A pair of rigid surfaces with $p_g = q = 2$ and $K^2 = 8$ whose universal cover is not the bidisk

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Abstract

We construct two complex-conjugated rigid surfaces with $p_g = q = 2$ and $K^2 = 8$ whose universal cover is not biholomorphic to the bidisk $\mathbb{H} \times \mathbb{H}$. We show that these are the unique surfaces with these invariants and Albanese map of degree 2, apart the family of product-quotient surfaces given in [**Pen11**]. This completes the classification of surfaces with $p_g = q = 2, K^2 = 8$ and Albanese map of degree 2.

0 Introduction

Despite the work of many authors, surfaces S of general type with the lowest possible value of the holomorphic Euler characteristic, namely such that $\chi(\mathcal{O}_S) = 1$, are far from being classified, see e.g. the survey papers [**BCP06**], [**BCP11**] and [**MLP12**] for a detailed bibliography on the subject. These surfaces satisfy the Bogomolov-Miyaoka-Yau inequality $K^2 \leq 9$.

The ones with $K^2 = 9$ are rigid, their universal cover is the unit ball in $\mathbb{C}^2$ and $p_g = q \leq 2$. The fake planes, i.e. surfaces with $p_g = q = 0$, have been classified in [**PY07**] and [**CS10**], the two Cartwright-Steger surfaces [**CS10**] satisfy $q = 1$, whereas no example is known for $p_g = q = 2$.

The next case is $K^2 = 8$. In this situation, Debarre's inequality for irregular surfaces [**Deb82**] implies

$$0 \leq p_g = q \leq 4.$$

The cases $p_g = q = 3$ and $p_g = q = 4$ are nowadays classified ([**HP02**], [**Pir02**], [**Deb82**, Beauville's appendix]), whereas for $p_g = q \leq 2$ some families are known ([**Sha78**], [**BCG05**], [**Pol06**], [**Pol08**], [**CP09**], [**Pen11**]) but there is no complete description yet.

All the examples of surfaces with $\chi = 1$ and $K^2 = 8$ known so far are uniformized by the bidisk $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ is the Poincaré upper half-plane; so the following question naturally arose:

Is there a smooth minimal surface of general type with invariants $\chi = 1$ and $K^2 = 8$ and whose universal cover is not biholomorphic to $\mathbb{H} \times \mathbb{H}$?

For general facts about surfaces uniformized by the bidisk, we refer the reader to [**CF09**]. One of the aims of this paper is to give an affirmative answer to the question above. In fact we construct two rigid surfaces with $p_g = q = 2$ and $K^2 = 8$ whose universal cover is not the bidisk. Moreover, we show that these surfaces are complex-conjugated and that they are the unique surfaces with these invariants and having Albanese map of degree 2, apart the family of product-quotient surfaces constructed in [**Pen11**]. This complete the classification of surfaces with $p_g = q = 2$, $K^2 = 8$ and Albanese map of degree 2.

Our results can be summarized as follows, see Proposition 1.3, Theorem 2.5, Theorem 4.8, Theorem 4.13 and Proposition 4.16.

Main Theorem. *Let S be a minimal smooth surface of general type with $p_g = q = 2$, $K^2 = 8$ and such that its Albanese map $\alpha \colon S \longrightarrow A := \operatorname{Alb}(S)$ is a generically finite double cover. Writing D_A for the branch locus of α, there are exactly two possibilities, both of which occur:*

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(I) *$D_A^2 = 32$ and D_A is an irreducible curve with one ordinary point of multiplicity 6 and no other singularities. These are the product-quotient surfaces constructed in* [**Pen11**];

(II) *$D_A^2 = 24$ and D_A has two ordinary points p_1, p_2 of multiplicity 4 and no other singularities. More precisely, in this case we can write*

$$D_A = E_1 + E_2 + E_3 + E_4,$$

where the E_i are elliptic curves intersecting pairwise transversally at p_1, p_2 and not elsewhere. Moreover, A is an étale double cover of the abelian surface $A' := E' \times E'$, where E' denotes the equianharmonic elliptic curve.

Up to isomorphism, there are exactly two such surfaces, which are complex-conjugate. Finally, the universal cover of these surfaces is not biholomorphic to the bidisk $\mathbb{H} \times \mathbb{H}$.

According to the dichotomy in the Main Theorem, we will use the terminology *surfaces of type I* and *surfaces of type II*, respectively. Besides answering the question above about the universal cover, the Main Theorem is also significant because

- it contains a new geometric construction of rigid surfaces, which is usually something hard to do;
- it provides a substantially new piece in the fine classification of minimal surfaces of general type with $p_g = q = 2$;
- it shows that surfaces of type *II* present the the so-called `Diff⇏Def` phenomenon, meaning that their diffeomorphism type does not determine their deformation class, see Remark 4.19.

Actually, the fact that there is exactly one surface of type *II* up to complex conjugation is a remarkable feature. The well-known Cartwright-Steger surfaces [**CS10**] share the same property, however our construction is of a different nature, more geometric and explicit.

The paper is organized as follows.

In Section 1 we provide a general result for minimal surfaces S with $p_g = q = 2$, $K^2 = 8$ and Albanese map $\alpha \colon S \longrightarrow A$ of degree 2, and we classify all the possible branch loci D_A for α (Proposition 1.3).

In Section 2 we consider surfaces of type *I*, showing that they coincide with the family of product-quotient surfaces constructed in [**Pen11**] (Theorem 2.5).

