. Fix a holomorphic line bundle Λ on X of degree $d \in \mathbb{Z}$. A *strongly parabolic* $\mathrm{SL}(n,\mathbb{C})$*-Higgs bundle with fixed determinant* Λ is a pair (V,φ), where V is a parabolic bundle of rank n such that $\Lambda^n V \cong \Lambda$, and where $\varphi \in H^0(X, \mathrm{SParEnd}(V) \otimes K(D))$ is such that $\mathrm{tr}(\varphi) \equiv 0$.

Note that, strictly speaking, "$\mathrm{SL}(n,\mathbb{C})$-bundle" should only refer to the case where the line bundle Λ is trivial, so we are committing a slight abuse of language here.

When there is no need to specify the structure group, or when it is clear from the context, we shall often make a further innocuous abuse of language and say simply (strongly) parabolic Higgs bundle.

In order to introduce strongly parabolic $\mathrm{PGL}(n,\mathbb{C})$-Higgs bundles, recall that any holomorphic $\mathrm{PGL}(n,\mathbb{C})$-bundle over the curve X lifts to a holomorphic vector bundle $V \to X$, and that two such lifts V and V' differ by tensoring by a line bundle.

Definition 2.7. A *strongly parabolic* $\mathrm{PGL}(n,\mathbb{C})$*-Higgs bundle* is an equivalence class $[(V,\varphi)]$ of strongly parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles, where (V,φ) and (V',φ') are considered equivalent if there is a line bundle L such that $V' \cong V \otimes L$, the parabolic structure of V' is the one obtained from V, and $\varphi' = \varphi \otimes \mathrm{Id}_L$.

Remark 2.8. Recall that $\mathrm{PGL}(n,\mathbb{C})$-bundles over the curve X are topologically classified by $\pi_1(\mathrm{PGL}(n,\mathbb{C})) \cong \mathbb{Z}_n$. Fixing a topological type $c \in \mathbb{Z}_n$ and a holomorphic line bundle Λ whose degree modulo n equals c, any holomorphic $\mathrm{PGL}(n,\mathbb{C})$-bundle of topological type c may be lifted to a holomorphic vector bundle whose determinant bundle is isomorphic to Λ. Moreover, two lifts with the same determinant bundle differ by tensoring by a line bundle which is a n-torsion point of the Jacobian $\mathrm{Jac}(X)$. These facts reflect the identifications $\mathrm{PGL}(n,\mathbb{C}) = \mathrm{GL}(n,\mathbb{C})/\mathbb{C}^* = \mathrm{SL}(n,\mathbb{C})/\mathbb{Z}_n = \mathrm{PSL}(n,\mathbb{C})$.

2.3. **Stability and moduli spaces.** In the following we recall the stability condition for parabolic Higgs bundles and introduce their moduli spaces.

Definition 2.9. Given a parabolic vector bundle V, a *parabolic subbundle* is a vector subbundle $V' \subseteq V$, with the parabolic structure defined as follows. For each $p \in D$, the quasi-parabolic structure is given by the flag

$$V'_p = V'_{p,1} \supsetneq V'_{p,2} \supsetneq \cdots \supsetneq V_{p,s_p} \supsetneq \{0\},$$

where $V'_{p,i} = V'_p \cap V_{p,i}$, discarding all the repetitions of subspaces in the filtration. Moreover, the weights $0 \leqslant \alpha'_1(p) < \cdots < \alpha'_{s_p}(p) < 1$ are taken to be the greatest possible among the corresponding original weights, meaning that

$$\alpha'_i(p) = \max_j \{\alpha_j(p) \mid V_{p,j} \cap V'_p = V'_{p,i}\} = \max_j \{\alpha_j(p) \mid V'_{p,i} \subseteq V_{p,j}\}. \tag{2.2}$$

In other words, the weight attached to $V'_{p,i}$ is the weight $\alpha_j(p)$ whose index j is such that $V'_{p,i} \subseteq V_{p,j}$ but $V'_{p,i} \not\subseteq V_{p,j+1}$.

Definition 2.10. The *degree* of a parabolic Higgs bundle (V, φ) is the degree of the underlying bundle, $\deg(V) \in \mathbb{Z}$. The *parabolic degree* $\operatorname{pardeg}(V)$ and *parabolic slope* $\operatorname{par}\mu(V)$ of (V, φ) are the parabolic degree and slope, respectively, of the underlying parabolic vector bundle, defined by

$$\operatorname{pardeg}(V) = \deg(V) + \sum_{p \in D} \sum_{i=1}^{s_p} m_i(p)\alpha_i(p) \qquad \text{and} \qquad \operatorname{par}\mu(V) = \frac{\operatorname{pardeg}(V)}{n}.$$

Definition 2.11. A strongly parabolic Higgs bundle (V, φ) is *semistable* if

$$\operatorname{par}\mu(V') \leqslant \operatorname{par}\mu(V)$$

for every non-zero parabolic subbundle $V' \subseteq V$ which is *φ-invariant*, that is, $\varphi(V') \subseteq V' \otimes K(D)$. It is *stable* if it is semistable and strict inequality holds above for all proper non-zero φ-invariant parabolic subbundles $V' \subseteq V$.

Consider now quasi parabolic Higgs bundles of rank n, degree d, and quasi-parabolic type $\boldsymbol{m}$. The space of compatible parabolic weights $\boldsymbol{\alpha} = (\alpha_1(p), \ldots, \alpha_{s_p}(p))_{p \in D}$ is a product $\mathcal{S}$ of simplices (excluding some boundaries) determined by the inequalities in (2.1), one simplex for each point of D. Let (V, φ) be a parabolic Higgs bundle of type $(\boldsymbol{m}, \boldsymbol{\alpha})$. If (V, φ) is semistable but not stable, then

$$n\Big(d' + \sum_{p \in D} \sum_{i=1}^{s'_p} m'_i(p)\alpha'_i(p)\Big) = n'\Big(d + \sum_{p \in D} \sum_{i=1}^{s_p} m_i(p)\alpha_i(p)\Big), \tag{2.3}$$

where d' and n' are the degree and rank of a destabilizing parabolic Higgs subbundle $(V', \varphi|_{V'})$ with multiplicities $m'_i(p)$ and weights $\alpha'_i(p)$. For given d', n' and $m'_i(p)$, equation (2.3) determines an intersection of a hyperplane with $\mathcal{S}$ which is called a *wall*. There are only finitely many possible values for n' and $m'_i(p)$ and, for each of these, there are also finitely many values of d' for which these hyperplanes intersect $\mathcal{S}$. Hence there are finitely many walls.

Definition 2.12. Fix n, d and a quasi-parabolic type $\boldsymbol{m}$. A weight $\boldsymbol{\alpha} \in \mathcal{S}$ is called *generic* if it does not belong to a wall. A connected component of the complement of the set of walls is called a *chamber*.

Remark 2.13. It is immediate from this definition that for generic weights, a semistable parabolic Higgs bundle is in fact stable. Moreover, for generic weights in the same chamber, the stability condition is unchanged, so the corresponding moduli spaces (to be introduced presently) will be isomorphic.

A GIT construction of the moduli space $\mathcal{M}_d^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{GL}(n,\mathbb{C}))$ of semistable parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles over X, of rank n, degree d and parabolic type $(\boldsymbol{m}, \boldsymbol{\alpha})$, was carried out by Yokogawa [41], and the deformation theory of parabolic Higgs bundles was also studied by Yokogawa [42] (cf. Boden–Yokogawa [10]). A gauge theoretic construction of the moduli space of (non-strongly) parabolic Higgs bundles was done by Konno [27]. It was proved by Yokogawa that the stable locus of the moduli space is smooth and quasi-projective. Thus we have the following result.

Proposition 2.14. *Assume that the weights $\boldsymbol{\alpha}$ are generic. Then the moduli space $\mathcal{M}_d^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{GL}(n,\mathbb{C}))$ is a smooth quasi-projective variety. Moreover, for generic weights in the same chamber of $\mathcal{S}$ the corresponding moduli spaces are isomorphic.* □

In order to obtain the moduli space of parabolic $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles over X, consider the determinant map

$$p : \mathcal{M}_d^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{GL}(n,\mathbb{C})) \to T^*\operatorname{Pic}^d(X) \cong \operatorname{Pic}^d(X) \times H^0(X,K),$$
$$(V,\varphi) \mapsto (\Lambda^n V, \operatorname{tr}(\varphi)),$$

with $\operatorname{Pic}^d(X)$ the component of the Picard variety of X of degree d line bundles. Notice that, since (V,φ) is strongly parabolic, the residue of the trace $\operatorname{tr}(\varphi)$ vanishes along D, and so it is in fact a section of K. Let

$$\mathcal{M}_\Lambda^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C})) = p^{-1}(\Lambda, 0).$$

For generic weights, this is again a smooth quasi-projective variety. Since this is the moduli space we shall mostly be working with, whenever there is no risk of confusion, we shall denote it simply by $\mathcal{M}$.

Next we want to introduce the moduli space of parabolic $\mathrm{PGL}(n,\mathbb{C})$-Higgs bundles. In view of Definition 2.7 and Remark 2.8 we consider the group

$$\Gamma_n = \operatorname{Jac}_{[n]}(X) = \{L \in \operatorname{Jac}(X) \mid L^n \cong \mathcal{O}_X\} \subset \operatorname{Jac}(X)$$

of n-torsion points of the Jacobian of X. Recall that $\Gamma_n \cong H^1(X,\mathbb{Z}_n) \cong \mathbb{Z}_n^{2g}$. It will be convenient to distinguish the elements of Γ_n as an abstract group and as line bundles; thus, if γ denotes an element of Γ_n, the corresponding line bundle will be denoted by L_γ. Fix a line bundle Λ and let $[d] \in \mathbb{Z}_n$ denote the reduction of $d = \deg(\Lambda)$ modulo n. The group Γ_n acts on $\mathcal{M}$ by

$$\gamma \cdot (V,\varphi) = (V \otimes L_\gamma, \varphi \otimes \mathrm{Id}_{L_\gamma}) \tag{2.4}$$

(note that Γ_n acts trivially on the parabolic structure). We take the moduli space of parabolic $\mathrm{PGL}(n,\mathbb{C})$-Higgs bundles of topological type $[d]$ to be

$$\mathcal{M}_{[d]}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{PGL}(n,\mathbb{C})) = \mathcal{M}/\Gamma_n.$$

We remark that this is consistent with the abstract definition of stability of parabolic G-Higgs bundles coming from [6]. As opposed to $\mathcal{M}$, the moduli space $\mathcal{M}/\Gamma_n$ is not smooth, but rather an orbifold, with singularities arising from the fixed points of the action of Γ_n.

Serre duality for a parabolic vector bundles (see [42, 10]) says that for a parabolic vector bundle V

$$H^1(\operatorname{ParEnd}(V)) \simeq H^0(\operatorname{SParEnd}(V) \otimes K(D))^*$$

(and analogously in the traceless case), in other words, the infinitesimal deformation space of V is dual to the space of Higgs fields on V. Thus, letting $\mathcal{N}$ denote the moduli space of parabolic vector bundles (with fixed determinant Λ), there is an embedding of the cotangent bundle $T^*\mathcal{N} \hookrightarrow \mathcal{M}$ as an open subset. The natural symplectic structure on the cotangent bundle extends to $\mathcal{M}$, which is thus a holomorphic symplectic manifold (see Bottacin [11, Sec. 5], Biswas–Ramanan [9, Sec. 6], Konno [27], Logares–Martens [29], and cf. Yokogawa [42] and Boden–Yokogawa [10]). Moreover, Konno's gauge theoretic construction (loc. cit.) shows that $\mathcal{M}$ has a compatible hyper-Kähler metric.

2.4. **Isomorphism between moduli spaces for different degrees and weights.** Let Λ and Λ' be line bundles on X, not necessarily of the same degree. In this section we prove that, under mild conditions on the parabolic structure $\boldsymbol{\alpha}$, one can find a parabolic structure $\boldsymbol{\alpha}'$ so that the moduli spaces $\mathcal{M}_\Lambda^{\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C}))$ and $\mathcal{M}_{\Lambda'}^{\boldsymbol{\alpha}'}(\mathrm{SL}(n,\mathbb{C}))$ are isomorphic. This expands on [14, Proposition 2.1] and marks a substantial difference to the non-parabolic case, where such an isomorphism can only exist if $\deg(\Lambda)$ and $\deg(\Lambda')$ are equal modulo n.

We need the notion of tensor product of parabolic (Higgs) bundles. This is better viewed in the more general context of parabolic (or filtered) sheaves (see Boden–Yokogawa [10], Yokogawa [42], and Simpson [37]) but we shall only need a few simple facts which we now review. In fact it suffices for us to consider the case when one of the bundles is a parabolic line bundle (with trivial Higgs field), so let $(V, \boldsymbol{\alpha})$ be a parabolic vector bundle bundle of rank n and let $(L, \boldsymbol{\beta})$ be a parabolic line bundle on X. There is a *parabolic tensor product* of the parabolic bundles V and L, denoted by $(V \otimes^P L, \boldsymbol{\alpha}')$. The parabolic weights $\boldsymbol{\alpha}'$ are given by

$$\alpha_i'(p) = \begin{cases} \alpha_i(p) + \beta(p) & \text{if } \alpha_i(p) + \beta(p) < 1, \\ \alpha_i(p) + \beta(p) - 1 & \text{if } \alpha_i(p) + \beta(p) \geqslant 1, \end{cases} \tag{2.5}$$

where for each $p \in D$ the correct ordering of the $\alpha_i'(p)$ by size corresponds to a cyclic permutation of the ordering of the indices $i = 1, \dots, s_p$. The multiplicity of the weight $\alpha_i'(p)$ is $m_i(p)$. The parabolic degree of the parabolic tensor product is given by the usual formula:

$$\begin{aligned} \operatorname{pardeg}(V \otimes^P L) &= \operatorname{pardeg}(V) + n \operatorname{pardeg}(L) \\ &= \deg(V) + \sum_{p,i} m_i(p)\alpha_i(p) + n(\deg(L) + \sum_p \beta(p)). \end{aligned}$$

In view of this we get the following formula for the (non-parabolic) degree of $V \otimes^P L$:

$$\begin{aligned} \deg(V \otimes^P L) &= \operatorname{pardeg}(V \otimes^P L) - \sum_{p,i} m_i(p)\alpha_i'(p) \\ &= \deg(V) + n\deg(L) + \sum_{p,i} m_i(p)(\alpha_i(p) - \alpha_i'(p) + \beta(p)) \\ &= \deg(V) + n\deg(L) + \# \bigcup_{p \in D} \{ i \mid \alpha_i(p) + \beta(p) \geqslant 1 \}. \end{aligned} \tag{2.6}$$

Finally we remark that if V underlies a parabolic Higgs bundle (V, φ), then $V \otimes^P L$ has a Higgs field induced by $\varphi \otimes \mathrm{Id}_L$ and that (V, φ) is stable if and only if $(V \otimes^P L, \varphi \otimes \mathrm{Id}_L)$ is (see Simpson [37]).

Theorem 2.15. *Let $\boldsymbol{\alpha} = (\alpha_i(p))_{p \in D}$ be a system of parabolic weights and let Λ and Λ' be line bundles of degrees d and d', respectively. Suppose there are $\boldsymbol{\beta} = (\beta(p))_{p \in D}$ in $[0,1)$ such that*

$$d' - d - \# \bigcup_{p \in D} \{ i \mid \alpha_i(p) + \beta(p) \geqslant 1 \} \equiv 0 \pmod{n}. \tag{2.7}$$

Then there is a parabolic line bundle $(L, \boldsymbol{\beta})$ inducing an isomorphism

$$\begin{aligned} \mathcal{M}_\Lambda^{\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C})) &\xrightarrow{\cong} \mathcal{M}_{\Lambda'}^{\boldsymbol{\alpha}'}(\mathrm{SL}(n,\mathbb{C})), \\ (V, \varphi) &\mapsto (V \otimes^P L, \varphi \otimes \mathrm{Id}_L), \end{aligned}$$

where the weights $\boldsymbol{\alpha}'$ are given by (2.5).

Proof. In view of (2.6) and the remarks preceding the statement of the theorem, we can find a line bundle L such that tensoring by L gives an isomorphism $\mathcal{M}_d^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{GL}(n,\mathbb{C})) \xrightarrow{\cong} \mathcal{M}_{d'}^{\boldsymbol{m},\boldsymbol{\alpha}'}(\mathrm{GL}(n,\mathbb{C}))$. In order to get the isomorphism between the fixed determinant moduli spaces it suffices to adjust L by tensoring by a suitable (non-parabolic) degree zero line bundle. □

The following corollary describes two situations where it is possible to find $\boldsymbol{\beta}$ satisfying (2.7). The conditions imposed are mild, and since we will at some point impose full flags at all points of D, we will be under the assumptions of this corollary.

Corollary 2.16. *Consider the moduli space* $\mathcal{M}_\Lambda^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C}))$.

(1) *If* Λ' *is any line bundle of degree* d' *such that* $d' \equiv d \pmod n$*, then* $\mathcal{M}_\Lambda^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C})) \cong \mathcal{M}_{\Lambda'}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C}))$.

(2) *Suppose the parabolic structure* $\boldsymbol{\alpha}$ *is such that for some point* $p_0 \in D$ *the flag is full (i.e.,* $s_{p_0} = n$*). Then, given any line bundle* Λ' *of any degree* d'*, there exists a parabolic structure* $\boldsymbol{\alpha}'$ *such that* $\mathcal{M}_{\Lambda'}^{\boldsymbol{m},\boldsymbol{\alpha}'}(\mathrm{SL}(n,\mathbb{C})) \cong \mathcal{M}_\Lambda^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C}))$.