In Section 3 we start the investigation of surfaces of type *II*. The technical core of this part is Proposition 3.8, showing that, in this situation, the pair $(A,\, D_A)$ can be realized as an étale double cover of the pair $(A',\, D'_A)$, where $D_{A'}$ is a configuration of four elliptic curves in $A' = E' \times E'$ intersecting pairwise and transversally only at the origin $o' \in A'$ (as far as we know, the existence of such a configuration was first remarked in [**Hir84**]). The most difficult part is to prove that we can choose the double cover $A \longrightarrow A'$ in such a way that the curve D_A becomes 2-divisible in the Picard group of A (Proposition 3.8). The rigidity of S then follows from a characterization of A' proven in [**KH05**] (cf. also [**Aid**]).

In Section 4 we show that there are precisely two surfaces of type *II* up to isomorphism, and that they are complex-conjugated (Theorem 4.13). In order to do this, we had to study the groups of automorphisms and anti-biholomorphisms of A that preserve the branch locus D_A, and their permutation action on the set of the sixteen square roots of $\mathcal{O}_A(D_A)$ in the Picard group of A (Proposition 4.15).

Finally, we show that the universal cover of the surfaces of type *II* is not biholomorphic to $\mathbb{H} \times \mathbb{H}$ (Proposition 4.16), we note that they can be given the structure of an open ball quotient in at least two different ways (Remark 4.18) and we sketch an alternative geometric construction for their Albanese variety A (Remark 4.20).

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Notation and conventions. We work over the field of complex numbers. All varieties are assumed to be projective. For a smooth surface S, K_S denotes the *canonical class*, $p_g(S) = h^0(S,\, K_S)$ is the *geometric genus*, $q(S) = h^1(S,\, K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1-q(S)+p_g(S)$ is the *Euler-Poincaré characteristic*.

Linear equivalence of divisors is denoted by $\simeq$. If D_1 is an effective divisor on S_1 and D_2 is an effective divisor on S_2, we say that the pair $(S_1,\, D_1)$ is an *étale double cover* of the pair $(S_2,\, D_2)$ if there exists an étale double cover $f\colon S_1 \longrightarrow S_2$ such that $D_1 = f^*D_2$.

If A is an abelian surface, we denote by $(-1)_A\colon A \longrightarrow A$ the involution $x \longrightarrow -x$. If $a \in A$, we write $t_a\colon A \longrightarrow A$ for the translation by a, namely $t_a(x) = x + a$. We say that a divisor $D \subset A$ (respectively, a line bundle $\mathcal{L}$ on A) is *symmetric* if $(-1)_A^*D = D$ (respectively, if $(-1)_A^*\mathcal{L} \simeq \mathcal{L}$).

1 The structure of the Albanese map

Let us denote by S a minimal surface of general type with $p_g = q = 2$ and maximal Albanese dimension, and by

$$\alpha\colon S \longrightarrow A = \mathrm{Alb}(S)$$

its Albanese map. It follows from [**Cat13**, Section 5] that $\deg\alpha$ is equal to the index of the image of $\wedge^4 H^1(S,\, \mathbb{Z})$ inside $H^4(S,\, \mathbb{Z}) = \mathbb{Z}[S]$, hence it is a topological invariant of S. So, one can search to classify these surfaces by looking at the pair of invariants $\big(K_S^2,\, \deg\alpha\big)$.

Lemma 1.1. *Let S be as above and assume that there is a generically finite double cover $\tilde{\alpha}\colon S \longrightarrow \widetilde{A}$, where $\widetilde{A}$ is an abelian surface. Then $\widetilde{A}$ can be identified with $A = \mathrm{Alb}(S)$ and there exists an automorphism $\psi\colon A \longrightarrow A$ such that $\tilde{\alpha} = \psi \circ \alpha$.*

Proof. The universal property of the Albanese map ([**Bea96**, Chapter V]) implies that the morphism $\widetilde{\alpha}\colon S \longrightarrow \widetilde{A}$ factors through a morphism $\psi\colon A \longrightarrow \widetilde{A}$. But $\widetilde{\alpha}$ and α are both generically of degree 2, so ψ must be a birational map between abelian varieties, hence an isomorphism. Thus we can identify $\widetilde{A}$ with A and with this identification ψ is an automorphism of A. ☐

Throughout the paper, we will assume $\deg\alpha = 2$, namely that $\alpha\colon S \longrightarrow A$ is a generically finite double cover. Let us denote by $D_A \subset A$ the branch locus of α and let

$$\begin{tikzcd} S \arrow[r, "c"] \arrow[dr, "\alpha"'] & X \arrow[d, "\alpha_X"] \\ & A \end{tikzcd} \tag{1}$$

be its Stein factorization. The map $\alpha_X\colon X \longrightarrow A$ is a finite double cover and the fact that S is smooth implies that X is normal, see [**BHPVdV04**, Chapter I, Theorem 8.2]. In particular X has at most isolated singularities, hence D_A is reduced. Moreover, D_A is 2-divisible in $\mathrm{Pic}(A)$, in other words there exists a divisor L_A on A such that $D_A \simeq 2L_A$.

We have a *canonical resolution diagram*

$$\begin{tikzcd} \bar{S} \arrow[r] \arrow[d, "\beta"'] & X \arrow[d, "\alpha_X"] \\ B \arrow[r, "\varphi"] & A, \end{tikzcd} \tag{2}$$

see [**BHPVdV04**, Chapter III, Section 7], [**PP13**, Section 2] and [**Rit10**]. Here $\beta\colon \bar{S} \longrightarrow B$ is a finite double cover, $\bar{S}$ is smooth, but not necessarily minimal, S is the minimal model of $\bar{S}$ and

$\varphi \colon B \longrightarrow A$ is composed of a series of blow-ups. Let $x_1, x_2, \dots, x_r$ be the centers of these blow-ups and let $\mathcal{E}_i$ be the inverse image of x_i in B such that

$$\mathcal{E}_i\mathcal{E}_j = -\delta_{ij}, \quad K_B = \varphi^*K_A + \sum_{i=1}^r \mathcal{E}_i.$$

Then the branch locus D_B of $\beta \colon \bar{S} \longrightarrow B$ is smooth and can be written as

$$D_B = \varphi^*D_A - \sum_{i=1}^r d_i\mathcal{E}_i, \tag{3}$$

where the d_i are even positive integers, say $d_i = 2m_i$. Let us introduce the following definitions:

- a *negligible singularity* of D_A is a point x_j such that $d_j = 2$, and $d_i \leq 2$ for any point x_i infinitely near to x_j;
- a $[2d+1,\, 2d+1]$-*singularity* of D_A is a pair $(x_i,\, x_j)$ such that x_j belongs to the first infinitesimal neighbourhood of x_i and $d_i = 2d+2$, $d_j = 2d$;
- a $[2d,\, 2d]$-*singularity* of D_A is a pair $(x_i,\, x_j)$ such that x_i belongs to the first infinitesimal neighbourhood of x_j and $d_i = 2d$, $d_j = 2d+2$;
- a *minimal singularity* of D_A is a point x_j such that its inverse image in $\bar{S}$ via the canonical resolution contains no (-1)-curves;

For instance, an ordinary double point and an ordinary triple point are both negligible minimal singularities, whereas a [3, 3]-point is neither negligible nor minimal. Every ordinary singularity is minimal, but the converse is not true: a [4, 4]-point is minimal, but not ordinary.

Lemma 1.2. *In our situation, the following holds*:

(a) *we have $S = \bar{S}$ in* (2) *if and only if all singularities of D_A are minimal*;

(b) *if S contains no rational curves, then D_A contains no negligible singularities.*

Proof. ***(a)*** If D_A contains a non-minimal singularity then, by definition, $\bar{S}$ is not a minimal surface, hence $\bar{S} \neq S$. Conversely, if all singularities of D_A are minimal then there are no (-1)-curves on $\bar{S}$ coming from the resolution of the singularities of D_A. Since the abelian surface A contains no rational curves, this implies that $\bar{S}$ contains no (-1)-curves at all, so $\bar{S} = S$.

(b) Any negligible singularity of D_A gives rise to some rational double point in X, and hence to some rational curve in $\bar{S}$ that cannot be contracted by the blow-down morphism $\bar{S} \longrightarrow S$ (again because A contains no rational curves, so all (-1)-curves in $\bar{S}$ come from the resolution of singularities of X). This is impossible because we are assuming that S contains no rational curves.

□

By using the formulae in [**BHPVdV04**, p. 237], we obtain

$$2 = 2\chi(\mathcal{O}_{\bar{S}}) = L_A^2 - \sum m_i(m_i - 1), \quad K_{\bar{S}}^2 = 2L_A^2 - 2\sum(m_i - 1)^2. \tag{4}$$

Notice that the sums only involve the non-negligible singularities of $D_A \simeq 2L_A$. The two equalities in (4) together imply

$$K_S^2 \geq K_{\bar{S}}^2 = 4 + 2\sum(m_i - 1). \tag{5}$$

We are now ready to analyse in detail the case $K_S^2 = 8$.

Proposition 1.3. *Let S be a minimal surface with $p_g = q = 2$ and $K_S^2 = 8$. Then S contains no rational curves, in particular K_S is ample. Using the previous notation, if the Albanese map $\alpha \colon S \longrightarrow A$ is a generically finite double cover then we are in one of the following cases*:

(I) $D_A^2 = 32$ *and* D_A *has one ordinary singular point of multiplicity* 6 *and no other singularities*;

(II) $D_A^2 = 24$ *and* D_A *has two ordinary singular points of multiplicity* 4 *and no other singularities.*

Proof. The non-existence of rational curves on S is a consequence of a general bound for the number of rational curves on a surface of general type, see [**Miy84**, Proposition 2.1.1].

Since $K_S^2 = 8$, inequality (5) becomes

$$\sum (m_i - 1) \leq 2. \tag{6}$$

By Lemma 1.2 there are no negligible singularities in D_A, so (6) implies that we have three possibilities:

- D_A contains precisely one singularity (which is necessarily ordinary) and $m_1 = 3$, that is $d_1 = 6$; this is case ***(I)***.
- D_A contains precisely two singularities and $m_1 = m_2 = 2$, that is $d_1 = d_2 = 4$. We claim that these two quadruple points cannot be infinitely near. In fact, the canonical resolution of a [4, 4]-point implies that $\bar{S}$ contains (two) rational curves and, since a [4, 4]-point is a minimal singularity, this would imply the existence of rational curves on $S = \bar{S}$, a contradiction. So we have two ordinary points of multiplicity 4, and we obtain case ***(II)***.
- D_A contains precisely one singularity (which is necessarily ordinary) and $m_1 = 2$, that is $d_1 = 4$. An ordinary singularity is minimal, hence we get equality in (5), obtaining $K_S^2 = 6$ (this situation is considered in [**PP13**]), which case is excluded.

□

Remark 1.4. Lemma 1.2 and Proposition 1.3 imply that for any surface S with $p_g = q = 2$, $K_S^2 = 8$ and Albanese map of degree 2, we have $\bar{S} = S$. Furthermore, referring to diagram (1), the following holds:

- in case ***(I)***, the birational morphism $c \colon S \longrightarrow X$ contracts precisely one smooth curve Z, such that $g(Z) = 2$ and $Z^2 = -2$. This means that the singular locus of X consists of one isolated singularity x, whose geometric genus is $p_g(X, x) = \dim_{\mathbb{C}} R^1 c_* \mathcal{O}_S = 2$;
- in case ***(II)***, the birational morphism $c \colon S \longrightarrow X$ contracts precisely two disjoint elliptic curves Z_1, Z_2 such that $(Z_1)^2 = (Z_2)^2 = -2$. This means that the singular locus of X consists of two isolated elliptic singularities x_1, x_2 of type $\widetilde{E}_7$, see [**Ish14**, Theorem 7.6.4].