Proof. For the first item, we just have to take L a nth root of $\Lambda'\Lambda^{-1}$ and use the usual tensor product $V \mapsto V \otimes L$. This is of course the generalization to the parabolic case of the classical isomorphism in the non-parabolic case.

For the second item, suppose that $d' - d \equiv k \pmod n$. Since the flag over p_0 is full, we can choose $\beta(p_0) \in [0, 1)$ such that $\#\{i \mid \alpha_i(p_0) + \beta(p_0) \geqslant 1\} = k$. If $p \in D \smallsetminus \{p_0\}$, take $\beta(p) = 0$, so $\#\{i \mid \alpha_i(p) + \beta(p) \geqslant 1\} = 0$. With these choices, (2.7) holds, and the conclusion follows by the theorem. □

2.5. Basic assumptions.

We now make two assumptions.

Assumption 2.17. We assume from now on that:

(1) the weights $\boldsymbol{\alpha}$ are generic;

(2) the flags over all points of D are full (i.e., $m_i(p) = 1$ for all i and all $p \in D$, thus $s_p = n$ for all p).

Since from now on $m_i(p) = 1$ for all i, p, we shall remove the $\boldsymbol{m}$ from the notation.

The first assumption is essential for us. It implies that any semistable parabolic Higgs bundle is stable and hence, as shown by Yokogawa [41], the moduli space is smooth.

The second assumption serves two purposes. Firstly, the SYZ mirror symmetry picture (outlined in the next section) has currently only been shown under this assumption. Secondly, it simplifies the formulas in our calculations of Hodge polynomials. We point out, however, that these calculations generalize without too much trouble to the case of general flags.

Summarizing, under Assumption 2.17, the moduli space of parabolic Higgs bundles $\mathcal{M}$ is a smooth quasi-projective hyper-Kähler manifold. Its dimension can be calculated using deformation theory (see, for example, [14, Proposition 2.4]) and is given by

$$\dim(\mathcal{M}) = 2(n^2 - 1)(g - 1) + |D|n(n - 1), \tag{2.8}$$

where we recall that $|D| = \deg(D)$ is the number of marked points on X.

3. Mirror symmetry

In this section we recall the Hausel–Thaddeus mirror symmetry proposal in the parabolic case. First, in Section 3.1 we treat the Hitchin system and mirror symmetry according to Strominger–Yau–Zaslow. Next, in Section 3.2, we recall the definition of the stringy E-polynomial and show, following Thaddeus, its independence of the parabolic weights. Finally, in Section 3.3 we state our main result and outline its proof.

3.1. The Hitchin map and SYZ mirror symmetry.

In this section we briefly describe how $\mathcal{M}$ and $\mathcal{M}/\Gamma_n$ are mirror partners in the sense of Strominger–Yau–Zaslow (SYZ) [39]. This has been shown in the parabolic case by Biswas–Dey [8] (following Hausel–Thaddeus [21]). The general version of SYZ mirror symmetry proved by these authors requires considering a naturally defined gerbe (or B-field) on the moduli spaces (see also Donagi–Pantev [13]). As explained below, the statement of SYZ mirror symmetry involves identifying fibers of the Hitchin maps of the two moduli spaces as dual abelian varieties. The need for introducing the B-field comes from the lack of a natural base point in these fibers. In the parabolic case there is a twist in the story: the moduli spaces $\mathcal{M}$ and $\mathcal{M}/\Gamma_n$ are also mirror partners in the original sense of SYZ.

Remark 3.1. We recall that the true mirror partners are in fact the *de Rham moduli spaces*; these are moduli spaces of local systems on X and are diffeomorphic to the Higgs bundle moduli spaces under the non-abelian Hodge correspondence. As explained in [21, Sec. 1] in the non-parabolic case, the statements on the de Rham side can be translated into statements on the Higgs bundle side through a hyper-Kähler rotation, and this works exactly the same way in the parabolic case. We refer the reader to Simpson [37] and Alfaya–Gómez [1] for details on the de Rham moduli spaces in the parabolic case.

We now introduce the Hitchin system in the parabolic setting. This goes back to Bottacin [11] and Nasatyr–Steer [33]. We start by defining the *Hitchin map* h on the moduli spaces $\mathcal{M}$ and $\mathcal{M}/\Gamma_n$: it takes a parabolic Higgs bundle (V, φ) to the coefficients of the characteristic polynomial of the twisted endomorphism $\varphi : V \to V \otimes K(D)$. Thus $h(V, \varphi) = (s_2, \ldots, s_n)$, with $s_i = \mathrm{tr}(\wedge^i \varphi)$. Since φ is strongly parabolic, its restriction to every $p \in D$ is nilpotent, and so all the corresponding coefficients $s_i(p)$ of the characteristic polynomial vanish. We therefore have

$$\begin{aligned} h(V,\varphi)\colon \mathcal{M} \to \mathcal{A} = \bigoplus_{i=2}^{n} H^0(X, K^i((i-1)D)),\\ \varphi \mapsto (s_2, \ldots, s_n), \end{aligned} \tag{3.1}$$

where $\mathcal{A}$ is the *Hitchin base*. It is clear that h factors through the quotient $\mathcal{M}/\Gamma_n$, so we also have a Hitchin map h' on this moduli space:

$$\begin{array}{ccc} \mathcal{M} & & \mathcal{M}/\Gamma_n \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle h'} & \\ & \mathcal{A} & \end{array}$$

Observe that

$$\dim(\mathcal{A}) = (n^2-1)(g-1) + \frac{n(n-1)|D|}{2} = \frac{\dim(\mathcal{M})}{2}.$$

By [41], the map h is proper, hence so is h'. The coordinate functions of h and h' are independent and Poisson commute, and these maps form the *Hitchin systems* for $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$, respectively. In particular, for $s \in \mathcal{A}'$, the fibers $h^{-1}(s)$ and $h'^{-1}(s)$ are complex Lagrangian subvarieties of $\mathcal{M}$ and $\mathcal{M}/\Gamma_n$.

To describe the generic Hitchin fibers more precisely, consider the quasi-projective surface given by the total space $|K(D)|$ of $K(D)$ and the projection $\pi\colon |K(D)| \to X$. Given a point $s = (s_2, \ldots, s_n)$ in the Hitchin base $\mathcal{A}$, there is a projective curve X_s, lying in $|K(D)|$, defined by the zeros of the section

$$\lambda^n + \pi^* s_2 \lambda^{n-2} + \cdots + \pi^* s_n \in H^0(|K(D)|, \pi^*(K^n(nD))) \tag{3.2}$$

where λ is the tautological section of $\pi^*(K(D))$ and $\lambda^{n-i}\pi^* s_i \in H^0(|K(D)|, \pi^*(K^n((n-1)D))) \subseteq H^0(|K(D)|, \pi^*(K^n(nD)))$. The curve X_s is called the *spectral curve* associated to $s \in \mathcal{A}$. The restriction of π to X_s gives an n-cover $\pi : X_s \to X$ which is ramified over the locus where (3.2) has multiple roots. This locus is always non-empty.

By Lemma 3.1 of [15], there is an open and dense subspace $\mathcal{A}' \subset \mathcal{A}$ such that X_s is smooth whenever $s \in \mathcal{A}'$. Moreover, for such generic s, Lemma 3.2 of [15] states that the fibre $h^{-1}(s)$ is naturally isomorphic to

$$P^{d'} = \{L \in \operatorname{Pic}^{d'}(X_s) \mid \det(\pi_* L) \cong \Lambda\}, \tag{3.3}$$

where $d' = d + n(n-1)(g - 1 + |D|/2)$.

Remark 3.2. Lemma 3.2 of [15] needs the full flags assumption on every point of the divisor D. This is one reason why we use confine ourselves to the full flag condition.

Definition 3.3. Consider a degree n cover $\pi : Y \to X$. The *norm map* between the groups of divisors $\mathrm{Nm}_\pi \colon \mathrm{Div}(Y) \to \mathrm{Div}(X)$ is the homomorphism taking a divisor $E = \sum E(p)p$ on Y to the divisor $\mathrm{Nm}_\pi(E) = \sum E(p)\pi(p)$ on X.

The norm map just defined factors through the norm map between the Picard groups $\mathrm{Nm}_\pi \colon \operatorname{Pic}^0(X_s) \to \operatorname{Pic}^0(X)$, by $\mathrm{Nm}_\pi(L) = L'$ where $L \cong \mathcal{O}_Y(E)$ and $L' \cong \mathcal{O}_X(\mathrm{Nm}_\pi(E))$.

Definition 3.4. The *Prym variety of* Y associated to π, denoted by $\mathrm{Prym}_\pi(Y)$, is the abelian subvariety of $\operatorname{Pic}^0(Y)$ defined as the identity component of the kernel of Nm_π.

The kernel of Nm_π is connected if and only if π is ramified and sometimes the term Prym variety is used for the full kernel of Nm_π. We have adopted Definition 3.4 in accordance with [21]. Since for $s \in \mathcal{A}'$ the cover $\pi : X_s \to X$ is ramified[1], we have

$$\mathrm{Prym}_\pi(X_s) = \ker(\mathrm{Nm}_\pi) = \{L \in \operatorname{Pic}^0(X_s) \mid \mathrm{Nm}_\pi(L) \cong \mathcal{O}_X\}. \tag{3.4}$$

Note that

$$\det(\pi_* L) \cong \mathrm{Nm}_\pi(L) \otimes \det(\pi_*(\mathcal{O}_{X_s})) \cong \mathrm{Nm}_\pi(L) \otimes (K(D))^{-n(n-1)/2}. \tag{3.5}$$

Thus we see that

$$P^{d'} \cong \{L \in \operatorname{Pic}^{d'}(X_s) \mid \mathrm{Nm}_\pi(L) \cong \Lambda(K(D))^{n(n-1)/2}\}$$

is a torsor for $\mathrm{Prym}_\pi(X_s)$.

It is also easy to see that $h'^{-1}(s)$ is isomorphic $P^{d'}/\Gamma_n$, hence it is a torsor for the quotient $\mathrm{Prym}_\pi(X_s)/\Gamma_n$, where Γ_n acts by tensoring by the pull-back via π. The quotient $\mathrm{Prym}_\pi(X_s)/\Gamma_n$ is an abelian variety, isogenous to $\mathrm{Prym}_\pi(X_s)$.

By dualising the short exact sequence coming from the norm map, one easily checks [21, Lemma 2.3] that these two abelian varieties are dual to each other, in the sense that

$$\operatorname{Pic}^0(\mathrm{Prym}_\pi(X_s)) \cong \mathrm{Prym}_\pi(X_s)/\Gamma_n \qquad \text{and} \qquad \operatorname{Pic}^0(\mathrm{Prym}_\pi(X_s)/\Gamma_n) \cong \mathrm{Prym}_\pi(X_s).$$

Theorem 3.5 (Hausel–Thaddeus [20])**.** *Assume that $s \in \mathcal{A}$ has simple zeros. Then the Hitchin fibers $h^{-1}(s)$ and $h'^{-1}(s)$ can be naturally identified with a pair of dual abelian varieties. Hence $\mathcal{M}$ and $\mathcal{M}/\Gamma_n$ are SYZ mirror partners.*

Proof. Assume first that $\Lambda \cong K(D)^{-n(n-1)/2}$ (so that $d = -n(n-1)(g - 1 + |D|/2)$ and $d' = 0$). In this case, (3.3), (3.4) and (3.5) show that the fibre of h over a generic point $s \in \mathcal{A}'$ is naturally identified with $\mathrm{Prym}_\pi(X_s)$ and not just a torsor over it, and analogously for the fibre of h'. Hence, in view of the observations preceding the statement of the theorem, we have the desired conclusion if we show that the base points of these Pryms form a Lagrangian section. When $\Lambda \cong \mathcal{O}_X$, Biswas–Ares–Govindarajan [7] constructed the parabolic "Hitchin component"[2], generalizing Hitchin's construction [24] in the non-parabolic case. The Hitchin component is the image of $\mathcal{A}$ under an explicitly constructed

[1]In Section 5 we shall need to consider the norm map and corresponding Prym of unramified covers.

[2]so called because under the non-abelian Hodge theory correspondence it corresponds to a special component of the character variety of representations in $\mathrm{SL}(n, \mathbb{R})$ of the fundamental group of the Riemann surface (with certain prescribed holonomies around the punctures).

section of the Hitchin map and it is a Lagrangian submanifold. Moreover, one easily sees that under the isomorphism $\mathcal{M}^{\boldsymbol{\alpha}'}_{\mathcal{O}_X}(\mathrm{SL}(n,\mathbb{C})) \cong \mathcal{M}^{\boldsymbol{\alpha}}_{K(D)^{-n(n-1)/2}}(\mathrm{SL}(n,\mathbb{C}))$ provided by Corollary 2.16, the Hitchin component maps to the distinguished base points in the Hitchin fibers.

Now, for any other line bundle Λ of any degree, use Corollary 2.16 to get an isomorphism $\mathcal{M}^{\boldsymbol{\alpha}'}_{\Lambda}(\mathrm{SL}(n,\mathbb{C})) \cong \mathcal{M}^{\boldsymbol{\alpha}}_{K(D)^{-n(n-1)/2}}(\mathrm{SL}(n,\mathbb{C}))$ (since we are assuming full flags, the hypotheses of Corollary 2.16 are satisfied). Clearly this map is in fact an isomorphism of the corresponding Hitchin systems, and descends to the $\mathrm{PGL}(n,\mathbb{C})$-Hitchin systems, giving us in particular the desired identifications of the Hitchin fibers as dual abelian varieties. □

Remark 3.6. This fits with a general phenomenon in SYZ-mirror symmetry, where if torus fibrations admit a Lagrangian section, then the B-field is unnecessary for the symmetry to work out; see for example Hitchin [25] or Polishchuk [35]. In the more general version involving a B-field, the identification of Hitchin fibers as dual abelian varieties comes about through a choice of trivialization of the restriction of the gerbe. Moreover, if there is a canonical coherent choice of trivialization of the gerbe in all fibers, the "gerby" duality (see Hausel–Thaddeus [21, p. 202] for its definition) follows from the usual one described here. This would be the case if, for example, the gerbe were known to be trivial.

3.2. **The (stringy) E-polynomial.** Let M be a *semiprojective* variety (see [19]). This means that M is quasi-projective and that (i) it carries an algebraic $\mathbb{C}^*$-action such that for any point p in M, the limit of the $\mathbb{C}^*$-orbit $(t\cdot p)_{t\in\mathbb{C}^*}$ when t goes to 0 exists in M, and (ii) the subvarieties of M of fixed points under $\mathbb{C}^*$ are compact. Then by [19, Corollary 1.3.2] if M is smooth, the (compactly supported) cohomology of M is pure. Hence its *E-polynomial* is given by

$$E(M) = \sum_{p,q=0}^{\dim(M)} (-1)^{p+q} h_c^{p,q}(M) u^p v^q,$$

where $h_c^{p,q}(M) = \dim H_c^{p,q}(M,\mathbb{C})$. Also, if M has an action of a group Γ, we let $E(M)^\Gamma$ denote the *Γ-invariant E-polynomial*, i.e.,

$$E(M)^{\Gamma_n} = \sum_{p,q=0}^{\dim(M)} (-1)^{p+q} \dim H_c^{p,q}(M)^\Gamma u^p v^q, \tag{3.6}$$

where $H_c^{p,q}(M)^\Gamma \subset H_c^{p,q}(M)$ is the Γ-invariant subspace.

The main motivational example for the definition of semiprojective varieties comes precisely from the moduli spaces of Higgs bundles. Indeed, they carry an algebraic $\mathbb{C}^*$-action, also in the strongly parabolic case, defined by

$$t\cdot(V,\varphi) = (V, t\varphi),\ t\in\mathbb{C}^*. \tag{3.7}$$

The following proposition describes two well-known fundamental properties of this action, which in particular show that $\mathcal{M}$ is semiprojective.

Proposition 3.7. *The $\mathbb{C}^*$-action on $\mathcal{M}$ verifies the following properties.*

(1) *For any point $(V,\varphi)\in\mathcal{M}$, the limit $\lim_{t\to 0}(V,t\varphi)$ exists in $\mathcal{M}$ and is a fixed point.*
(2) *The subvarieties of $\mathcal{M}$ of fixed points are compact.*

Therefore, $\mathcal{M}$ is semiprojective.

Proof. This is well known. The first item follows from the properness of the Hitchin map (3.1), just as in [38, Corollary 9.20]. Regarding the second item, the $\mathbb{C}^*$-fixed points are precisely the critical points of the real function $f:\mathcal{M}\to\mathbb{R}_{\geqslant 0}$ given by the L^2 of the

Higgs field (see Proposition 3.3 of [14]). By Theorem 4.1 (a) of [10], f is proper, hence (2) also follows. □

The moduli space $\mathcal{M}/\Gamma_n$ has orbifold singularities and, following Hausel-Thaddeus [21] (in turn based on Batyrev–Dais [5] and Batyrev [4]), we consider the *stringy E-polynomial* of $\mathcal{M}/\Gamma_n$, defined as

$$E_{\mathrm{st}}(\mathcal{M}/\Gamma_n) = \sum_{\gamma\in\Gamma_n} E(\mathcal{M}^\gamma)^{\Gamma_n}(uv)^{F(\gamma)}, \tag{3.8}$$

where the various objects on the right hand side will be defined in the following. We note in passing that the E-polynomial encodes the Chen–Ruan cohomology of $\mathcal{M}/\Gamma_n$ as an orbifold [12].

The subspace $\mathcal{M}^\gamma \subset \mathcal{M}$ is the locus pointwise fixed by γ. Since it is abelian, Γ_n preserves $\mathcal{M}^\gamma$ and then $E(\mathcal{M}^\gamma)^{\Gamma_n}$ is defined as in (3.6). The *fermionic shift* $F(\gamma)$ is defined as follows: given $p \in \mathcal{M}^\gamma$, the element γ acts on the tangent space $T_p\mathcal{M}$ with eigenvalues $\lambda_1, \dots, \lambda_{\dim(\mathcal{M})}$. Since Γ_n is finite, these are roots of the unity, hence we can write $\lambda_j = e^{2\pi i w_j}$, with $0 \leqslant w_j < 1$ and $w_j \in \mathbb{Q}$. The fermionic shift is the number

$$F(p,\gamma) = \sum_{j=1}^{\dim(\mathcal{M})} w_j. \tag{3.9}$$

Clearly it is constant along the connected component of $\mathcal{M}^\gamma$ containing p. In general the fermionic shift is just a rational number but, in our case, we can be much more precise. Recall that γ acts by $L_\gamma \cdot (V,\varphi) = (V \otimes L_\gamma, \varphi \otimes \mathrm{Id}_{L_\gamma})$, where L_γ is the corresponding n-torsion line bundle. Recall that $\mathcal{N}$ denotes the moduli space of parabolic vector bundles with fixed determinant Λ and the same parabolic structure as the one considered in $\mathcal{M}$. Then γ acts by diffeomorphisms on $\mathcal{N}$, hence acts by symplectomorphisms on the cotangent bundle $T^*\mathcal{N}$, which is an open and dense subspace of $\mathcal{M}$, so γ acts by symplectomorphisms on $\mathcal{M}$. It follows that for each eigenvalue λ_j, λ_j^{-1} is also an eigenvalue. Since γ acts trivially on the subspace $T_p\mathcal{M}^\gamma \subseteq T_p\mathcal{M}$, we conclude that

$$F(p,\gamma) = \sum_{l=1}^{\dim(N_p\mathcal{M}^\gamma)/2} (w_l + 1 - w_l) = \frac{\dim(N_p\mathcal{M}^\gamma)}{2}. \tag{3.10}$$

where $N_p\mathcal{M}^\gamma \subseteq T_p\mathcal{M}$ denotes the normal bundle to $\mathcal{M}^\gamma$ at p. We have already observed that, in general, $F(p,\gamma)$ only depends on the connected component of $\mathcal{M}^\gamma$ containing p. We shall in fact see in Section 6 that $\mathcal{M}^\gamma$ is non-connected, but we shall also conclude directly (see (6.19) below) that the value of $F(p,\gamma)$ is really independent of p, thus independent of the component where it lies. That is the reason why we just wrote $F(\gamma)$ in the definition (3.8) of the stringy E-polynomial.

Remark 3.8. Note that $F(e) = 0$ where e is the trivial element of Γ_n. Thus the stringy E-polynomial of a smooth variety coincides with the usual one. In particular we have that $E(\mathcal{M}) = E_{\mathrm{st}}(\mathcal{M})$.

We conclude this section by pointing out that the E-polynomials are independent of the weights $\boldsymbol{\alpha}$, as long as these are generic. This will be useful in our later calculations (specifically, in the proof of Proposition 4.5) since it allows us to make simplifying assumptions on the weights. Everything follows from the work of Thaddeus [40], who studied how the moduli space of parabolic Higgs bundles changes under wall crossing of the parabolic weights. It is immediate from his description that the Hodge numbers of the moduli space are unchanged under wall crossing. We shall need a Γ_n-equivariant version of this result. This also follows from Thaddeus' description, which we now briefly recall.

Let $\boldsymbol{\alpha}$ belong to only one wall in the space of parabolic weights and consider weights $\boldsymbol{\alpha}^-$ and $\boldsymbol{\alpha}^+$ in the two adjacent chambers. For brevity write $\mathcal{M}^\pm$ for either of the moduli spaces $\mathcal{M}_\Lambda^{\boldsymbol{\alpha}^\pm}(\mathrm{SL}(n,\mathbb{C}))$ and $\mathcal{M}_d^{\boldsymbol{\alpha}^\pm}(\mathrm{GL}(n,\mathbb{C}))$ (everything in this section applies to both of these). There are *flip loci* $\mathcal{S}^\pm \subset \mathcal{M}^\pm$ which correspond to those parabolic Higgs bundles which are $\boldsymbol{\alpha}_\pm$-stable and $\boldsymbol{\alpha}_\mp$-unstable. In the following, write $\mathbf{V} = (V,\varphi)$ for a parabolic Higgs bundle. Points of $\mathcal{S}^-$ correspond to parabolic Higgs bundles $\mathbf{V}$ which are non-split extensions

$$0 \to \mathbf{V}^+ \to \mathbf{V} \to \mathbf{V}^- \to 0, \tag{3.11}$$

of parabolic Higgs bundles, where $\mathbf{V}^\pm$ are stable with respect to the parabolic weights induced by $\boldsymbol{\alpha}^\pm$. There is an analogous description of $\mathcal{S}^+$. Thus there is a natural identification

$$g\colon \mathcal{M}^- \smallsetminus \mathcal{S}^- \xrightarrow{\cong} \mathcal{M}^+ \smallsetminus \mathcal{S}^+.$$

Denote by $\pi^\pm\colon \tilde{\mathcal{M}}^\pm \to \mathcal{M}^\pm$ the blow-ups of $\mathcal{M}^\pm$ along $\mathcal{S}^\pm$ and by $\mathcal{E}^\pm \subset \tilde{\mathcal{M}}^\pm$ the exceptional divisors. Thaddeus [40, (6.2)] proves that there is an isomorphism

$$\tilde{g}\colon \tilde{\mathcal{M}}^- \xrightarrow{\cong} \tilde{\mathcal{M}}^+ \tag{3.12}$$

which restricts to an isomorphism $\mathcal{E}^- \xrightarrow{\cong} \mathcal{E}^+$ of the exceptional divisors and coincides with g on their complement. It is a standard fact about blow-ups that the cohomology groups of $\mathcal{M}^\pm$ inject into the cohomology groups of $\tilde{\mathcal{M}}^\pm$ and from (3.12) it follows that $\tilde{g}$ induces isomorphisms

$$H_c^{p,q}(\mathcal{M}^-) \cong H_c^{p,q}(\mathcal{M}^+), \tag{3.13}$$

considering these cohomology groups as subspaces of $H_c^{p,q}(\tilde{\mathcal{M}}^\pm)$. Thus, for generic $\boldsymbol{\alpha}$, the E-polynomials of $\mathcal{M}_d^{\boldsymbol{\alpha}}(\mathrm{GL}(n,\mathbb{C}))$ and $\mathcal{M}_\Lambda^{\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C}))$ are independent of $\boldsymbol{\alpha}$.

In view of what we have said so far, it is now easy to prove the following.

Proposition 3.9. *Let $\Gamma_n = \mathrm{Jac}_{[n]}(X)$ act on the moduli space of parabolic Higgs bundles by the action defined in* (2.4). *The isomorphism $\tilde{g}$ of* (3.12) *is equivariant with respect to this action. Consequently, the isomorphism* (3.13) *is also Γ_n-equivariant.*

Proof. The basic observation is that the action of Γ_n preserves $\mathcal{S}^\pm \subset \mathcal{M}^\pm$; this follows from the description of $\mathcal{S}^-$ (and the analogous description of $\mathcal{S}^+$) as corresponding to extensions of the form (3.11). Hence the Γ_n-actions lift to the blow-ups $\tilde{\mathcal{M}}^\pm$ (as follows from the universal property of the blow-up, Hartshorne [18, Cor. II.7.15]). Moreover, the restriction of $\tilde{g}$ to the open dense subset $\tilde{\mathcal{M}}^- \smallsetminus \mathcal{E}^- \subset \tilde{\mathcal{M}}^-$ is just g, which is Γ_n-equivariant by our initial basic observation. It follows that $\tilde{g}$ is Γ_n-equivariant as claimed. □

Corollary 3.10. *Assume that $\boldsymbol{\alpha}$ is generic and let $\mathcal{M}^{\boldsymbol{\alpha}}$ denote either $\mathcal{M}_d^{\boldsymbol{\alpha}}(\mathrm{GL}(n,\mathbb{C}))$ or $\mathcal{M}_\Lambda^{\boldsymbol{\alpha}}(\mathrm{SL}(n,\mathbb{C}))$. Then the compactly supported Dolbeault cohomology of $\mathcal{M}^{\boldsymbol{\alpha}}$ is independent of $\boldsymbol{\alpha}$ as a Γ_n-module. Thus the E-polynomial $E(\mathcal{M}^{\boldsymbol{\alpha}})$ and the Γ_n-invariant E-polynomial $E(\mathcal{M}^{\boldsymbol{\alpha}})^{\Gamma_n}$ are both independent of $\boldsymbol{\alpha}$.* □

Remark 3.11. We shall see that the *stringy* E-polynomial of the moduli space of parabolic $\mathrm{PGL}(n,\mathbb{C})$-Higgs bundles is also independent of $\boldsymbol{\alpha}$. Indeed, it will follow from the description given in Theorem 6.3 below that for any $e \neq \gamma \in \Gamma_n$, the parabolic Higgs bundles in the fixed locus $(\mathcal{M}^{\boldsymbol{\alpha}})^\gamma \subset \mathcal{M}^{\boldsymbol{\alpha}}$ are $\boldsymbol{\alpha}$-semistable for any value of $\boldsymbol{\alpha}$. In other words $(\mathcal{M}^{\boldsymbol{\alpha}})^\gamma$ does not intersect the flip locus $\mathcal{S}^\pm$ and thus $\tilde{g}$ from (3.12) restricts to a Γ_n-equivariant isomorphism. Thus all the terms in the definition (3.8) of the stringy E-polynomial are independent of $\boldsymbol{\alpha}$.

3.3. **Topological mirror symmetry and the main result.** The topological mirror symmetry conjecture of Hausel–Thaddeus says that the stringy E-polynomials of the mirror partners $\mathcal{M}$ and $\mathcal{M}/\Gamma_n$ should agree. Since the SYZ mirror symmetry statement is really about the de Rham moduli spaces, rather than the Dolbeault moduli spaces, so is the topological mirror symmetry conjecture (see Remark 3.1). On the other hand, it is the rich algebraic geometry of the Higgs bundle moduli spaces and, in particular, the fact that it carries a $\mathbb{C}^*$-action which allows Hausel and Thaddeus [21] to prove the equality of the E-polynomials in the non-parabolic case. This suffices because they also prove that the de Rham and Dolbeault moduli spaces have the same E-polynomials. The proof of this latter result uses that the Dolbeault and de Rham moduli spaces live in a family, the *Hodge moduli space*, which parametrizes so-called λ-connections. This moduli space fibers over the affine line $\mathbb{C}$ with fibers away from zero all isomorphic to the de Rham moduli space and degenerating to the Dolbeault moduli space over zero. Parabolic λ-connections and the corresponding moduli spaces were constructed and studied by Alfaya–Gómez [1], and their results provide the necessary input for applying the arguments of Hausel–Thaddeus [21, Sec. 6] (cf. Hausel–Rodriguez-Villegas [19, Cor. 1.3.3]) directly in the parabolic situation. Thus the parabolic de Rham moduli spaces have the same E-polynomials as the moduli spaces of parabolic Higgs bundles, and we can exclusively work with the latter for the remainder of the paper.

The topological mirror symmetry conjecture can now be stated in terms of the Higgs bundle moduli spaces as follows.

Conjecture 3.12 (Hausel–Thaddeus [20, 21])**.** *For any rank n, any line bundle Λ and any system of generic weights $\boldsymbol{\alpha}$, the equality of E-polynomials*

$$E(\mathcal{M}_\Lambda^{\boldsymbol{\alpha}}) = E_{\mathrm{st}}(\mathcal{M}_\Lambda^{\boldsymbol{\alpha}}/\Gamma_n) \tag{3.14}$$

holds.

Our main result states that this is true for $n = 2, 3$.

Theorem 3.13. *If $n = 2, 3$, then Conjecture 3.12 holds.*

Proof. We follow the strategy of [21] which we now explain. From the definition of the stringy E-polynomial (3.8) of $\mathcal{M}/\Gamma_n$ and from Remark 3.8, we have that

$$E_{\mathrm{st}}(\mathcal{M}/\Gamma_n) = E(\mathcal{M})^{\Gamma_n} + \sum_{\gamma \neq e} E(\mathcal{M}^\gamma)^{\Gamma_n}(uv)^{F(\gamma)}.$$

On the other hand, let $E(\mathcal{M})^{\mathrm{var}}$ denote the *variant* part of $E(\mathcal{M})$ in Hausel and Thaddeus' terminology. It is defined analogously to $E(\mathcal{M})$ but the coefficients are given by subtracting the dimensions of the Γ_n-invariant subspaces, i.e.,

$$E(\mathcal{M}) = E(\mathcal{M})^{\Gamma_n} + E(\mathcal{M})^{\mathrm{var}}.$$

Hence (3.14) is equivalent to

$$E(\mathcal{M})^{\mathrm{var}} = \sum_{\gamma \neq e} E(\mathcal{M}^\gamma)^{\Gamma_n}(uv)^{F(\gamma)}. \tag{3.15}$$

Now, Theorems 3.14 and 3.15 below imply that (3.15) holds for any $n = 2, 3$, proving Theorem 3.13. □

Thus the following two theorems complete the proof of Theorem 3.13. Here $\mathcal{F}_{(1,1,\dots,1)}$ denotes the subspace of $\mathcal{M}$ consisting of subvarieties of fixed points of the $\mathbb{C}^*$-action (3.7) of type $(1, 1, \dots, 1)$, to be properly defined in the following section (see in particular (4.1)), and $E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}}$ is the variant part of the corresponding E-polynomial.

Theorem 3.14. *Let $n = 2, 3$. For any system of generic weights $\boldsymbol{\alpha}$, and any line bundle Λ, we have $E(\mathcal{M})^{\mathrm{var}} = (uv)^{\dim(\mathcal{M})/2} E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}}$.*

Proof. For $n = 2$, this follows from comparing Holla [26, Theorem 5.23] and Nitsure [34, Proposition 3.11] and using the argument of Atiyah–Bott [3, Prop. 9.7]; see also [34, Remark 3.11] and [14, Remark 10.1]. For $n = 3$, it follows from [14, Theorem 12.22] for $n = 3$; see also Remarks 12.17 and 12.19 of loc. cit.. □

Theorem 3.15. *For any n prime, any system of generic weights $\boldsymbol{\alpha}$, and any line bundle Λ, we have*

$$(uv)^{\dim(\mathcal{M})/2} E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}} = \sum_{\gamma \neq e} E(\mathcal{M}^{\gamma})^{\Gamma_n} (uv)^{F(\gamma)}$$

and both sides are equal to

$$\frac{n^{2g}-1}{n} (n!)^{|D|} (uv)^{(n^2-1)(g-1)+|D|n(n-1)/2} ((1-u)(1-v))^{(n-1)(g-1)}. \tag{3.16}$$

Remark 3.16. When $n = 2$, the polynomial (3.16) is equivalent to the one which appears in [20], the difference in sign being due to different conventions.

Remark 3.17. Since Theorem 3.15 is valid for any prime n, we see that the only obstacle for having a proof of Theorem 3.13 for any such n is the fact that Theorem 3.14 is not known to hold for any n prime.

The remaining part of the paper will be dedicated to the proof of Theorem 3.15. Again we follow the arguments of [21]. We shall prove that both $(uv)^{\dim(\mathcal{M})/2} E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}}$ and $\sum_{\gamma \neq e} E(\mathcal{M}^{\gamma})^{\Gamma_n} (uv)^{F(\gamma)}$ are equal to the given polynomial. The proofs of these equalities are completely independent of each other. The case of $(uv)^{\dim(\mathcal{M})/2} E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}}$ will be treated in Section 4, while the case of $\sum_{\gamma \neq e} E(\mathcal{M}^{\gamma})^{\Gamma_n} (uv)^{F(\gamma)}$ is going to be dealt with in Section 6. Section 5 is an independent section, containing some results on Prym varieties of unramified covers, which are needed in Section 6.

4. The polynomial $(uv)^{\dim(\mathcal{M})/2} E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}}$

The $\mathbb{C}^*$-action (3.7) on the moduli space $\mathcal{M}$ is a fundamental tool on the study of its geometry and topology. In particular the cohomology of $\mathcal{M}$ is completely determined by the cohomology of the subvarieties of fixed points, hence so is the E-polynomial of $\mathcal{M}$. Here we aim to compute the E-polynomial of a certain subspace of the fixed point loci of the $\mathbb{C}^*$-action, relevant for Theorem 3.15. In the next subsections, we describe these fixed point locus.

4.1. The fixed points of the $\mathbb{C}^*$-action.

Here we shall consider the fixed point subvarieties of the $\mathbb{C}^*$-action (3.7). From Proposition 3.7 we know that these are compact, but now we need a more explicit description of the fixed points. This is provided by the following result due to Simpson (see [37, Theorem 8]).