Definition 1.5. According to the dichotomy in Proposition 1.3, we will use the terminology *surfaces of type I* and *surfaces of type II*, respectively.

Proposition 1.6. *Let us denote as above by* D_A *the branch locus of the Albanese map* $\alpha \colon S \longrightarrow A$*. Then*:

- *if* S *is of type* I, *the curve* D_A *is irreducible*;
- *if* S *is of type* II, *the curve* D_A *is of the form* $D_A = E_1 + E_2 + E_3 + E_4$, *where the* E_i *are elliptic curves meeting pairwise transversally at two points* p_1, p_2 *and not elsewhere. In particular, we have* $E_iE_j = 2$ *for* $i \neq j$.

Proof. Suppose first that S is of type I and consider the blow-up $\varphi \colon B \longrightarrow A$ at the singular point $p \in D_A$. Let $C_1, \ldots, C_r$ be the irreducible components of the strict transform of D_A and $\mathcal{E} \subset B$ the exceptional divisor. The curve D_A only contains the ordinary singularity p, so the C_i are pairwise disjoint; moreover, the fact that

$$\sum_{i=1}^{r} C_i = \varphi^* D_A - 6\mathcal{E}$$

is 2-divisible in $\mathrm{Pic}(B)$ implies that $C_i^2 = C_i\big(\sum_{i=1}^r C_i\big)$ is an even integer. Now recall that the abelian surface A contains no rational curves and that every curve of geometric genus 1 on it is smooth and has self-intersection 0; then we infer $g(C_i) \geq 2$ and we can write

$$\begin{aligned} 6-4 &= \mathcal{E}(\varphi^* D_A - 6\mathcal{E}) + (\varphi^* D_A - 6\mathcal{E})^2 \\ &= K_B\Big(\sum_{i=1}^r C_i\Big) + \Big(\sum_{i=1}^r C_i\Big)^2 = \sum_{i=1}^r (2g(C_i)-2) \geq 2r, \end{aligned}$$

that is $r=1$ and D_A is irreducible.

Assume now that S is of type II, and write $D_A = E_1 + \cdots + E_r$, where each E_i is an irreducible curve. Denote by m_i and n_i the multiplicities of E_i at the two ordinary singular points p_1 and p_2 of D_A, and let $p_a(E_i)$ and g_i be the arithmetic and the geometric genus of E_i, respectively. We have $\sum_{i=1}^r m_i = \sum_{i=1}^r n_i = 4$ and

$$E_i^2 = 2p_a(E_i) - 2 = 2g_i - 2 + m_i(m_i-1) + n_i(n_i-1).$$

Using this, we can write

$$\begin{aligned} 24 = D_A^2 &= \sum_{i=1}^r E_i^2 + 2\sum_{j<k} E_j E_k \\ &= 2\sum_{i=1}^r g_i - 2r + \sum_{i=1}^r m_i(m_i-1) + \sum_{i=1}^r n_i(n_i-1) + 2\sum_{j<k}(m_j m_k + n_j n_k) \\ &= 2\sum_{i=1}^r g_i - 2r + \Big(\sum_{i=1}^r m_i\Big)^2 + \Big(\sum_{i=1}^r n_i\Big)^2 - \sum_{i=1}^r m_i - \sum_{i=1}^r n_i \\ &= 2\sum_{i=1}^r g_i - 2r + 24, \end{aligned}$$

that is $\sum_{i=1}^r g_i = r$. Since A contains no rational curves we have $g_i \geq 1$, and we conclude that

$$g_1 = \cdots = g_r = 1. \tag{7}$$

But every curve of geometric genus 1 on A is smooth, so (7) implies that D_A is the sum of r elliptic curves E_i passing through the singular points p_1 and p_2. Therefore $r=4$, because these points have multiplicity 4 in the branch locus D_A. □

2 Surfaces of type I

2.1 The product-quotient examples

The following family of examples, whose construction can be found in [**Pen11**], shows that surfaces of type I do actually exist. Let C' be a curve of genus $g(C') \geq 2$ and let G be a finite group that acts freely on $C' \times C'$. We assume moreover that the action is *mixed*, namely that there exists an element in G exchanging the two factors; this means that

$$G \subset \mathrm{Aut}(C' \times C') \simeq \mathrm{Aut}(C')^2 \rtimes \mathbb{Z}/2\mathbb{Z}$$

is not contained in $\mathrm{Aut}(C')^2$. Then the quotient $S := (C' \times C')/G$ is a smooth surface with

$$\chi(\mathcal{O}_S) = (g-1)^2/|G|, \quad K_S^2 = 8\chi(\mathcal{O}_S). \tag{8}$$

The intersection $G^0 := G \cap \mathrm{Aut}(C')^2$ is an index 2 subgroup of G, fitting into the non-split extension

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

and such that the genus of the curve $C := C'/G^0$ equals $q(S)$, see [**Fra13**, Lemma 2.9].

We have a commutative diagram

$$\begin{tikzcd}
C' \times C' \arrow[r, "t"] \arrow[d] & C \times C \arrow[d, "u"] \\
S \arrow[r, "\beta"] & \mathrm{Sym}^2(C),
\end{tikzcd}$$

where $t\colon C' \times C' \longrightarrow C \times C$ is a $(G^0 \times G^0)$-cover, $u\colon C \times C \longrightarrow \mathrm{Sym}^2(C)$ is the natural projection onto the second symmetric product and $\beta\colon S \longrightarrow \mathrm{Sym}^2(C)$ is a finite cover of degree $|G^0|$.