Proposition 4.1. *A stable parabolic $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle $(V, \varphi) \in \mathcal{M}$ is a fixed point under $\mathbb{C}^*$ if and only if either*

(1) $\varphi \equiv 0$, *or*
(2) *V admits a decomposition $V \cong \bigoplus_{j=1}^{l} V_j$ such that the following hold:*
  - *the subbundles V_j are parabolic and the decomposition $V \cong \bigoplus_{j=1}^{l} V_j$ is compatible with the parabolic structure, i.e., at every point $p \in D$, every subspace $V_{p,i}$ is a direct sum of fibers at p of certain subbundles V_j.*

- *the Higgs field splits as* $\varphi = \sum_{j=1}^{l} \varphi_j$*, with* $\varphi_j : V_j \to V_{j+1} \otimes K(D)$ *non-zero for all* $j = 1, \ldots, l-1$*, and* $\varphi_l \equiv 0$.

A parabolic Higgs bundle of the kind described in the preceding proposition is called a *Hodge bundle.* Note that we can include the ones of the form $(V, 0)$ in point (2) by taking $l = 1$, however it will be convenient for us to distinguish the two kinds of fixed points notationally.

Definition 4.2. A fixed point with non-vanishing Higgs field is said to be of *type* $(n_1, n_2, \ldots, n_l)$, with $\sum n_j = n$, if $\mathrm{rk}(V_j) = n_j$, for all j. Denote by $\mathcal{F}_{(n_1,n_2,\ldots,n_l)}$ the union of the subvarieties of $\mathcal{M}$ of all fixed points of type $(n_1, n_2, \ldots, n_l)$.

As is is well known, it follows from Bialynicki-Birula stratification associated to the $\mathbb{C}^*$-action that the cohomology of $\mathcal{M}$ is determined by the cohomology of all fixed point subvarieties of the $\mathbb{C}^*$-action. Indeed, the $\mathbb{C}^*$-flows gives rise to Zariski locally trivial affine bundles, with fibre $\mathbb{C}^{\dim(\mathcal{M})/2}$, over the disjoint union of all $\mathcal{F}_{(n_1,n_2,\ldots,n_l)}$ together with $\mathcal{N}$. This follows by Proposition 3.7 (1), and the projection of these affine bundles is just taking the limit of the flow when t goes to 0. Since the E-polynomial is additive with respect to disjoint unions and multiplicative with respect to locally trivial fibrations in the Zariski topology, we consequently have that

$$E(\mathcal{M}) = (uv)^{\dim(\mathcal{M})/2} \left(E(\mathcal{N}) + \sum_{(n_1,n_2,\ldots,n_l)} E(\mathcal{F}_{(n_1,n_2,\ldots,n_l)}) \right). \tag{4.1}$$

All $\mathcal{F}_{(n_1,n_2,\ldots,n_l)}$ and $\mathcal{N}$ are smooth and projective so we can consider their usual E-polynomials.

According to Theorem 3.15, the relevant subvarieties to be considered are the ones corresponding to type $(1, 1, \ldots, 1)$, that is $\mathcal{F}_{(1,1,\ldots,1)}$.

4.2. **The subvarieties $\mathcal{F}_{(1,1,\ldots,1)}$.** Let n be a prime number. Our next task is to obtain a geometric description of the subspace $\mathcal{F}_{(1,1,\ldots,1)}$. If (V, φ) represents a fixed point of the $\mathbb{C}^*$-action of type $(1, 1, \ldots, 1)$ then

$$V = \bigoplus_{j=1}^{n} L_j \qquad \text{and} \qquad \varphi = \sum_{j=1}^{n-1} \varphi_j, \quad \varphi_j : L_j \to L_{j+1} \otimes K(D), \ \varphi_n \equiv 0. \tag{4.2}$$

Since in $\mathcal{M}$ we always have fixed determinant Λ, then

$$\prod L_j \cong \Lambda. \tag{4.3}$$

The subspace $\mathcal{F}_{(1,1,\ldots,1)}$ is decomposed into connected components which can be labelled by the topological data coming from decomposition (4.2), namely the degrees of the bundles L_j and the way the weights are distributed among them at each point of D. Actually, instead of using the degrees of the bundles L_j, we shall opt for a slight variation of this.

Over each $p \in D$, we have the corresponding parabolic structure

$$V_p = V_{p,1} \supsetneq V_{p,2} \supsetneq \cdots \supsetneq V_{p,n} \supsetneq \{0\}, \qquad 0 \leqslant \alpha_1(p) < \cdots < \alpha_n(p) < 1. \tag{4.4}$$

By Proposition 4.1, each L_j is a parabolic subbundle of V and the decomposition (4.2) is compatible with the parabolic structure (4.4). The filtration of the fibre $L_{j,p}$ of L_j at p is of course trivial

$$L_{j,p} \supsetneq \{0\}, \tag{4.5}$$

and the corresponding weight $\beta_j(p)$ assigned to $L_{j,p}$ is $\beta_j(p) = \alpha_i(p)$ where i is such that $L_{j,p} \subseteq V_{p,i}$ but $L_{j,p} \nsubseteq V_{p,i+1}$; this is precisely the condition coming from (2.2). Since there are n line subbundles and the filtration (4.4) has length n, we see that (4.4) is determined by a distribution of the weights at p among the fibers of the line subbundles L_j at p. Precisely, $V_{p,n} = L_{j,p}$ where j is such that $\beta_j(p) = \alpha_n(p)$ and, for $i < n$, $V_{p,i} = V_{p,i+1} \oplus L_{j',p}$ with j' such that $\beta_{j'}(p) = \alpha_i(p)$. Such distribution of the n weights at p is provided by a permutation of the set $\{1, \dots, n\}$, so by an element $\varpi_n(p)$ of the symmetric group S_n. Write such permutation by a word

$$\varpi_n(p) = a_1(p)a_2(p)\dots a_n(p) \in S_n$$

with $a_j(p) \in \{1, \dots, n\}$, where this means that we assign the weight $\alpha_{a_j(p)}(p)$ to the fibre $L_{j,p}$. The conclusion is that the parabolic structure on $V = \bigoplus_{j=1}^n L_j$ is determined by an element

$$\varpi_n = (\varpi_n(p_1), \dots, \varpi_n(p_{|D|})) \in S_n^{|D|}. \tag{4.6}$$

Now we have to see how the Higgs field comes into play. It is given by (4.2), so $\varphi_j \in H^0(X, \mathrm{SParHom}(L_j, L_{j+1}) \otimes K(D))$ for every j. The residue of φ at $p \in D$ is given, according to the decomposition (4.2) of V, by

$$\varphi_p = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \varphi_{1,p} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varphi_{n-1,p} & 0 \end{pmatrix}.$$

Suppose $\varpi_n(p) = a_1(p)a_2(p)\dots a_n(p)$. Since φ is strongly parabolic, it follows from (4.5) that if $a_j(p) > a_{j+1}(p)$ then $\varphi_{j,p} = 0$. Thus

$$a_j(p) > a_{j+1}(p) \Longrightarrow \varphi_j \in H^0(X, \mathrm{Hom}(L_j, L_{j+1}) \otimes K(D-p)).$$

For each $j = 1, \dots, n-1$, define the subdivisor of D

$$S_j(\varpi_n) = \{p \in D \mid a_j(p) > a_{j+1}(p)\} \subseteq D,$$

so that

$$\varphi_j \in H^0(X, \mathrm{Hom}(L_j, L_{j+1}) \otimes K(D - S_j(\varpi_n))).$$

Let

$$M_j = L_j^{-1} L_{j+1} K(D - S_j(\varpi_n))$$

and write

$$m_j = \deg(M_j) = -d_j + d_{j+1} + 2g - 2 + |D| - s_j(\varpi_n) \geqslant 0. \tag{4.7}$$

with $d_j = \deg(L_j)$ and $s_j(\varpi_n) = |S_j(\varpi_n)|$, the cardinal of $S_j(\varpi_n)$. By (4.3),

$$\prod_{j=1}^{n-1} M_j^j \cong L_n^n \Lambda^{-1} K^{\frac{n(n-1)}{2}} \left(\frac{n(n-1)}{2} D - \sum_{j=1}^{n-1} j S_j(\varpi_n) \right) \tag{4.8}$$

and this implies

$$\begin{cases} d + \sum_{j=1}^{n-1} j(m_j + s_j(\varpi_n)) \equiv 0 \pmod{n}, & \text{if} \quad n \geqslant 3 \\ d + m_1 + s_1 - |D|(\varpi_2) \equiv 0 \pmod{2}, & \text{if} \quad n = 2. \end{cases} \tag{4.9}$$

Clearly the collection $(m_j)_j$ determines the collection $(d_j)_j$ and vice-versa through (4.7) and (4.8).

The proper φ-invariant subbundles of V are the ones of the form $V_l = \bigoplus_{j=l}^n L_j$, for $2 \leqslant l \leqslant n$. The stability condition $\mathrm{par}\mu(V_l) < \mathrm{par}\mu(V)$ (cf. Definition 2.11) for the subbundle V_l reads as

(4.10)

$$(n-l+1)\sum_{j=1}^{l-1} jm_j + (l-1)\sum_{j=l}^{n-1}(n-j)m_j < \sum_{p\in D}\left(\sum_{i=1}^{n}(n-l+1)\alpha_i(p) - n\sum_{j=l}^{n}\alpha_{a_j(p)}(p)\right) + \\ + (g-1+|D|/2)n(n-l+1)(l-1) - \\ - (n-l+1)\sum_{j=1}^{l-1} js_j(\varpi_n) - (l-1)\sum_{j=l}^{n-1}(n-j)s_j(\varpi_n).$$

Given ϖ_n as in (4.6) and $m_1, \dots, m_{n-1}$ non-negative integers such that (4.9) and (4.10) hold, denote by $\mathcal{F}_{(1,1,\dots,1)}(\varpi_n, m_1, \dots, m_{n-1})$ be the subspace of $\mathcal{F}_{(1,1,\dots,1)}$ determined by the given numerical/topological data. So we can write the decomposition of $\mathcal{F}_{(1,1,\dots,1)}$ as

$$\mathcal{F}_{(1,1,\dots,1)} = \bigsqcup_{\varpi_n\in S_n^{|D|}} \bigsqcup_{\substack{m_1,\dots,m_{n-1}\geqslant 0\\ \text{such that (4.9), (4.10) hold}}} \mathcal{F}_{(1,1,\dots,1)}(\varpi_n, m_1, \dots, m_{n-1}), \tag{4.11}$$

and, from what we have done so far, the following is clear.

Proposition 4.3. *Let $\varpi_n \in S_n^{|D|}$ as in (4.6) and $m_1, \dots, m_{n-1}$ non-negative integers verifying (4.9) and (4.10) for every $l = 2, \dots, n$. Then the critical subvariety $\mathcal{F}_{(1,1,\dots,1)}(\varpi_n, m_1, \dots, m_{n-1})$ is given by the pull-back diagram*

$$\begin{array}{ccc} \mathcal{F}_{(1,1,\dots,1)}(\varpi_n, m_1, \dots, m_{n-1}) & \longrightarrow & \mathrm{Jac}^{d_n}(X) \\ \downarrow & & \downarrow \\ \prod_{j=1}^{n-1}\mathrm{Sym}^{m_j}(X) & \longrightarrow & \mathrm{Jac}^{\sum_j jm_j}(X)\ , \end{array}$$

where:

- *the top map is $(V,\varphi) = (\bigoplus_j L_j, \sum_j \varphi_j) \mapsto L_n$;*
- $d_n = \frac{1}{n}\sum_{j=1}^{n-1} j(m_j + s_j(\varpi_n)) - (n-1)(g-1+|D|/2)$;
- *the vertical map on the left is given by $(V,\varphi) = (\bigoplus_j L_j, \sum_j \varphi_j) \mapsto (\mathrm{div}(\varphi_1), \dots, \mathrm{div}(\varphi_{n-1}))$;*
- *the map on the bottom is $(D_1, \dots, D_{n-1}) \mapsto \mathcal{O}_X(\sum_j jD_j)$;*
- *the vertical map on the right is $L_n \mapsto L_n^n \Lambda^{-1} K^{\frac{n(n-1)}{2}}\left(\frac{n(n-1)}{2}D - \sum_{j=1}^{n-1} jS_j(\varpi_n)\right)$.*

4.3. **The E-polynomial of the variant part.** The proof of the next result uses the description $\mathcal{F}_{(1,1,\dots,1)}(\varpi_n, m_1, \dots, m_{n-1})$ given in Proposition 4.3. It can be found essentially in [22, Theorem 7.6 (iv)], [16, Proposition 3.11] and [21, Proposition 10.1]. Again it is essential that n is prime.

Recall that the group Γ_n acts on $\mathcal{M}$ by (2.4). This action clearly preserves each component $\mathcal{F}_{(1,1,\dots,1)}(\varpi_n, m_1, \dots, m_{n-1})$ of $\mathcal{F}_{(1,1,\dots,1)}$.

Proposition 4.4. *Let n be prime. The variant part of the cohomology of $\mathcal{F}_{(1,1,\dots,1)}(\varpi_n, m_1, \dots, m_{n-1})$ is non-trivial only in degree $m_1 + \cdots + m_{n-1}$. More precisely,*

$$H^*(\mathcal{F}_{(1,1,\dots,1)}(\varpi_n, m_1, \dots, m_{n-1}), \mathbb{C})^{\mathrm{var}} \cong \bigoplus_{\gamma\in\Gamma_n\setminus\{e\}} \bigotimes_{j=1}^{n-1} \Lambda^{m_j} H^1(X, L_\gamma^j),$$

where $H^1(X, L_\gamma^j)$ denotes twisted cohomology with values in the local system L_γ^j, and L_γ is the flat line bundle corresponding to γ.

We are now in position to determine the variant part of the E-polynomial of $\mathcal{M}$.

Proposition 4.5. *For any n prime, the following holds:*

$$(uv)^{\dim(\mathcal{M})/2}E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}} = \frac{n^{2g}-1}{n}(n!)^{|D|}(uv)^{(n^2-1)(g-1)+|D|n(n-1)/2}((1-u)(1-v))^{(n-1)(g-1)}.$$

Proof. By (4.11),

$$E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}}(u,v) = \sum_{\varpi_n\in S_n^{|D|}} \sum_{\substack{m_1,\dots,m_{n-1} \\ \text{such that (4.9), (4.10) hold}}} E(\mathcal{F}_{(1,1,\dots,1)}(\varpi_n,m_1,\dots,m_{n-1}))^{\mathrm{var}}(u,v), \tag{4.12}$$

and then multiply it by the factor $(uv)^{\dim(\mathcal{M})/2}$.

For any non-trivial $\gamma\in\Gamma_n$ and any j, $\dim H^1(X,L_\gamma^j)=g-1$, i.e., $\dim H^{1,0}(X,L_\gamma^j)=g-1$. Thus by Proposition 4.4, we find that

$$E(\mathcal{F}_{(1,1,\dots,1)}(\varpi_n,m_1,\dots,m_{n-1}))^{\mathrm{var}}(u,v) = (n^{2g}-1)\prod_{j=1}^{n-1}\sum_{\substack{p+q=m_j\\0\leqslant p,q\leqslant g-1}}(-1)^{p+q}\binom{g-1}{p}\binom{g-1}{q}u^pv^q.$$

We need to sum this expression over all k-tuples of permutations ϖ_n and over all non-negative integers m_j such that (4.9) and (4.10) hold.

Regarding the summation over the m_j, note that the right hand side is zero whenever there is an $m_j>2g-2$.

Next we make an assumption on the (generic) weights. Suppose they are such that the summand $\sum_{p\in D}\left(\sum_{i=1}^n(n-l+1)\alpha_i(p)-n\sum_{j=l}^n\alpha_{a_j(p)}(p)\right)$ in (4.10) is very close to zero. This is of course possible, for instance by imposing that the all the weights are also very to zero, thus very small comparing to 1:

$$\alpha_i(p)\ll 1,\ \text{for every } p\in D. \tag{4.13}$$

With these weights, and using the fact that $s_j(\varpi_n)\leqslant|D|$ for all j, one shows that (4.10) holds for $m_1=\dots=m_{n-1}=2g-2$, hence holds for any choice of m_j between 0 and $2g-2$ for every j. Therefore we can sum over all $0\leqslant m_1,\dots,m_{n-1}\leqslant 2g-2$ subject to condition (4.9). This is done by taking $\xi=\exp(2\pi i/n)$ and since, for a given integer $\nu\in\mathbb{Z}$, the sum $\sum_{l=0}^{n-1}\xi^{l\nu}$ equals n if $\nu\equiv 0 \pmod n$ and zero otherwise, we have that $E(\mathcal{F}_{(1,1,\dots,1)})^{\mathrm{var}}(u,v)$ in (4.12) equals, if $n\geqslant 3$,

$$\begin{aligned}
&\frac{n^{2g}-1}{n}\sum_{\varpi_n\in S_n^{|D|}}\sum_{m_1,\dots,m_{n-1}=0}^{2g-2}\sum_{l=0}^{n-1}\xi^{l\sum_{j=1}^{n-1}j(m_j+s_j(\varpi_n))}\prod_{j=1}^{n-1}\sum_{\substack{p+q=m_j\\0\leqslant p,q\leqslant g-1}}(-1)^{p+q}\binom{g-1}{p}\binom{g-1}{q}u^pv^q\\
&=\frac{n^{2g}-1}{n}\sum_{\varpi_n\in S_n^{|D|}}\sum_{l=0}^{n-1}\xi^{l\sum_{j=1}^{n-1}js_j(\varpi_n)}\prod_{j=1}^{n-1}\sum_{m_j=0}^{2g-2}\sum_{\substack{p+q=m_j\\0\leqslant p,q\leqslant g-1}}(-1)^{p+q}\binom{g-1}{p}\binom{g-1}{q}u^pv^q\xi^{jlm_j}\\
&=\frac{n^{2g}-1}{n}\sum_{\varpi_n\in S_n^{|D|}}\sum_{l=0}^{n-1}\xi^{l\sum_{j=1}^{n-1}js_j(\varpi_n)}\prod_{j=1}^{n-1}(1-\xi^{jl}u)^{g-1}(1-\xi^{jl}v)^{g-1}\\
&=\frac{n^{2g}-1}{n}(n!)^{|D|}((1-u)(1-v))^{(n-1)(g-1)}+\frac{n^{2g}-1}{n}\left(\frac{(1-u^n)(1-v^n)}{(1-u)(1-v)}\right)^{g-1}(n\mathcal{S}(n,d)-n!)^{|D|},
\end{aligned}$$

where in the last equality we used the fact that n is prime, and where

$$\mathcal{S}(n,d)=\#\left\{\varpi_n(p)\in S_n \mid \sum_{j=1}^{n-1}js_j(\varpi_n(p))\equiv 0 \pmod n\right\}.$$

For $n = 2$ we perform the precise same computation, except that we use the expression corresponding to $n = 2$ in (4.9), yielding

$$E(\mathcal{F}_{(1,1)})^{\mathrm{var}} = 2^{|D|-1}(2^{2g}-1)((1-u)(1-v))^{g-1}+\frac{2^{2g}-1}{2}((1+u)(1+v))^{g-1}(2\mathcal{S}(2,d)-2)^{|D|},$$

where $\mathcal{S}(2,d) = \#\{\varpi_2(p) \in S_2 \mid |D| + s_1(\varpi_2(p)) \equiv 0 \pmod 2\}$.

It is clear that that the values of both $\mathcal{S}(n,d)$ and $\mathcal{S}(2,d)$ are independent of $p \in D$. It is also clear that $\mathcal{S}(2,d) = 1$. Actually by Lemma 4.6 below, we have $\mathcal{S}(n,d) = (n-1)!$, hence, for any $n \geqslant 2$ prime, $(uv)^{\dim(\mathcal{M})/2}E(\mathcal{F}_{(1,1,\ldots,1)})^{\mathrm{var}}(u,v)$ equals

$$\frac{n^{2g}-1}{n}(n!)^{|D|}(uv)^{(n^2-1)(g-1)+|D|n(n-1)/2}((1-u)(1-v))^{(n-1)(g-1)}.$$

This completes the proof for the moduli space with weights under the assumption (4.13). But by Corollary 3.10 we know that the E-polynomials $E(\mathcal{M})$ and $E(\mathcal{M})^{\Gamma_n}$ are independent of the weights, hence so is $E(\mathcal{F}_{(1,1,\ldots,1)})^{\mathrm{var}}$, therefore the formula we reached is valid for any generic weights. □

The next lemma completes the proof of Proposition 4.5.

Lemma 4.6. *For any $n \geqslant 2$ and d, $\mathcal{S}(n,d) = (n-1)!$.*

Proof. This is a purely combinatorial proof. Since the number $\mathcal{S}(n,d)$ is obviously independent of $p \in D$, we will remove it from the notation. Any permutation $\varpi_n \in S_n$ is obtained from a permutation $\varpi_{n-1} \in S_{n-1}$ by inserting n in the appropriate position. Conversely, any $\varpi_{n-1} = a_1\, a_2 \ldots a_{n-1} \in S_{n-1}$ produces n distinct permutations in S_n, by inserting n in ϖ_{n-1} in each one of the possible j positions of ϖ_{n-1}, where $j \in \{0,\ldots,n-1\}$. Write $\varpi_{n-1}(j)$ for such permutation in S_n, so that

$$\begin{aligned}\varpi_{n-1}(0) &= n\, a_1 \ldots a_{n-1},\\ \varpi_{n-1}(j) &= a_1 \ldots a_j\, n\, a_{j+1} \ldots a_{n-1} \quad 1 \leqslant j \leqslant n-2,\\ \varpi_{n-1}(n-1) &= a_1 \ldots a_{n-1}\, n.\end{aligned}$$

Fix any $\varpi_{n-1} \in S_{n-1}$. Let A be the ordered set of the indexes i between 1 and $n-2$ where $s_i(\varpi_{n-1}) = 1$. In other words,

$$A = \{i_1,\ldots,i_s \mid i_\ell < i_{\ell+1},\ a_{i_\ell} > a_{i_\ell+1}\},$$

for some $s \in \{0,\ldots,n-2\}$ (where $A = \emptyset \Leftrightarrow s = 0$). Notice that we always have $i_\ell \geqslant \ell$. Let

$$\sigma = \sum_{i=1}^{n-2} i s_i(\varpi_{n-1}) = \sum_{i\in A} i = i_1 + \cdots + i_s.$$

For each $j = 0,\ldots,n-1$, let $\sigma_n(j) \in \mathbb{Z}_n$ be the class modulo n of the difference

$$\sum_{i=1}^{n-1} i s_i(\varpi_{n-1}(j)) - \sigma.$$

We claim that for any $\varpi_{n-1} \in S_{n-1}$,

$$\{\sigma_n(j) \mid j \in \{0,\ldots,n-1\}\} = \{0,\ldots,n-1\}. \tag{4.14}$$

Note that this proves that $\mathcal{S}(n,d)$ is in bijection with S_{n-1} for any d, hence proves the lemma.

To prove (4.14), we shall explicitly give for each $k \in \{0,\ldots,n-1\}$, the corresponding $j \in \{0,\ldots,n-1\}$ such that $\sigma_{n-1}(j) = k$.

- If $k = 0$, then we have obviously to take $j = n-1$. Indeed, $\sum_{i=1}^{n-1} i s_i(\varpi_{n-1}(n-1)) = \sigma$, hence $\sigma_n(n-1) = 0$.

- If $1 \leqslant k \leqslant s$, take $j = i_\ell \in A$, where $\ell = s - k + 1$. Then $\sum_{i=1}^{n-1} is_i(\varpi_{n-1}(i_\ell)) = \sigma + s - \ell + 1 = \sigma + k$, so $\sigma_n(i_\ell) = k$
- If $s + 1 \leqslant k \leqslant s + i_1$, then take $j = k - s - 1$. In fact, since $j < i_1$, we have that $\sum_{i=1}^{n-1} is_i(\varpi_{n-1}(j)) = j + 1 + \sigma + s = \sigma + k$, i.e., $\sigma_n(k - s - 1) = k$.
- Suppose now that $i_\ell + s - \ell + 2 \leqslant k \leqslant i_{\ell+1} + s - \ell$, for some $\ell \in \{1, \dots, s-1\}$. Note that these situations are possible only if $i_{\ell+1} \geqslant i_\ell + 2$. By taking $j = k - s + \ell - 1$, one checks that $\sum_{i=1}^{n-1} is_i(\varpi_{n-1}(j)) = \sigma + j + s - \ell + 1 = \sigma + k$, and $\sigma_n(k - s + \ell - 1) = k$.
- Finally, if $i_s + 2 \leqslant k \leqslant n - 1$, choose $j = k - 1$. Indeed, $\sum_{i=1}^{n-1} is_i(\varpi_{n-1}(j)) = \sigma + j + 1 = \sigma + k$. Hence, $\sigma_n(k - 1) = k$.

In these items we ran through all the possible values of $k \in \{0, \dots, n-1\}$, exactly once each, and we found a bijection with the positions $j \in \{0, \dots, n-1\}$ such that $\sigma_n(j) = k \in \mathbb{Z}_n$. This proves (4.14) and thus the lemma. □

5. Unramified cyclic covers, norm maps and Pryms

The purpose of the following section is to recall some classical results about Prym varieties of unramified coverings, essentially going back to Narasimhan–Ramanan [32] and Mumford [31], and corresponding to Section 7 of [21]. For the benefit of the interested reader, we have included complete proofs.

5.1. **Connected components of the kernel of a norm map.** In Section 3.1, we considered the Prym variety of a ramified cover in the context of the Hitchin fibration. In the case of an unramified cover the structure of the kernel of the norm map turns out to be quite different.

Let n be a prime number. Fix $\gamma \in \Gamma_n$ and let L_γ be the corresponding n-torsion line bundle on X. Denote the associated unramified regular n-cover by

$$\pi : X_\gamma \to X. \tag{5.1}$$

Recall that X_γ is the spectral cover of X defined as the curve in the total space $|L_\gamma|$ of L_γ defined by the equation $\lambda^n - 1 = 0$, where $\lambda \in H^0(|L_\gamma|, p^* L_\gamma)$ is the tautological section, and $p : |L_\gamma| \to X$ is the projection. Then π is the restriction of p to X_γ. The line bundle $\pi^* L_\gamma$ is trivial over X_γ since the nowhere vanishing section $\lambda : \mathcal{O}_{X_\gamma} \to \pi^* L_\gamma$ gives a canonical trivialization.

Let $\mathrm{Pic}(X)$ be Picard group of X and $\mathrm{Pic}^i(X)$ be the component corresponding to line bundles of degree i, so that $\mathrm{Pic}^0(X) \cong \mathrm{Jac}(X)$, and

$$\mathrm{Pic}(X) = \bigsqcup_{i \in \mathbb{Z}} \mathrm{Pic}^i(X).$$

Consider the same notations for the curve X_γ. The dimension of $\mathrm{Pic}(X)$ is g while the dimension of $\mathrm{Pic}(X_\gamma)$ is the genus of X_γ, given by $n(g-1)+1$.

The pullback map $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_\gamma)$ is not injective neither surjective. The non-surjectivity of π^* is clear by dimensional reasons and also because $\pi^*(\mathrm{Pic}^i(X)) \subset \mathrm{Pic}^{ni}(X_\gamma)$. The next proposition provides the description of the image. Consider the Galois group of the covering $\pi : X_\gamma \to X$. It is isomorphic to $\mathbb{Z}_n$, which we consider as the group of the n-th roots of unity. Let $\xi = \exp(2\pi i/n)$ be the standard generator. The Galois group $\mathbb{Z}_n$ acts on $\mathrm{Pic}^i(X_\gamma)$ by pullback and obviously a line bundle over X_γ is fixed by $\mathbb{Z}_n$ if and only if it is fixed by ξ.

Proposition 5.1. *The kernel of π^* is the finite free abelian group generated by L_γ, that is $\ker(\pi^*) \cong \mathbb{Z}_n$. The image coincides with $\mathrm{Pic}(X_\gamma)^{\mathbb{Z}_n}$, i.e. the fixed point subvariety of $\mathrm{Pic}(X_\gamma)$ under the Galois group. So π^* yields an isomorphism $\mathrm{Pic}^i(X)/\mathbb{Z}_n \cong \mathrm{Pic}^{ni}(X_\gamma)^{\mathbb{Z}_n}$.*

Proof. We already know that $\pi^* L_\gamma^j \cong \mathcal{O}_{X_\gamma}$, for any $j = 0, \dots, n-1$. For the converse, take L a degree 0 line bundle on X whose pullback is trivial. Then

$$\mathcal{O}_X \oplus L_\gamma^{-1} \oplus \cdots \oplus L_\gamma^{-(n-1)} \cong \pi_* \mathcal{O}_{X_\gamma} \cong \pi_* \pi^* L \cong L \otimes (\mathcal{O}_X \oplus L_\gamma^{-1} \oplus \cdots \oplus L_\gamma^{-(n-1)})$$

which implies that L must be some power of L_γ.

Regarding the image of π^*, since $\pi \circ \xi = \pi$, it is clear that $\xi^* \pi^* L = \pi^* L$ for any $L \in \mathrm{Pic}(X)$. Conversely, if $F \in \mathrm{Pic}(X_\gamma)$ is fixed by ξ, then F descends to a line bundle L in X so that $F = \pi^* L$. □

We shall now consider the norm map associated to the unramified cover $\pi : X_\gamma \to X$. There are several incarnations of this map, all of them compatible with each other. We will consider three of them and use the same notation for all. The context will clarify the ones we are using. The norm map on divisors is given by

$$\mathrm{Nm}_\pi : \mathrm{Div}(X_\gamma) \to \mathrm{Div}(X), \quad E = \sum_p E(p) p \mapsto \mathrm{Nm}_\pi(E) = \sum_p E(p) \pi(p).$$

(This has already been defined before, for more general coverings, in Definition 3.3.) We also have the norm map on the fields of non-zero meromorphic functions, given by

$$\mathrm{Nm}_\pi : \mathcal{M}(X_\gamma)^* \to \mathcal{M}(X)^*, \quad \mathrm{Nm}_\pi(f)(p) = \prod_{q \in \pi^{-1}(p)} f(q). \tag{5.2}$$

It is clear that $\mathrm{Nm}_\pi(\mathrm{div}(f)) = \mathrm{div}(\mathrm{Nm}_\pi(f))$, for any $f \in \mathcal{M}(X_\gamma)^*$, hence the norm map on divisors induces the norm map on the Picard groups, i.e., on line bundles:

$$\mathrm{Nm}_\pi : \mathrm{Pic}(X_\gamma) \to \mathrm{Pic}(X), \quad \mathcal{O}_{X_\gamma}(E) \mapsto \mathcal{O}_X(\mathrm{Nm}_\pi(E)). \tag{5.3}$$

Let $\ker(\mathrm{Nm}_\pi)$ be the subvariety of $\mathrm{Jac}(X_\gamma)$ defined as the kernel of (5.3), and consider the group homomorphism

$$p : \mathrm{Pic}(X_\gamma) \to \ker(\mathrm{Nm}_\pi), \qquad L \mapsto L^{-1} \otimes \xi^* L. \tag{5.4}$$

It is well-defined since $\mathrm{Nm}_\pi(L^{-1} \otimes \xi^* L) = \mathrm{Nm}_\pi(L^{-1}) \otimes \mathrm{Nm}_\pi(\xi^* L) = \mathrm{Nm}_\pi(L)^{-1} \otimes \mathrm{Nm}_\pi(L) = \mathcal{O}_X$.

The following is a generalization to $n \geqslant 2$ of Lemma 1 of Mumford [31].

Proposition 5.2. *The homomorphism p is surjective and the same holds for the restriction of p to the disjoint union* $\bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma)$.

Proof. Let $M \in \ker(\mathrm{Nm}_\pi) \subset \mathrm{Jac}(X_\gamma)$. Then M must be isomorphic to $\mathcal{O}_{X_\gamma}(F)$, for some degree 0 divisor F, such that $\mathrm{Nm}_\pi(F) = \mathrm{div}(f)$, for some non-zero meromorphic function f on X. But the norm map (5.2) on function fields is surjective (see [28] and also [2, p. 282]), hence $f = \mathrm{Nm}_\pi(g)$ for some $g \in \mathcal{M}(X_\gamma)^*$. Let $G = \mathrm{div}(g)$. Then $\mathrm{Nm}_\pi(G) = \mathrm{Nm}_\pi(\mathrm{div}(g)) = \mathrm{div}(\mathrm{Nm}_\pi(g)) = \mathrm{div}(f) = \mathrm{Nm}_\pi(F)$. Define $\bar{F} = F - G$. Then $\mathrm{Nm}_\pi(\bar{F}) = 0$, hence $\bar{F}$ must be of the form $\bar{F} = \sum_{p \in X_\gamma} \bar{F}(p) p$ with

$$\sum_{i=0}^{n-1} \bar{F}(\xi^i(p)) = 0. \tag{5.5}$$

Now choose one and only one element in the support of $\bar{F}$ in each fibre of Nm_π over the support of $\mathrm{Nm}_\pi(\bar{F})$. This yields a collection of points $p_1, \dots, p_m$ in the support of $\bar{F}$ such that $\pi(p_i) \neq \pi(p_j)$, for $i \neq j$. For each $l = 1, \dots, m$, choose integers $k_1(l), \dots, k_n(l)$ such that

$$\bar{F}(\xi^i(p_l)) = -k_{i+1}(l) + k_i(l) \tag{5.6}$$

for every $i = 0, \ldots, n$ and where, by definition, $k_0(l) = k_n(l)$. This is possible (in an infinite number of ways) due to (5.5). Define the divisor

$$\tilde{F} = \sum_{l=1}^{m} \sum_{i=0}^{n-1} k_{i+1}(l)\xi^i(p_l)$$

and the corresponding line bundle $\tilde{L} \cong \mathcal{O}_{X_\gamma}(\tilde{D})$. It follows from (5.6) that $\mathcal{O}_{X_\gamma}(\bar{F}) \cong \tilde{L}^{-1} \otimes \xi^*\tilde{L}$, that is,

$$M \cong \tilde{L}^{-1} \otimes \xi^*\tilde{L} = p(\tilde{L}),$$

because $M \cong \mathcal{O}_{X_\gamma}(F)$, $\bar{F} = F - G$ and $\mathcal{O}_{X_\gamma}(G)$ is trivial. Hence p is surjective.

To show that its restriction to $\bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma)$ is also surjective, consider the same line bundle $M = p(\tilde{L})$ and let $\tilde{d}$ be the degree of $\tilde{L}$. Let $a \in \{0, \ldots, n-1\}$ be the reduction of $\tilde{d}$ modulo n and choose a line bundle M' on X, of degree $(\tilde{d}-a)/n$. Then $\deg(\tilde{L}\otimes\pi^*M') = a$ and

$$M = \tilde{L}^{-1} \otimes \pi^*M'^{-1} \otimes \xi^*(\tilde{L} \otimes \pi^*M') = p(\tilde{L} \otimes \pi^*M'),$$

completing the proof. □

Proposition 5.3. *The kernel of p equals the image of π^*. Hence*

$$\ker(p) \cap \bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma) = \pi^*(\mathrm{Pic}^0(X)) \cong \mathrm{Pic}^0(X)/\mathbb{Z}_n.$$

Proof. The kernel of p is precisely given by the fixed points under ξ (thus under $\mathbb{Z}_n$), hence the result follows immediately from Proposition 5.1. The second part follows because $\pi^*(\mathrm{Pic}^i(X)) \subset \mathrm{Pic}^{ni}(X_\gamma)$. □

The previous propositions can be summarized in the next corollary:

Corollary 5.4. *The following sequence of groups is exact:*

$$0 \to \mathbb{Z}_n \to \mathrm{Pic}(X) \xrightarrow{\pi^*} \mathrm{Pic}(X_\gamma) \xrightarrow{p} \ker(\mathrm{Nm}_\pi) \to 0.$$

Moreover, the restriction of p to $\bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma)$,

$$p : \bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma) \to \ker(\mathrm{Nm}_\pi) \tag{5.7}$$

is a holomorphic $\mathrm{Pic}^0(X)/\mathbb{Z}_n$-principal bundle.