Assume now that C' has genus 3 and that $G^0 \simeq \mathbb{Z}/2\mathbb{Z}$. Then $G \simeq \mathbb{Z}/4\mathbb{Z}$ and C has genus 2. Denoting by $\Delta \subset C \times C$ the diagonal and by $\Gamma \subset C \times C$ the graph of the hyperelliptic involution $\iota\colon C \longrightarrow C$, we see that Δ and Γ are smooth curves isomorphic to C and satisfying

$$\Delta\Gamma = 6, \quad \Delta^2 = \Gamma^2 = -2.$$

The ramification divisor of u is precisely Δ, so $u(\Delta)^2 = -4$, whereas $u(\Gamma)$ is a (-1)-curve. The corresponding blow-down morphism $\varphi\colon \mathrm{Sym}^2(C) \longrightarrow A$ is the Abel-Jacobi map, and A is an abelian surface isomorphic to the Jacobian variety $J(C)$. The composed map

$$\alpha = \varphi \circ \beta\colon S \longrightarrow A$$

is a generically finite double cover, that by the universal property coincides, up to automorphisms of A, with the Albanese morphism of S. Such a morphism is branched over $D_A := (\varphi \circ u)(\Delta)$, which is a curve with $D_A^2 = 32$ and containing an ordinary sextuple point and no other singularities: in fact, the curves $u(\Delta)$ and $u(\Gamma)$ intersect transversally at precisely six points, corresponding to the six Weierstrass points of C.

From this and (8), it follows that S is a surface with $p_g = q = 2$, $K_S^2 = 8$ and of type I. Note that, with the notation of Section 1, we have $B = \mathrm{Sym}^2(C)$ and $D_B = u(\Delta)$.

Remark 2.1. Here is a different construction of the singular curve D_A considered in the previous example. Let $A := J(C)$ be the Jacobian of a smooth genus 2 curve and let us consider a symmetric theta divisor $\Theta \subset A$. Then the Weierstrass points of Θ are six 2-torsion points of A, say $p_0, \ldots, p_5$, and D_A arises as the image of Θ via the multiplication map $2_A\colon A \longrightarrow A$ given by $x \mapsto 2x$. Note that D_A is numerically equivalent to 4Θ.

Remark 2.2. Recently, R. Pignatelli and the first author studied some surfaces with $p_g = q = 2$ and $K_S^2 = 7$, originally constructed in [**CF15**] and arising as *triple* covers $S \longrightarrow A$ branched over D_A, where $(A,\, D_A)$ is as in the previous example. We refer the reader to [**PP16**] for more details.

2.2 The classification

The aim of this subsection is to show that every surface of type I is a product-quotient surface of the type described in Subsection 2.1.

Lemma 2.3. *Let D be an irreducible curve contained in an abelian surface A, with $D^2 = 32$ and having an ordinary point p of multiplicity 6 and no other singularities. Then, up to translations, we can suppose $p = 0$ and D symmetric, namely $(-1)_A^* D = D$.*

Proof. Since $D^2 > 0$, the line bundle $\mathcal{L} = \mathcal{O}_A(D)$ is non-degenerate and so, by [**BL04**, Chapter 4, §6], there exists a line bundle $\mathcal{L}'$ which is algebraically equivalent to $\mathcal{L}$ and symmetric, i.e. $(-1)_A^* \mathcal{L}' \simeq \mathcal{L}'$. Since algebraically equivalent line bundles differ by a translation, see [**BL04**, Section 4.6], up to translations we can assume that $\mathcal{L}$ itself is symmetric. Then $D' := (-1)_A^* D$ is an effective divisor linearly equivalent to D, hence the two translates

$$D_p := t_{-p}^* D \quad \text{and} \quad D'_p := t_p^* D'$$

are algebraically equivalent irreducible divisors, both having a sextuple point at 0. If D_p and D'_p were distinct, we would have $D_p D'_p \geq 36$, a contradiction because $D^2 = 32$; so $D_p = D'_p$. But $D'_p = (-1)_A^* D_p$, hence D_p is a symmetric translate of D having its sextuple point at 0, as desired. $\square$

Proposition 2.4. *If $D \subset A$ is as in* Lemma 2.3*, then there exists a smooth genus* 2 *curve C such that $A = J(C)$. Furthermore, up to translations, the curve D can be obtained as in* Remark 2.1*, namely as the image of a symmetric theta divisor $\Theta \subset A$ via the multiplication map $2_A \colon A \longrightarrow A$.*

Proof. By Lemma 2.3, we can assume that D is a symmetric divisor and that its sextuple point is the origin $0 \in A$. The geometric genus of D is 2, hence its normalization $C \longrightarrow D$ is a smooth genus 2 curve. By the universal property of the Jacobian, the composed map $C \longrightarrow D \hookrightarrow A$ factors through an isogeny

$$\eta \colon J(C) \longrightarrow A,$$

where we can assume, up to translations, that the image Θ of the embedding $C \hookrightarrow J(C)$ is a theta divisor containing the origin $0 \in J(C)$. Thus, the abelian surface A is isomorphic to $J(C)/T$, where $T := \ker \eta$ is a torsion subgroup whose order $|T|$ equals the degree d of η. The group T contains the group generated by the six points

$$0 = p_0,\, p_1, \ldots, p_5$$

corresponding to the six distinct points of C over $0 \in D$. The restriction of η to C is birational, so we have

$$\eta^* D = \Theta_0 + \cdots + \Theta_{d-1},$$

where $\Theta_0 = \Theta$ and the Θ_j are translates of Θ_0 by the elements of T. Since $D^2 = 32$, we obtain $(\eta^* D)^2 = 32d$. On the other hand, all the curves Θ_j are algebraically equivalent, hence $\Theta_i \Theta_j = 2$ for all pairs $(i,\, j)$ and we infer $(\eta^* D)^2 = (\Theta_0 + \cdots + \Theta_{d-1})^2 = 2d^2$. So $32d = 2d^2$, that is $d = 16$.

This shows that the reducible curve $\eta^* D$ has sixteen sextuple points $p_0, \ldots, p_{15}$, such that every curve Θ_j contains six of them; conversely, since all the Θ_j are smooth, through any of the p_k pass exactly six curves. We express these facts by saying that the sixteen curves Θ_j and the sixteen points p_k form a *symmetric* (16_6)-*configuration.* The involution $(-1)_A$ acts on D, so the involution $(-1)_{J(C)}$ acts on Θ, that is Θ is a symmetric divisor on $J(C)$. Furthermore, the action of $(-1)_A$ induces the multiplication by -1 on the tangent space $T_{A,0}$, hence it preserves the six tangent directions of D at 0; this means that $p_0, \ldots, p_5$ are fixed points for the restriction of $(-1)_{J(C)}$ to Θ. But a non-trivial involution with six fixed points on a smooth curve of genus 2 must be the hyperelliptic involution, so $p_0, \ldots, p_5$ are the Weierstrass points of Θ. By [**Mum84**, Chapter 3.2, pp. 28-39], these six points generate the (order 16) subgroup $J(C)[2]$ of points of order 2 in $J(C)$, thus $T = J(C)[2]$.

Summing up, our symmetric (16_6)-configuration coincides with the so-called *Kummer configuration*, see [**BL04**, Chapter 10]; moreover, A is isomorphic to $J(C)$ and the map η coincides with the multiplication map $2_A \colon A \longrightarrow A$. □

Theorem 2.5. *Surfaces of type I are precisely the product-quotient surfaces described in Section* 2.1*, in particular they form a family of dimension* 3*. More precisely, denoting by $\mathcal{M}_I$ their Gieseker moduli space and by $\mathcal{M}_2$ the moduli space of curves of genus* 2*, there exists a quasi-finite morphism $\mathcal{M}_I \longrightarrow \mathcal{M}_2$ of degree* 15*.*

Proof. Given any surface S of type I, by Proposition 2.4 there exists a smooth curve C of genus 2 such that S is the canonical desingularization of the double cover $\alpha \colon X \longrightarrow A$, where $A = J(C)$, branched over the singular curve D_A described in the example of Section 2.1 and in Remark 2.1. Equivalently, S arises as a double cover $\beta \colon S \longrightarrow B$, where $B = \mathrm{Sym}^2(C)$, branched over the smooth diagonal divisor D_B. There are sixteen distinct covers, corresponding to the sixteen square roots of D_B in $\mathrm{Pic}(B)$. One of them is the double cover $u \colon C \times C \longrightarrow B$, whereas the others are fifteen surfaces S with $p_g(S) = q(S) = 2$ and Albanese variety isomorphic to $J(C)$; for a general choice of C, such surfaces are pairwise non-isomorphic.

On the other hand, once fixed a curve C of genus 2, the product-quotient construction uniquely depends on the choice of the étale double cover $C' \longrightarrow C$, that is on the choice of a non-trivial 2-torsion element of $J(C)$. There are precisely fifteen such elements, that necessarily correspond to the fifteen surfaces with $p_g(S) = q(S) = 2$ and $\mathrm{Alb}(S) \simeq J(C)$ found above.

Therefore every surface of type I is a product-quotient example, and the map $\mathcal{M}_I \longrightarrow \mathcal{M}_2$ defined by $[S] \mapsto [C]$ is a quasi-finite morphism of degree 15. □

Remark 2.6. The moduli space of genus 2 curves C with a non-trivial 2 torsion point in $J(C)$ is rational (see [**Dol08**]). According to the description of $\mathcal{M}_I$ in the proof of Theorem 2.5, we see that $\mathcal{M}_I$ is rational.

Theorem 2.5 in particular implies that the universal cover of S coincides with the universal cover of $C' \times C'$, so we obtain

Corollary 2.7. *Let S be a surface of type I and $\widetilde{S} \longrightarrow S$ its universal cover. Then $\widetilde{S}$ is biholomorphic to the bidisk $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ is the Poincaré upper half-plane.*

3 Surfaces of type II : construction

3.1 Line bundles on abelian varieties and the Appell-Humbert theorem

In this subsection we shortly collect some results on abelian varieties that will be used in the sequel, referring the reader to [**BL04**, Chapters 1-4] for more details. Let $A = V/\Lambda$ be an abelian variety, where V is a finite-dimensional $\mathbb{C}$-vector space and $\Lambda \subset V$ a lattice. Then the *Appell-Humbert Theorem*, see [**BL04**, Theorem 2.2.3], implies that

- the Néron-Severi group $\operatorname{NS}(A)$ can be identified with the group of hermitian forms $h \colon V \times V \longrightarrow \mathbb{C}$ whose imaginary part $\operatorname{Im} h$ takes integral values on Λ;
- the Picard group $\operatorname{Pic}(A)$ can be identified with the group of pairs $(h,\, \chi)$, where $h \in \operatorname{NS}(A)$ and χ is a *semicharacter*, namely a map

  $$\chi \colon \Lambda \longrightarrow U(1), \quad \text{where } U(1) = \{z \in \mathbb{C} \mid |z| = 1\},$$

  such that

  $$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\pi i \operatorname{Im} h(\lambda,\, \mu)} \quad \text{for all } \lambda,\, \mu \in \Lambda. \tag{9}$$
- with these identifications, the first Chern class map $c_1 \colon \operatorname{Pic}(A) \longrightarrow \operatorname{NS}(A)$ is nothing but the projection to the first component, i.e. $(h,\, \chi) \mapsto h$.