The following is now immediate from the stated property of the map (5.7).

Corollary 5.5. *The kernel $\ker(\mathrm{Nm}_\pi)$ of the norm map (5.3) has n connected components, which are labelled by the n connected components of $\bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma)$ via the group homomorphism (5.4).*

Recall from Definition 3.4 that the Prym variety of X_γ associated to the covering $\pi : X_\gamma \to X$ is the abelian variety defined as the connected component of $\ker(\mathrm{Nm}_\pi)$ containing the identity:

$$\mathrm{Prym}_\pi(X_\gamma) = \ker(\mathrm{Nm}_\pi)_0.$$

Note that now we do not have the equality corresponding to (3.4).

Proposition 5.6. *Two line bundles M_1 and M_2 are in the same connected component of the kernel of Nm_π if and only if $M_1 = L_1^{-1} \otimes \xi^*L_1$ and $M_2 = L_2^{-1} \otimes \xi^*L_2$, with $\deg(L_1) = \deg(L_2)$. In particular $\mathrm{Prym}_\pi(X_\gamma)$ is the subspace of $\ker(\mathrm{Nm}_\pi)$ consisting of those line bundles of the form $L^{-1} \otimes \xi^*L$, with $\deg(L) = 0$.*

Thus the following sequence of groups is exact:

$$0 \to \mathbb{Z}_n \to \mathrm{Pic}^0(X) \xrightarrow{\pi^*} \mathrm{Pic}^0(X_\gamma) \xrightarrow{p} \mathrm{Prym}_\pi(X_\gamma) \to 0,$$

hence

$$\dim(\mathrm{Prym}_\pi(X_\gamma)) = (n-1)(g-1). \tag{5.8}$$

We shall need a generalization of Proposition 5.6 to any fibre of the norm map, and not only its kernel. That is easily achieved since such fibre is a torsor for the kernel, hence being isomorphic to $\ker(\mathrm{Nm}_\pi)$ although not canonically. Let then Λ be a degree d, holomorphic line bundle over X. Choose an arbitrary line bundle $L_0 \in \mathrm{Nm}_\pi^{-1}(\Lambda)$. Given this choice, we have the obvious isomorphism

$$\ker(\mathrm{Nm}_\pi) \xrightarrow{\cong} \mathrm{Nm}_\pi^{-1}(\Lambda), \qquad M \mapsto M \otimes L_0.$$

Consider the union $\bigsqcup_{i=d}^{d+n-1} \mathrm{Pic}^i(X_\gamma)$. The same kind of isomorphism holds,

$$\bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma) \xrightarrow{\cong} \bigsqcup_{i=d}^{d+n-1} \mathrm{Pic}^i(X_\gamma), \qquad L \mapsto L \otimes L_0$$

and we have the analogue of the restriction of the map p to $\bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma)$,

$$p_{L_0} : \bigsqcup_{i=d}^{d+n-1} \mathrm{Pic}^i(X_\gamma) \to \mathrm{Nm}_\pi^{-1}(\Lambda), \qquad p_{L_0}(L) = L^{-1} \otimes \xi^* L \otimes L_0. \tag{5.9}$$

Hence the following diagram commutes:

$$\begin{array}{ccc} \bigsqcup_{i=0}^{n-1} \mathrm{Pic}^i(X_\gamma) & \xrightarrow{\cong} & \bigsqcup_{i=d}^{d+n-1} \mathrm{Pic}^i(X_\gamma) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_{L_0}} \\ \ker(\mathrm{Nm}_\pi) & \xrightarrow[\cong]{} & \mathrm{Nm}_\pi^{-1}(\Lambda) \end{array} ,$$

The twisted version of the previous results reads then as follows. This result (and the particular case of Corollary 5.5) goes back at least to Narasimhan–Ramanan [32].

Proposition 5.7. *The map p_{L_0} is a holomorphic $\mathrm{Pic}^0(X)/(\mathbb{Z}_n)$-principal bundle and the n connected components of $\mathrm{Nm}_\pi^{-1}(\Lambda)$ are labeled by the degree $i \in \{d, \dots, d+n-1\}$. Moreover, $\mathrm{Pic}^i(X)$ is a holomorphic $\mathrm{Pic}^0(X)/\mathbb{Z}_n$-principal bundle over a connected component of $\mathrm{Nm}_\pi^{-1}(\Lambda)$.*

Thus each connected component of $\mathrm{Nm}_\pi^{-1}(\Lambda)$ is a torsor for $\mathrm{Prym}_\pi(X_\gamma)$.

5.2. **The action of the Galois group.** We now wish to see how the Galois group $\mathbb{Z}_n$ acts on the on components of the fibre of the norm map. This is not strictly necessary for what follows but we include it for completeness.

We continue with an unramified n-cover (5.1) and a line bundle Λ over X of degree d. Let $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$ be the set consisting of the n connected components of the fibre of the norm map of over Λ. Let (n,d) denote the greatest common divisor of n and d.

Proposition 5.8. *The $\mathbb{Z}_n$-orbit of any element of $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$ has $n/(n,d)$ elements. In particular, $\mathbb{Z}_n$ acts trivially on $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$ if and only if d is a multiple of n and acts transitively if and only if n and d are coprime.*

Proof. Let M be a line bundle of degree d such that $\mathrm{Nm}_\pi(M) = \Lambda$. Then $M = p_{L_0}(L)$ for some degree $i \in \{d, \dots, d+n-1\}$ line bundle L over X_γ, where p_{L_0} is defined in (5.9),

$$M = L^{-1} \otimes \xi^* L \otimes L_0.$$

By Proposition 5.7, the component of $\mathrm{Nm}_\pi^{-1}(\Lambda)$ where M lies is determined by the degree i of L. Since

$$\xi^* M = \xi^* L^{-1} \otimes \xi^* \xi^* L \otimes \xi^* L_0 = (\xi^* L \otimes L_0)^{-1} \otimes \xi^*(\xi^* L \otimes L_0) \otimes L_0 = p_{L_0}(\xi^* L \otimes L_0)$$

then $\xi^* M$ lies in the component determined by the degree of $\xi^* L \otimes L_0$, which is $i + d$. So $\xi^* L \otimes L_0$ is in the same connected component as M is and only if $i + d$ is equal to i modulo n, that is d is a multiple of n.

If n does not divide d then, from what we saw, the orbit of M on $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$ is determined by class in $\mathbb{Z}_n$ of the numbers $i+jd$, with $j = 0, \dots, n-1$. Hence we conclude that the orbit of M runs over $n/(n,d)$ different connected components of $\mathrm{Nm}_\pi^{-1}(\Lambda)$. □

We know that $\mathrm{Pic}(X_\gamma)^{\mathbb{Z}_n}$ is the image of π^*. Let us now see what is its intersection with a fibre of Nm_π.

Proposition 5.9. *The intersection* $\mathrm{Pic}(X_\gamma)^{\mathbb{Z}_n} \cap \mathrm{Nm}_\pi^{-1}(\Lambda)$ *is the image of* $\pi^*|_{\Gamma_n}$. *In particular, it is empty if* d *is not a multiple of* n. *If not empty, it has* n^{2g-1} *elements.*

Proof. Since $\mathbb{Z}_n$ is generated by ξ, it is enough to consider a line bundle $L \in \mathrm{Nm}_\pi^{-1}(\Lambda)$ such that $L \cong \xi^* L$. By Proposition 5.1, we have that $L \cong \pi^* N$ for some line bundle N over X. Then $\Lambda \cong \mathrm{Nm}_\pi(\pi^* N) \cong N^n$. This is only possible if $d = n \deg(N)$, showing that otherwise the intersection is empty.

This proves that the intersection is the image, under π^*, of the n^{2g} nth-roots of Λ. But this image only has n^{2g-1} elements since two such roots are pulled-back to the same element whenever they differ by a power of L_γ. □

5.3. **The action of Γ_n and the Weil pairing.** Now we consider the action of Γ_n. An element $\delta \in \Gamma_n$ acts on $\mathrm{Nm}_\pi^{-1}(\Lambda)$ by

$$M \mapsto M \otimes \pi^* L_\delta, \tag{5.10}$$

where L_δ is the n-torsion line bundle on X corresponding to δ. Indeed, $\mathrm{Nm}_\pi(\pi^* L_\delta) = L_\delta^n \cong \mathcal{O}_X$, therefore $\mathrm{Nm}_\pi(M \otimes \pi^* L_\delta) = \Lambda$.

Proposition 5.10. *The* $\mathbb{Z}_n = \langle\gamma\rangle \subset \Gamma_n$ *acts trivially on* $\mathrm{Nm}_\pi^{-1}(\Lambda)$. *The* Γ_n*-action on* $\mathrm{Nm}_\pi^{-1}(\Lambda)$ *induces a free action of* $\Gamma_n/\mathbb{Z}_n \cong \mathbb{Z}_n^{2g-1}$ *on* $\mathrm{Nm}_\pi^{-1}(\Lambda)$.

Proof. The elements $\delta \in \Gamma_n$ which fix some element in $\mathrm{Nm}_\pi^{-1}(\Lambda)$ are the ones such that $\pi^* L_\delta \cong \mathcal{O}_{X_\gamma}$, and thus they fix every point in $\mathrm{Nm}_\pi^{-1}(\Lambda)$. By Proposition 5.1, δ is such an element if and only if L_δ is a power of L_γ, i.e., $\delta \in \langle\gamma\rangle \cong \mathbb{Z}_n$. So the Γ_n-action factors through a free $\Gamma_n/(\mathbb{Z}_n) \cong (\mathbb{Z}_n)^{2g-1}$-action on $\mathrm{Nm}_\pi^{-1}(\Lambda)$. □

We must especially study the action of Γ_n on the set of connected components of $\mathrm{Nm}_\pi^{-1}(\Lambda)$. Since $\Gamma_n \cong H^1(X, \mathbb{Z}_n)$, we have a pairing on Γ_n given by cup product followed by evaluation on the fundamental class:

$$\langle\, ,\rangle : \Gamma_n \times \Gamma_n \to \mathbb{Z}_n, \tag{5.11}$$

where $\mathbb{Z}_n$ is given the multiplicative structure. This is a symplectic pairing, called the *Weil pairing*.

It will be convenient to give a different (but equivalent) definition of the Weil pairing. First, given a meromorphic function f on X and a divisor D on X whose support is disjoint from the support of the divisor of f, define

$$f(D) = \prod_{p \in X} f(p)^{D(p)}. \tag{5.12}$$

Weil reciprocity (see for instance [2, p. 283]) states that $f(\mathrm{div}(g)) = g(\mathrm{div}(f))$ for any pair of meromorphic functions f, g on X. Using this, the Weil pairing (5.11) can also be defined as follows. Take two n-torsion line bundles L_1, L_2 on X. Let D_1 and D_2 be divisors, with disjoint support, so that $L_i \cong \mathcal{O}_X(D_i)$. Then $nD_i = \mathrm{div}(f_i)$ for some meromorphic function $f_i : X \to \mathbb{C}$, and

$$\langle L_1, L_2 \rangle = \frac{f_1(D_2)}{f_2(D_1)} \in \mathbb{Z}_n. \tag{5.13}$$

Remark 5.11. Sometimes (cf. [2]), $\langle L_1, L_2 \rangle$ is defined as $\frac{n}{2\pi i} \log \frac{f_1(D_2)}{f_2(D_1)}$, but this is by considering $\mathbb{Z}_n$ with the additive structure.

Recall that $\xi = \exp(2\pi i/n) \in \mathbb{Z}_n$ denotes the standard generator of the multiplicative group $\mathbb{Z}_n$. Recall also that if L_δ is a n-torsion line bundle on X, then $\pi^* L_\delta$ lies in the kernel of the norm map, so by Proposition 5.2 it is of the form $F_\delta^{-1} \otimes \xi^* F_\delta$ for some line bundle F_δ on X_γ of degree between 0 and $n-1$. The next result generalizes Mumford [31, Lemma 2] to $n \geqslant 2$.

Proposition 5.12. *Let L_γ be the line bundle corresponding to $\gamma \in \Gamma_n \smallsetminus \{e\}$ and let L_δ be any n-torsion line bundle on X which is not a power of L_γ. Let F_δ be a line bundle on X_γ, of degree between 0 and $n-1$, such that $\pi^* L_\delta \cong F_\delta^{-1} \otimes \xi^* F_\delta$. Then there exists a non-zero integer $l(\gamma)$ between 1 and $n-1$, depending only on γ, such that*

$$\langle L_\delta, L_\gamma \rangle = \xi^{l(\gamma) \deg(F_\delta)}.$$

Proof. It will be convenient to use definition (5.13). Let D_γ and D_δ be divisors on X, with disjoint support, such that $L_\gamma \cong \mathcal{O}_X(D_\gamma)$ and $L_\delta \cong \mathcal{O}_X(D_\delta)$ and such that $nD_\gamma = \mathrm{div}(f_\gamma)$ and $nD_\delta = \mathrm{div}(f_\delta)$ for some non-zero meromorphic functions f_γ, f_δ. Since $\pi^* L_\gamma$ is trivial, then $\pi^* D_\gamma = \mathrm{div}(g)$ for some meromorphic function $g \in \mathcal{M}(X_\gamma)^*$. From this it follows that $f_\gamma = \mathrm{Nm}_\pi(g)$. On the other hand, if $F_\delta \cong \mathcal{O}_{X_\gamma}(D)$ for some divisor D on X_γ, then there is a non-zero meromorphic function h on X_γ such that $\pi^* D_\delta = -D + \xi D + \mathrm{div}(h)$. It also follows that $f_\delta = \mathrm{Nm}_\pi(h)$.

Since $\mathrm{div}(\xi^* g) = \xi^* \pi^* D_\gamma = \pi^* D_\gamma = \mathrm{div}(g)$, there is a non-zero complex number $\lambda(\gamma)$, depending only on g, i.e. only on γ, such that

$$\xi^* g = \lambda(\gamma) g. \tag{5.14}$$

Note that $\lambda(\gamma) \neq 1$, since otherwise that would be saying that L_γ was trivial. Furthermore, $g^n = \pi^* f_\gamma = \xi^* \pi^* f_\gamma = (\xi^* g)^n$, hence $\lambda(\gamma)^n = 1$, thus $\lambda(\gamma) = \xi^{l(\gamma)}$ for some integer $l(\gamma) \in \{0, \ldots, n-1\}$ depending only on γ.

Recall that $\mathrm{Nm}_\pi(g)(p) = \prod_{\tilde{p} \in \pi^{-1}(p)} g(\tilde{p})$. Then it is easy to see directly from the definition (5.12), that $\mathrm{Nm}_\pi(g)(D_\delta) = g(\pi^* D_\delta)$. Similarly for $\mathrm{Nm}_\pi(h)(D_\gamma)$. Then, using

(5.14), we get

$$\begin{aligned}\langle L_\delta, L_\gamma\rangle &= \frac{\mathrm{Nm}_\pi(h)(D_\gamma)}{\mathrm{Nm}_\pi(g)(D_\delta)} = \frac{h(\pi^* D_\gamma)}{g(\pi^* D_\delta)}\\ &= \frac{h(\mathrm{div}(g))}{g(\mathrm{div}(h) - D + \xi D)} = \frac{g(D)}{g(\xi D)}\\ &= \frac{\prod_{p\in X_\gamma} g(p)^{D(p)}}{\prod_{p\in X_\gamma} g(p)^{(\xi D)(p)}} = \frac{\prod_{p\in X_\gamma} g(p)^{D(p)}}{\prod_{p\in X_\gamma} (\lambda(\gamma)^{-1} g(\xi\cdot p))^{D(\xi\cdot p)}}\\ &= \lambda(\gamma)^{\sum_{p\in X_\gamma} D(\xi\cdot p)} = \lambda(\gamma)^{\deg(F_\delta)}\\ &= \xi^{l(\gamma)\deg(F_\delta)}\end{aligned}$$

as claimed. □

If $n = 2$ we have $l(\gamma) = 1$ for every non-trivial γ, recovering Mumford's result.

The Weil pairing corresponds to intersection form in homology $H_1(X, \mathbb{Z}_n)$. From this definition it is clear that, for each $\gamma \in \Gamma_n \smallsetminus \{e\}$, it is possible to choose a basis

$$(\gamma, \delta_0, \delta_1, \ldots, \delta_{2g-2}) \tag{5.15}$$

of Γ_n, including γ and such that

$$\langle L_{\delta_i}, L_\gamma\rangle = \begin{cases}\xi^{l(\gamma)} & i = 0\\ 1 & i \neq 0.\end{cases} \tag{5.16}$$

We assume from now on that such a basis has been chosen.

Now we describe the Γ_n-action on the n connected components of $\mathrm{Nm}_\pi^{-1}(\Lambda)$.