We will write $\mathcal{L} = \mathcal{L}(h,\, \chi)$, so that we have $\mathcal{L}(H,\, \chi) \otimes \mathcal{L}(H',\, \chi') = \mathcal{L}(H+H',\, \chi\chi')$. The line bundle $\mathcal{L}(H,\, \chi)$ is symmetric if and only if the semicharacter χ has values in $\{\pm 1\}$, see [**BL04**, Corollary 2.3.7]. Furthermore, for any $\bar{v} \in A$ with representative $v \in V$, we have

$$t_{\bar{v}}^* \mathcal{L}(h,\, \chi) = \mathcal{L}(h,\, \chi\, e^{2\pi i \operatorname{Im} h(v,\, \cdot)}), \tag{10}$$

see [**BL04**, Lemma 2.3.2].

Remark 3.1. Assume that the class of $\mathcal{L} = \mathcal{L}(h,\, \chi)$ is 2-divisible in $\operatorname{NS}(A)$, that is $h = 2h'$. Then $\operatorname{Im} h(\Lambda,\, \Lambda) \subseteq 2\mathbb{Z}$ and moreover formula (9) implies that $\chi \colon \Lambda \longrightarrow U(1)$ is a character, namely $\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)$. In particular, $\mathcal{L}$ belongs to $\operatorname{Pic}^0(A)$ if and only if there exists a character χ such that $\mathcal{L} = \mathcal{L}(0,\, \chi)$.

Proposition 3.2 ([**BL04**], Lemma 2.3.4)**.** *Let $A_1 = V_1/\Lambda_1$ and $A_2 = V_2/\Lambda_2$ be two abelian varieties, and let $f \colon A_2 \longrightarrow A_1$ be a homomorphism with analytic representation $F \colon V_2 \longrightarrow V_1$ and rational representation $F_\Lambda \colon \Lambda_2 \longrightarrow \Lambda_1$. Then for any $\mathcal{L}(h,\, \chi) \in \operatorname{Pic}(A_1)$ we have*

$$f^* \mathcal{L}(h,\, \chi) = \mathcal{L}(F^* h,\, F_\Lambda^* \chi), \tag{11}$$

Given a point $x \in A$ and a divisor $D \subset A$, let us denote by $m(D,\, x)$ the multiplicity of D at x.

Lemma 3.3 ([**BL04**], Proposition 4.7.2)**.** *Let $\mathcal{L} = \mathcal{L}(h,\, \chi)$ be a symmetric line bundle on A and D a symmetric effective divisor such that $\mathcal{L} = \mathcal{O}_A(D)$. For every 2-torsion point $x \in A[2]$ with representative $\frac{1}{2}\lambda$, where $\lambda \in \Lambda$, we have*

$$\chi(\lambda) = (-1)^{m(D,\, 0) + m(D,\, x)}.$$

3.2 The equianharmonic product

Let $\zeta := e^{2\pi i/6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, so that $\zeta^2 - \zeta + 1 = 0$, and consider the *equianharmonic elliptic curve*

$$E' := \mathbb{C}/\Gamma_\zeta, \quad \Gamma_\zeta := \mathbb{Z}\zeta \oplus \mathbb{Z}. \tag{12}$$

Setting $V := \mathbb{C}^2$, we can define

$$A' := E' \times E' = V/\Lambda_{A'}, \quad \Lambda_{A'} := \Gamma_\zeta \times \Gamma_\zeta.$$

Then A' is a principally polarized abelian surface, that we will call the *equianharmonic product.* Denoting by $(z_1,\, z_2)$ the coordinates of V and by $e_1 = (1,\, 0)$, $e_2 = (0,\, 1)$ its standard basis, the four vectors

$$\lambda_1 := \zeta e_1, \quad \lambda_2 := \zeta e_2, \quad e_1, \quad e_2 \tag{13}$$

form a basis for the lattice $\Lambda_{A'}$.

We now consider the four 1-dimensional complex subspaces of V defined as

$$\begin{aligned} &V_1 := \operatorname{span}(e_1) = \{z_2 = 0\}, && V_2 := \operatorname{span}(e_2) = \{z_1 = 0\}, \\ &V_3 := \operatorname{span}(e_1 + e_2) = \{z_1 - z_2 = 0\}, && V_4 := \operatorname{span}(e_1 + \zeta e_2) = \{\zeta z_1 - z_2 = 0\}. \end{aligned} \tag{14}$$

For each $k \in \{1,\, 2,\, 3,\, 4\}$, the subspace V_k contains a rank 1 sublattice $\Lambda_k \subset \Lambda_{A'}$ isomorphic to Γ_ζ, where

$$\begin{aligned} &\Lambda_1 := \mathbb{Z}\lambda_1 \oplus \mathbb{Z}e_1, && \Lambda_2 := \mathbb{Z}\lambda_2 \oplus \mathbb{Z}e_2, \\ &\Lambda_3 := \mathbb{Z}(\lambda_1 + \lambda_2) \oplus \mathbb{Z}(e_1 + e_2), && \Lambda_4 := \mathbb{Z}(\lambda_1 + \lambda_2 - e_2) \oplus \mathbb{Z}(\lambda_2 + e_1). \end{aligned} \tag{15}$$

Consequently, in A' there are four elliptic curves isomorphic to E', namely

$$E'_k := V_k/\Lambda_k, \quad k \in \{1,\, 2,\, 3,\, 4\}. \tag{16}$$

Proposition 3.4 ([**Hir84**] Section 1)**.** *The four curves E'_k only intersect* (*pairwise transversally*) *at the origin $o' \in A'$. Consequently, the reducible divisor*

$$D_{A'} := E'_1 + E'_2 + E'_3 + E'_4$$

has an ordinary quadruple point at o' and no other singularities.