Proposition 5.13. *Every element of Γ_n which is not in the subgroup $\langle\delta_0\rangle$ generated by δ_0 acts trivially on $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$, and $\langle\delta_0\rangle \cong \mathbb{Z}_n$ acts freely and transitively on $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$. In particular, Γ_n acts transitively on $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$.*

Proof. We will use Proposition 5.7. Let $M \in \mathrm{Nm}_\pi^{-1}(\Lambda)$, so that $M = p_{L_0}(L)$ for some degree $i \in \{d, \ldots, d + n - 1\}$ line bundle L over X_γ, i.e.

$$M = L^{-1} \otimes \xi^* L \otimes L_0.$$

Since $\pi^* L_\delta \in \ker(\mathrm{Nm}_\pi)$ then, by Proposition 5.2, $\pi^* L_\delta = p(F_\delta) = F_\delta^{-1} \otimes \xi^* F_\delta$ for some line bundle F_δ on X_γ of degree between 0 and $n - 1$. So

$$M \otimes \pi^* L_\delta = (L \otimes F_\delta)^{-1} \otimes \xi^*(L \otimes F_\delta) \otimes L_0.$$

If $\delta \notin \langle\delta_0\rangle$, then by (5.16) $\langle L_\delta, L_\gamma\rangle = 1$ hence, by Proposition 5.12, $\deg(F_\delta) = 0$, so that $\deg(L \otimes F_\delta) = i$ and therefore $M \otimes \pi^* L_\delta$ is in the same connected component as M by Proposition 5.7. For δ_0, for the same reasons we have $\deg(F_\delta) = 1$, so $\deg(L \otimes F_\delta) = i + 1$, thus the powers of δ_0 act transitively on $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$. □

6. The polynomial $E_{\mathrm{st}}(\mathcal{M}/\Gamma_n) - E(\mathcal{M})^{\Gamma_n}$

6.1. **The subvarieties $\mathcal{M}^\gamma$.** The stringy E-polynomial of $\mathcal{M}/\Gamma_n$ is defined by (3.8), but the statement of Theorem 3.15 is only about

$$\sum_{\gamma\neq e} E(\mathcal{M}^\gamma)^{\Gamma_n}(uv)^{F(\gamma)}.$$

This is the polynomial we aim to compute in the present section. This requires the study of the subvarieties $\mathcal{M}^\gamma$ of fixed points under the action of each nontrivial $\gamma \in \Gamma_n$. This study is an adaptation to parabolic Higgs bundles of the corresponding result for vector bundles studied by Narasimhan and Ramanan in [32] (cf. Hausel and Thaddeus [21, Sec. 7] for the case of Higgs bundles in the non-parabolic situation).

Recall that to each non-trivial $\gamma \in \Gamma_n$ we associate an unramified cyclic n-cover $\pi : X_\gamma \to X$ as in (5.1), with Galois group isomorphic to $\mathbb{Z}_n$ and $\mathrm{Nm}_\pi : \mathrm{Pic}^d(X_\gamma) \to \mathrm{Pic}^d(X)$ is the corresponding norm map.

Let $D_\gamma = \pi^{-1}(D)$ be the inverse image in X_γ of our fixed divisor $D = p_1 + \cdots + p_{|D|}$ in X, from which we have our fixed generic parabolic type $\boldsymbol{\alpha}$,

$$0 \leqslant \alpha_1(p) < \cdots < \alpha_n(p) < 1,$$

for each $p \in D$. It is important to note that our genericity assumption on the parabolic type $\boldsymbol{\alpha}$ still holds, but that we no longer need.

Given

$$\varpi_n = (\varpi_n(p_1), \ldots, \varpi_n(p_{|D|})) \in S_n^{|D|}, \tag{6.1}$$

we naturally construct a parabolic type of rank 1, denoted by $\boldsymbol{\alpha}_\gamma(\varpi_n)$ on D_γ as follows. For each $p \in D$, write

$$\varpi_n(p) = a_1(p)\ a_2(p) \ldots a_n(p) \in S_n.$$

Then attach the weights $0 \leqslant \alpha_1(p) < \cdots < \alpha_n(p) < 1$ to the set $\pi^{-1}(p) = \{q_1, \ldots, q_n\} \subset D_\gamma$ so that the point $q_i \in \pi^{-1}(p)$ is given the weight $\alpha_{a_i(p)}(p)$. This yields the parabolic type $\boldsymbol{\alpha}_\gamma(\varpi_n)$,

$$0 \leqslant \alpha_{a_i(p)}(p) < 1,$$

at each $q_i \in \pi^{-1}(p) \subset D_\gamma$. Strictly speaking, this depends on a choice of an ordering of the points in $\pi^{-1}(p)$ for each p. This ordering was implicitly chosen when we wrote $\pi^{-1}(p) = \{q_1, \ldots, q_n\}$. So, without the choice of that ordering, ϖ_n in (6.1) belongs to a torsor for the group $S_n^{|D|}$. In any case, any ordering is valid for our purposes.

For each $\varpi_n \in S_n^{|D|}$, consider the moduli space $\mathcal{M}_d^{\boldsymbol{\alpha}_\gamma(\varpi_n)}(\mathbb{C}^*)$ of strongly parabolic Higgs line bundles (i.e., $\mathbb{C}^*$-Higgs bundles) of degree $d \in \mathbb{Z}$, over X_γ, of parabolic type $\boldsymbol{\alpha}_\gamma(\varpi_n)$ over D_γ. Let $K_\gamma = \pi^* K$ be the canonical line bundle of X_γ.

Lemma 6.1.

1. *For every $\varpi_n \in S_n^{|D|}$, the moduli space $\mathcal{M}_d^{\boldsymbol{\alpha}_\gamma(\varpi_n)}(\mathbb{C}^*)$ is isomorphic to the moduli space of Higgs line bundles of degree d over X_γ, i.e., to the cotangent bundle $T^*\mathrm{Pic}^d(X_\gamma) \cong \mathrm{Pic}^d(X_\gamma) \times H^0(X_\gamma, K_\gamma)$.*
2. *The disjoint union $\bigsqcup_{\varpi_n \in S_n^{|D|}} \mathcal{M}_d^{\boldsymbol{\alpha}_\gamma(\varpi_n)}(\mathbb{C}^*)$ is isomorphic to $T^*\mathrm{Pic}^d(X_\gamma) \times S_n^{|D|}$.*

Proof. Take a strongly parabolic Higgs line bundle $(F, \phi) \in \mathcal{M}_d^{\boldsymbol{\alpha}_\gamma(\varpi_n)}(\mathbb{C}^*)$. Then $F \in \mathrm{Pic}^d(X_\gamma)$ and $\phi \in H^0(X_\gamma, K_\gamma(D_\gamma))$. But the fact that ϕ is strongly parabolic and F is a line bundle implies that actually $\phi \in H^0(X, K_\gamma)$, thus the map that forgets the parabolic structure yields the isomorphism stated in (1). Then (2) follows from (1). □

Proposition 6.2. *Let $\boldsymbol{\alpha} = (\alpha_1(p), \ldots, \alpha_n(p))_{p \in D}$ be any generic parabolic type and let $\varpi_n \in S_n^{|D|}$. If $(F, \phi) \in \mathcal{M}_d^{\boldsymbol{\alpha}_\gamma(\varpi_n)}(\mathbb{C}^*)$, then $(\pi_* F, \pi_* \phi)$ is a semistable strongly parabolic $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle of degree d and parabolic type $\boldsymbol{\alpha}$.*

Proof. Let $V = \pi_* F$ and $\varphi = \pi_* \phi : V \to V \otimes K$. Since π is unramified and F has degree d, then so has the rank n vector bundle V. Note that K is a subsheaf of $K(D)$ so φ is also a section of $\mathrm{End}(V) \otimes K(D)$. Let us see that V has a parabolic structure at D of type $\boldsymbol{\alpha}$. Let $p \in D$ and $\{q_1, \ldots, q_n\} = \pi^{-1}(p) \subset D_\gamma$. The filtration of V_p is given as

$$V_p = V_{p,1} = \bigoplus_{i=1}^{n} F_{q_i} \supsetneq V_{p,2} \supsetneq \cdots \supsetneq V_{p,n} \supsetneq \{0\}, \qquad 0 \leqslant \alpha_1(p) < \cdots < \alpha_n(p) < 1, \tag{6.2}$$

where, for $2 \leqslant j \leqslant n$,

$$V_{p,j} = V_{p,j-1}/F_{q'} \tag{6.3}$$

with q' being the point q_i attached with the weight $\alpha_{j-1}(p)$ i.e. $q' = q_i$ such that $a_i(p) = j-1$. So $V_{p,n} = F_{q_i}$ such that $a_i(p) = n$, $F_{p,n-1} = F_{q_j} \oplus F_{q_i}$ with j such that $a_j = n-1$, and so on. Doing this for every point of D determines the parabolic structure of V of type $\boldsymbol{\alpha}$.

Notice that, conversely, the parabolic structure of V, given at each point of D by (6.2) and (6.3) determines the element (6.1) of $S_n^{|D|}$.

Over each $p \in D$, $\varphi_p = \varphi|_{V_p}$ is diagonal with respect to the decomposition $V_p = \bigoplus F_{q_i}$ of V_p. Look at $\phi \in H^0(X_\gamma, K_\gamma)$ as a section of $K_\gamma(D_\gamma)$ which vanishes at every $q_i \in D_\gamma$. Then it is clear that φ is strongly parabolic with respect to (6.2). So (V, φ) is a strongly parabolic Higgs bundle of rank n, degree d and parabolic type $\boldsymbol{\alpha}$.

It remains to check semistability. For that, recall that $\xi = \exp(2\pi i/n)$ denotes the standard generator of the Galois group $\mathbb{Z}_n$, and note that $(\pi^*V, \pi^*\varphi) \cong (F \oplus \xi^*F \oplus \cdots \oplus \xi^{*n-1}F, \phi \oplus \xi^*\phi \oplus \cdots \oplus \xi^{*n-1}\phi)$ is a strongly parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle on X_γ with parabolic structure over D_γ induced from (6.2). Since

$$\mathrm{par}\mu(\pi^*\pi_*F) = \frac{nd + \sum_{q\in D_\gamma}\sum_{i=1}^n \alpha_i(\pi(q))}{n} = d + \sum_{p\in D}\sum_{i=1}^n \alpha_i(p) = \mathrm{par}\mu(\xi^{j*}F),$$

for every $j = 0, \ldots, n-1$, then $\pi^*V \cong \bigoplus_{j=1}^n \xi^{j*}F$ is a direct sum of parabolic line bundles, all of the same slope, and therefore it is semistable [36, Corollaire 10, p. 71]. Take a φ-invariant subbundle $V' \subset V$ of degree d' and rank n'. Then π^*V' is a $\pi^*\varphi$-invariant subbundle of π^*V. By semistability of π^*V, we must have $\mathrm{par}\mu(\pi^*V') \leqslant \mathrm{par}\mu(\pi^*V)$, that is

$$\frac{nd' + n\sum_{p\in D}\sum_{i=1}^n \alpha_i(p)}{n'} \leqslant \frac{nd + n\sum_{p\in D}\sum_{i=1}^n \alpha_i(p)}{n}.$$

But this is equivalent to

$$\frac{d' + \sum_{p\in D}\sum_{i=1}^n \alpha_i(p)}{n'} \leqslant \frac{d + \sum_{p\in D}\sum_{i=1}^n \alpha_i(p)}{n},$$

that is, to $\mathrm{par}\mu(V') \leqslant \mathrm{par}\mu(V)$, proving semistability of (V, φ). □

So the pushforward gives a map from $\bigsqcup_{\varpi_n \in S_n^{|D|}} \mathcal{M}_d^{\boldsymbol{\alpha}_\gamma(\varpi_n)}(\mathbb{C}^*) \cong T^*\operatorname{Pic}^d(X_\gamma) \times S_n^{|D|}$ to the moduli space $\mathcal{M}_d^{\boldsymbol{\alpha}}(\mathrm{GL}(n,\mathbb{C}))$ of degree d, strongly parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles, with parabolic type $\boldsymbol{\alpha}$. If we want the determinant to be Λ, we have to restrict this map to the subspace $T^*\operatorname{Nm}_\pi^{-1}(\Lambda) \times S_n^{|D|}$ if $n \geqslant 3$ is prime (or to $T^*\operatorname{Nm}_\pi^{-1}(\Lambda L_\gamma) \times S_2^{|D|}$ if $n = 2$), where $T^*\operatorname{Nm}_\pi^{-1}(\Lambda)$ denotes the cotangent bundle to $\operatorname{Nm}_\pi^{-1}(\Lambda)$. Indeed, $\det(\pi_*F) \cong \operatorname{Nm}_\pi(F)L_\gamma^{-n(n-1)/2}$, so if $n \geqslant 3$ is prime, $\operatorname{Nm}_\pi(F) \cong \Lambda$ (and if $n = 2$, $\operatorname{Nm}_\pi(F) \cong \Lambda L_\gamma$). Thus $(\pi_*F, \pi_*\phi)$ is such that $\det(\pi_*F) \cong \Lambda$ and $\operatorname{tr}(\pi_*\phi) = 0$ if and only if $(F, \phi) \in T^*\operatorname{Nm}_\pi^{-1}(\Lambda) \times S_n^{|D|}$, if $n \geqslant 3$ prime, or $(F, \phi) \in T^*\operatorname{Nm}_\pi^{-1}(\Lambda L_\gamma) \times S_2^{|D|}$ if $n = 2$. In any case, $\operatorname{Nm}_\pi^{-1}(\Lambda)$ is a torsor for $\ker(\operatorname{Nm}_\pi)$ thus $T^*\operatorname{Nm}_\pi^{-1}(\Lambda)$ is also a torsor for $T^*\ker(\operatorname{Nm}_\pi)$. In turn, from Proposition 5.7, each connected component of $T^*\ker(\operatorname{Nm}_\pi)$ is a torsor for $T^*\operatorname{Prym}_\pi(X_\gamma)$ which is isomorphic to $\operatorname{Prym}_\pi(X_\gamma) \times \mathbb{C}^{(n-1)(g-1)}$, since the Prym is an abelian variety of dimension $(n-1)(g-1)$, by (5.8). Hence

$$T^*\operatorname{Nm}_\pi^{-1}(\Lambda) \cong \operatorname{Nm}_\pi^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)}.$$

Therefore, Proposition 6.2 gives a map

$$\pi_* : \operatorname{Nm}_\pi^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_n^{|D|} \to \mathcal{M}, \tag{6.4}$$

with the obvious modification when $n = 2$.

Now we can describe the locus $\mathcal{M}^\gamma$ of points in $\mathcal{M}$ fixed by a non-trivial element $\gamma \in \Gamma_n$. This locus is going to be the image of π_*, which is isomorphic to the quotient of its domain by a natural action of the Galois group of $\pi : X_\gamma \to X$.

Theorem 6.3. *Let $\boldsymbol{\alpha}$ be any generic parabolic type and let $n \geqslant 3$ be prime. For every $\gamma \in \Gamma_n \setminus \{e\}$, the map (6.4) induces an isomorphism*

$$\mathcal{M}^\gamma \simeq (\mathrm{Nm}_\pi^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_n^{|D|})/\mathbb{Z}_n,$$

with $\mathbb{Z}_n$ acting diagonally, by pullback on $T^* \mathrm{Nm}_\pi^{-1}(\Lambda)$ and cyclically on each factor of $S_n^{|D|}$. If $n = 2$, replace $\mathrm{Nm}_\pi^{-1}(\Lambda)$ by $\mathrm{Nm}_\pi^{-1}(\Lambda L_\gamma)$.

Proof. Let $\gamma \in \Gamma_n \setminus \{e\}$ and (V, φ) represent a point in $\mathcal{M}$. Since (V, φ) is stable then its only parabolic endomorphisms are the scalars [40, (3.3)], hence the same argument as in Proposition 2.6 of [32] shows that $V \cong V \otimes L_\gamma$ if and only if isomorphic to the pushforward of a line bundle F over X_γ,

$$V \cong \pi_* F.$$

Moreover, the isomorphism $V \cong V \otimes L_\gamma$ is given by

$$\pi_*(\mathrm{Id}_F \otimes \lambda) : \pi_* F \xrightarrow{\cong} \pi_* F \otimes L_\gamma \tag{6.5}$$

where we recall that $\lambda : \mathcal{O}_{X_\gamma} \to \pi^* L_\gamma$ denotes the tautological section. From here one sees that the Higgs fields on $V \cong \pi_* F$ which are compatible under the isomorphism (6.5) are the ones which are pushforward of Higgs fields on F.

Consider now the parabolic structure on $V \cong \pi_* F$,

$$V_p = V_{p,1} \supsetneq V_{p,2} \supsetneq \cdots \supsetneq V_{p,n} \supsetneq \{0\}, \qquad 0 \leqslant \alpha_1(p) < \cdots < \alpha_n(p) < 1 \tag{6.6}$$

over each $p \in D$, so that the one on $V \otimes L_\gamma$ is

$$V_p \otimes L_\gamma = V_{p,1} \otimes L_{\gamma,p} \supsetneq V_{p,2} \otimes L_{\gamma,p} \supsetneq \cdots \supsetneq V_{p,n} \otimes L_{\gamma,p} \supsetneq \{0\}, \qquad 0 \leqslant \alpha_1(p) < \cdots < \alpha_n(p) < 1. \tag{6.7}$$

For $p \in D$, let $\pi^{-1}(p) = \{q_1, \ldots, q_n\}$. Then the isomorphism (6.5) over p is

$$V_p = (\pi_* F)_p = \bigoplus_{i=1}^n F_{q_i} \xrightarrow{\bigoplus_{i=1}^n (\mathrm{Id}_{F_{q_i}} \otimes \lambda_{q_i})} \bigoplus_{i=1}^n F_{q_i} \otimes L_{\gamma,p} = V_p \otimes L_{\gamma,p}. \tag{6.8}$$

Since $q_i \neq q_j$ then also $\lambda(q_i) \neq \lambda(q_j)$ for $i \neq j$. Hence the only non-trivial subspaces $0 \neq V_p'$ of V_p which are preserved under (6.8) those of the form

$$V_p' = \bigoplus_{j=1}^{\dim(V_p')} F_{q_{i_j}}. \tag{6.9}$$

So the isomorphism (6.5) respects the filtrations (6.6) and (6.7) if and only if each $V_{p,i}$ is of the form (6.9).

So, after providing an element $\varpi_n \in S_n^{|D|}$, this parabolic structure of V determines a parabolic structure on F over $D_\gamma = \pi^{-1}(D)$, by reversing the construction carried in (6.2) and (6.3).

The conclusion is that $(V, \varphi) \in \mathcal{M}^\gamma$, i.e., $(V, \varphi) \cong (V \otimes L_\gamma, \varphi \otimes \mathrm{Id}_{L_\gamma})$ if and only if $(V, \varphi) \cong \pi_*((F, \phi), \varpi_n)$ for some $((F, \phi), \varpi_n) \in \mathrm{Nm}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_n^{|D|}$ (with the obvious modification if $n = 2$).

It turns out that there are redundancies coming precisely from the action of the Galois group $\mathbb{Z}_n$ of $\pi : X_\gamma \to X$. Then clearly, for any $j = 0, \ldots, n-1$, we have $\pi_*(F, \phi) \cong \pi_*(\xi^{j*} F, \xi^{j*} \phi)$ as (non-parabolic) Higgs bundles and these are the only redundancies.

Now we take into account the parabolic structure. A parabolic Higgs bundle is defined by $((F,\phi),\varpi_n)\in \mathrm{Nm}_\pi^{-1}(\Lambda)\times\mathbb{C}^{(n-1)(g-1)}\times S_n^{|D|}$. Consider the action of the Galois group $\mathbb{Z}_n$ given by

$$\xi^j\cdot((F,\phi),\varpi_n)=((\xi^{j*}F,\xi^{j*}\phi),\xi^{-j}\cdot\varpi_n) \tag{6.10}$$

where each $\xi^j\in\mathbb{Z}_n$ acts diagonally on ϖ_n such that on each factor $\varpi_n(p)$ it acts as a cyclic permutation of length i (hence acts freely). Precisely, ξ^j acts diagonally on ϖ_n as

$$\xi^{-j}\cdot\varpi_n(p)=a_{n-j+1}(p)\ \dots\ a_n(p)\ a_1(p)\ \dots\ a_{n-j}(p), \tag{6.11}$$

for each $p\in D$. Thus the orbit of ϖ_n is given by the set of all the $n^{|D|}$ permutations which differ from the given one at each point by a cyclic permutation.

It is easy to check, by following again the construction in (6.2) and (6.3), that two elements of $\mathrm{Nm}_\pi^{-1}(\Lambda)\times\mathbb{C}^{(n-1)(g-1)}\times S_n^{|D|}$ give rise to isomorphic parabolic Higgs bundles if and only if they are in the same orbit under $\mathbb{Z}_n$. In other words, $(\pi_*F,\pi_*\phi,\varpi_n)$ and $(\pi_*\xi^{j*}F,\pi_*\xi^{j*}\phi,\xi^{-j}\cdot\varpi_n)$ determine isomorphic parabolic Higgs bundles, for each $j=0,\dots,n-1$, and that is the only way one can obtain isomorphic parabolic Higgs bundles under our construction. We conclude that

$$\mathcal{M}^\gamma\simeq(\mathrm{Nm}_\pi^{-1}(\Lambda)\times\mathbb{C}^{(n-1)(g-1)}\times S_n^{|D|})/\mathbb{Z}_n$$

with $\mathbb{Z}_n$ acting diagonally as in (6.10) and (6.11). □

Remark 6.4. As mentioned in Remark 3.11, our description of $\mathcal{M}^\gamma$ implies that the corresponding parabolic Higgs bundles are $\boldsymbol{\alpha}$-semistable for any value of $\boldsymbol{\alpha}$.

In particular, it follows from this theorem that, for any non-trivial $\gamma\in\Gamma_n$,

$$\dim(\mathcal{M}^\gamma)=2(n-1)(g-1). \tag{6.12}$$

It turns out that the parabolic structure will now make our life easier by allowing a slightly different description of the fixed point locus $\mathcal{M}^\gamma$, from which the calculation of the Γ_n-invariant E-polynomial $E(\mathcal{M}^\gamma)^{\Gamma_n}$ is simpler than in the non-parabolic case.

First, choose a section

$$s: S_n^{|D|}/\mathbb{Z}_n\to S_n^{|D|} \tag{6.13}$$

of the projection $S_n^{|D|}\to S_n^{|D|}/\mathbb{Z}_n$. Of course it corresponds to the choice of a representative of each class in $S_n^{|D|}/\mathbb{Z}_n$. There is no canonical choice of such s, but all of them are obviously algebraic.

Recall that Γ_n acts on $\mathcal{M}^\gamma$, by acting trivially on the Higgs field and on the weights and by pullback and tensor product on the factor $\mathrm{Nm}_\pi^{-1}(\Lambda)$ of $\mathcal{M}^\gamma$; cf. (5.10).

Proposition 6.5. *There is a Γ_n-equivariant isomorphism (depending on the choice of the section s in* (6.13)*)*

$$\mathcal{M}^\gamma\simeq\mathrm{Nm}_\pi^{-1}(\Lambda)\times\mathbb{C}^{(n-1)(g-1)}\times(S_n^{|D|}/\mathbb{Z}_n),$$

where Γ_n acts on the first factor as stated in (5.10) *and trivially on the other two factors.*

Proof. From Theorem 6.3, we know that $\mathcal{M}^\gamma\simeq(\mathrm{Nm}_\pi^{-1}(\Lambda)\times\mathbb{C}^{(n-1)(g-1)}\times S_n^{|D|})/\mathbb{Z}_n$. Consider the map

$$f_s:\mathrm{Nm}_\pi^{-1}(\Lambda)\times\mathbb{C}^{(n-1)(g-1)}\times(S_n^{|D|}/\mathbb{Z}_n)\to(\mathrm{Nm}_\pi^{-1}(\Lambda)\times\mathbb{C}^{(n-1)(g-1)}\times S_n^{|D|})/\mathbb{Z}_n$$

defined by

$$f_s(F,\phi,[\varpi_n])=[(F,\phi,s([\varpi_n]))].$$

Its inverse g_s is defined as follows: take $[(F,\phi,\varpi_n)] \in (\mathrm{Nm}_\pi^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_n^{|D|})/\mathbb{Z}_n$. Since $\mathbb{Z}_n$ acts freely on $S_n^{|D|}$, there is a unique i such that $\xi^i \cdot \varpi_n = s([\varpi_n])$. Then $[(F,\phi,\varpi_n)] = [(\xi^i \cdot F, \xi^i \cdot \phi, \xi^i \cdot \varpi_n)]$, and hence take

$$g_s([(F,\phi,\varpi_n)]) = (\xi^i \cdot F, \xi^i \cdot \phi, [\varpi_n]).$$

It is clear that indeed $g_s = f_s^{-1}$. It is clear that both f_s and its inverse are algebraic, yielding the stated isomorphism.

To see that it is Γ_n-equivariant, is just a matter of noticing that, for each $\delta \in \Gamma_n$,

$$\delta \cdot f_s(F,\phi,[\varpi_n]) = [F \otimes \pi^* L_\delta, \phi, \varpi_n] = f_s(\delta \cdot (F,\phi,[\varpi_n]))$$

because $\mathbb{Z}_n$ acts trivially on $\pi^* L_\delta$. □

The action of the Galois group $\mathbb{Z}_n$ on the product $\mathrm{Nm}_\pi^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_n^{|D|}$ can therefore be absorbed in the $S_n^{|D|}$ factor.

6.2. **Calculation of the polynomial.**

Proposition 6.6. *For any non-trivial $\gamma \in \Gamma_n$, we have the following isomorphism regarding the Γ_n-invariant part of $H_c^*(\mathcal{M}^\gamma, \mathbb{C})$:*

$$H_c^*(\mathcal{M}^\gamma,\mathbb{C})^{\Gamma_n} \cong H^*(\mathrm{Prym}_\pi(X_\gamma),\mathbb{C}) \otimes H_c^*(\mathbb{C}^{(n-1)(g-1)},\mathbb{C}) \otimes H^*(S_n^{|D|}/\mathbb{Z}_n,\mathbb{C}).$$

Proof. By Proposition 6.5, we can consider the Γ_n-action on $\mathrm{Nm}_\pi^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times (S_n^{|D|}/\mathbb{Z}_n)$, where Γ_n acts trivially on the second and third factors, hence the corresponding cohomologies are Γ_n-invariant. It then suffices to prove that $H^*(\mathrm{Nm}_\pi^{-1}(\Lambda),\mathbb{C})^{\Gamma_n} \cong H^*(\mathrm{Prym}_\pi(X_\gamma),\mathbb{C})$.

Consider the symplectic basis of Γ_n given by (5.15). By Proposition 5.13, the subgroup generated by δ_0 acts freely and transitively on $\pi_0(\mathrm{Nm}_\pi^{-1}(\Lambda))$, while any $\delta \notin \langle\delta_0\rangle$ acts trivially on these components. Write the decomposition of $\mathrm{Nm}_\pi^{-1}(\Lambda)$ into connected components as

$$\mathrm{Nm}_\pi^{-1}(\Lambda) = N_1 \sqcup \cdots \sqcup N_n, \tag{6.14}$$

where the indices are chosen so that if $x \in N_i$ then $\delta_0(x) \in N_{i+1}$ (where we assume $n+1=1$). Each N_i is a torsor for the Prym variety of X_γ, hence their cohomologies are the same.

Take a cohomology class in $H^k(\mathrm{Nm}_\pi^{-1}(\Lambda),\mathbb{C})$ represented by a k-form $\omega \in \mathcal{A}^*(\mathrm{Nm}_\pi^{-1}(\Lambda),\mathbb{C})$. Write

$$\omega = (\omega_1,\ldots,\omega_i,\ldots,\omega_n) \tag{6.15}$$

according to (6.14), where each ω_i represents a cohomology class in $H^k(N_i,\mathbb{C})$. The action of $\delta \in \Gamma_n$ on the cohomology class represented by ω is given by pullback

$$\delta \cdot \omega = \delta^* \omega. \tag{6.16}$$

So the decomposition of $\delta_0 \cdot \omega$ in (6.14) is given by

$$\delta_0 \cdot \omega = (\delta_0^*\omega_2,\ldots,\delta_0^*\omega_{i+1},\ldots,\delta_0^*\omega_1). \tag{6.17}$$

By (6.15) and (6.17) we see that the class represented by ω is invariant by the subgroup $\langle\delta_0\rangle$ if and only if the forms ω_i are such that $\omega_i = \delta_0^*\omega_{i+1}$, that is, if and only if ω is given by

$$\omega = ((\delta_0^{n-1})^*\omega_n,\ldots,(\delta_0^{n-i})^*\omega_n,\ldots,\omega_n). \tag{6.18}$$

Notice that this makes sense because δ_0 has order n.

Consider now an element $\delta \in \Gamma_n$ which is not in the subgroup generated by δ_0. Then δ preserves the connected components N_i of $\mathrm{Nm}_\pi^{-1}(\Lambda)$, acting hence on each $H^*(N_i,\mathbb{C})$.

Each N_i is a torsor for the Prym of X_γ and δ acts on N_i by translations by an element of the Prym:

$$\delta \cdot M = M \otimes \pi^* L_\delta.$$

To see that indeed $\pi^* L_\delta \in \mathrm{Prym}_\pi(X_\gamma)$, note first that it is in the kernel of Nm_π. Hence it is of the form $\pi^* L_\delta = F^{-1} \otimes \xi^* F$, with $\xi^{\deg(F)} = \langle \delta, \gamma \rangle$, by Proposition 5.12. But $\langle \delta, \gamma \rangle = 1$ i.e. $\deg(F) = 0$ and $\pi^* L_\delta \in \mathrm{Prym}_\pi(X_\gamma)$ by Proposition 5.6. Since $\mathrm{Prym}_\pi(X_\gamma)$ is an abelian variety, every class in $H^*(\mathrm{Prym}_\pi(X_\gamma), \mathbb{C})$ contains a unique representative which is invariant under translations. This property goes through $H^*(N_n, \mathbb{C}) \cong H^*(\mathrm{Prym}_\pi(X_\gamma), \mathbb{C})$ considering the torsor structure of N_n. This means that we can assume that the form ω_n in (6.18) is invariant under translations, so is δ-invariant, i.e., $\delta^* \omega_n = \omega_n$. Hence the action (6.16) of δ on ω given by (6.18) is

$$\delta \cdot \omega = ((\delta_0^{n-1})^* \omega_n, \ldots, (\delta_0^{n-i})^* \omega_n, \ldots, \omega_n).$$

We thus conclude that $H^*(\mathrm{Nm}_\pi^{-1}(\Lambda), \mathbb{C})^{\Gamma_n}$ is given precisely by the classes represented by the forms of type (6.18). Mapping those to $[\omega_n]$ gives an isomorphism with $H^*(N_n, \mathbb{C})$, hence also with $H^*(\mathrm{Prym}_\pi(X_\gamma), \mathbb{C})$. □

Now we can finally compute the sum of the stringy E-polynomial of $\mathcal{M}/\Gamma_n$ corresponding to non-trivial elements of Γ_n.

Proposition 6.7. *For any n prime, the following holds:*

$$\sum_{\gamma \neq e} E(\mathcal{M}^\gamma)^{\Gamma_n} (uv)^{F(\gamma)} = \frac{n^{2g}-1}{n} (n!)^{|D|} (uv)^{(n^2-1)(g-1)+|D|n(n-1)/2} ((1-u)(1-v))^{(n-1)(g-1)}.$$

Proof. By Proposition 6.6, and since $E(\mathbb{C}^{(n-1)(g-1)}) = (uv)^{(n-1)(g-1)}$,

$$E(\mathcal{M}^\gamma)^{\Gamma_n} = (uv)^{(n-1)(g-1)} E(\mathrm{Prym}_\pi(X_\gamma)) E(S_n^{|D|}/\mathbb{Z}_n),$$

for each $\gamma \in \Gamma_n \setminus \{e\}$.

Being an abelian variety, the cohomology of the Prym of X_γ is the alternating algebra on $H^1(\mathrm{Prym}_\pi(X_\gamma), \mathbb{C}) = H^{0,1}(\mathrm{Prym}_\pi(X_\gamma)) \oplus H^{1,0}(\mathrm{Prym}_\pi(X_\gamma))$. Write $V = H^{0,1}(\mathrm{Prym}_\pi(X_\gamma))$ and note that $\dim(V) = \dim(\mathrm{Prym}_\pi(X_\gamma)) = (n-1)(g-1)$. Thus $H^k(\mathrm{Prym}_\pi(X_\gamma), \mathbb{C}) = \Lambda^k(V \oplus \bar{V})$, therefore

$$H^{p,q}(\mathrm{Prym}_\pi(X_\gamma)) = \Lambda^p V \otimes \Lambda^q \bar{V},$$

whose dimension is $\binom{(n-1)(g-1)}{p}\binom{(n-1)(g-1)}{q}$. Hence

$$\begin{aligned} E(\mathrm{Prym}_\pi(X_\gamma)) &= \sum_{p,q=0}^{(n-1)(g-1)} (-1)^{p+q} \binom{(n-1)(g-1)}{p} \binom{(n-1)(g-1)}{q} u^p v^q \\ &= ((1-u)(1-v))^{(n-1)(g-1)}. \end{aligned}$$

The polynomial $E(S_n^{|D|}/\mathbb{Z}_n)$ is just the constant $\frac{1}{n}(n!)^{|D|}$, i.e., the number of elements of the space $S_n^{|D|}/\mathbb{Z}_n$.

We are now left to the computation of the fermionic shift $F(\gamma)$ as defined in (3.9). From (3.10), we know that $F(\gamma) = \dim(N_p \mathcal{M}^\gamma)/2$, but from (2.8) and (6.12), we conclude that

$$F(\gamma) = n(n-1)(g-1+|D|/2). \tag{6.19}$$

Therefore, for each $\gamma \neq e$,

$$E(\mathcal{M}^\gamma)^{\Gamma_n} (uv)^{F(\gamma)} = \frac{1}{n} (n!)^{|D|} (uv)^{(n-1)(g-1)+n(n-1)(g-1+|D|/2)} ((1-u)(1-v))^{(n-1)(g-1)}.$$

This is independent of $\gamma \in \Gamma_n \setminus \{e\}$, thus summing up this expression for all non-trivial elements of Γ_n, yields

$$\sum_{\gamma \neq e} E(\mathcal{M}^\gamma)^{\Gamma_n}(uv)^{F(\gamma)} = \frac{n^{2g}-1}{n}(n!)^{|D|}(uv)^{(n^2-1)(g-1)+|D|n(n-1)/2}((1-u)(1-v))^{(n-1)(g-1)},$$

as claimed. □

Propositions 4.5 and 6.7 together prove Theorem 3.15 and therefore, using Theorem 3.14, prove our main result, Theorem 3.13.

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