By the Appell-Humbert Theorem, the Néron-Severi group $\operatorname{NS}(A')$ of A' can be identified with the group of hermitian forms h on V whose imaginary part takes integral values on $\Lambda_{A'}$. We will use the symbol H for the 2×2 hermitian matrix associated to h with respect to the standard basis of V so that, thinking of $v,\, w \in V$ as column vectors, we can write $h(v,\, w) = {}^t v H \bar{w}$. We want now to identify those hermitian matrices $H_1, \ldots, H_4$ that correspond to the classes of the curves $E'_1, \ldots, E'_4$, respectively.

Proposition 3.5. *We have*

$$\begin{aligned} H_1 &= \frac{2}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & H_2 &= \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ H_3 &= \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, & H_4 &= \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & -\zeta \\ -\bar{\zeta} & 1 \end{pmatrix}, \end{aligned}$$

so that the hermitian matrix representing in $\operatorname{NS}(A')$ *the class of the divisor $D_{A'}$ is*

$$H := H_1 + H_2 + H_3 + H_4 = \frac{2}{\sqrt{3}} \begin{pmatrix} 3 & -1-\zeta \\ -1-\bar{\zeta} & 3 \end{pmatrix}.$$

Moreover, setting $\lambda = (a_1 + \zeta a_2,\, a_3 + \zeta a_4) \in \Lambda_{A'}$, the semicharacter $\chi_{D_{A'}}$ corresponding to the line bundle $\mathcal{O}_{A'}(D_{A'})$ can be written as

$$\chi_{D_{A'}}(\lambda) = (-1)^{a_1+a_2+a_3+a_4+a_1(a_2+a_3+a_4)+(a_2+a_3)a_4}.$$

Proof. The hermitian form $\tilde{h}$ on $\mathbb{C}$ given by $\tilde{h}(z_1,\, z_2) = \frac{2}{\sqrt{3}} z_1 \bar{z}_2$ is positive definite and its imaginary part is integer-valued on Γ_ζ, so it defines a positive class in $\mathrm{NS}(E')$. Moreover, in the ordered basis $\{\zeta,\, 1\}$ of Γ_ζ, the alternating form $\mathrm{Im}\, \tilde{h}$ is represented by the skew-symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, whose Pfaffian equals 1, so $\tilde{h}$ corresponds to the ample generator of the Néron-Severi group of E', see [**BL04**, Corollary 3.2.8]. In other words, $\tilde{h}$ is the Chern class of $\mathcal{O}_{E'}(0)$, where 0 is the origin of E'. Write $\mathcal{O}_{E'}(0) = \mathcal{L}(\tilde{h},\, \nu)$ for a suitable semicharacter $\nu\colon \Gamma_\zeta \longrightarrow \mathbb{C}$; since $\mathcal{O}_{E'}(0)$ is a symmetric line bundle, the values of ν at the generators of Γ_ζ can be computed by using Lemma 3.3, obtaining $\nu(1) = -1, \quad \nu(\zeta) = -1$. Consequently, for all $a,\, b \in \mathbb{Z}$ we get

$$\nu(a + b\zeta) = \nu(a)\nu(b\zeta)e^{\pi i \, \mathrm{Im}\, \tilde{h}(a,\, b\zeta)} = (-1)^a(-1)^b(-1)^{ab} = (-1)^{a+b+ab}. \tag{17}$$

For any $k \in \{1, \ldots, 4\}$ let us define a group homomorphism $F_k\colon A' \longrightarrow E'$ as follows:

$$F_1(z_1,\, z_2) = z_2, \quad F_2(z_1,\, z_2) = z_1, \quad F_3(z_1,\, z_2) = z_1 - z_2, \quad F_4(z_1,\, z_2) = \zeta z_1 - z_2.$$

By (14) we have $E'_k = F_k^*(0)$ and so, setting $\mathcal{O}_{A'}(E'_k) = \mathcal{L}(h_k,\, \chi'_k)$, by (11) we deduce

$$h_k = F_k^* \tilde{h}, \quad \chi'_k = F_k^* \nu. \tag{18}$$

This gives immediately the four matrices $H_1, \ldots, H_4$. Moreover, by using (17) and (18), we can write down the semicharacters $\chi'_1, \ldots, \chi'_4$; in fact, for any $\lambda = (a_1 + \zeta a_2,\, a_3 + \zeta a_4) \in \Lambda_{A'}$, we obtain

$$\begin{aligned}
\chi'_1(\lambda) &= (-1)^{a_3+a_4+a_3a_4} \\
\chi'_2(\lambda) &= (-1)^{a_1+a_2+a_1a_2} \\
\chi'_3(\lambda) &= (-1)^{a_1+a_2+a_3+a_4+(a_1+a_3)(a_2+a_4)} \\
\chi'_4(\lambda) &= (-1)^{a_1+a_2+a_3+a_4+(a_1+a_4)(a_2+a_3)+a_2a_3}.
\end{aligned}$$

The semicharacter $\chi_{D_{A'}}$ can be now computed by using the formula $\chi_{D_{A'}} = \chi'_1\chi'_2\chi'_3\chi'_4$. □

Remark 3.6. The hermitian matrix

$$H_1 + H_2 = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

represents in $\mathrm{NS}(A')$ the class of the principal polarization of product type

$$\Theta := E' \times \{0\} + \{0\} \times E'.$$

Remark 3.7. The free abelian group $\mathrm{NS}(A')$ is generated by the classes of the elliptic curves E'_1, E'_2, E'_3, E'_4. In fact, since $A' = E' \times E'$ and E' has complex multiplication, it is well-known that $\mathrm{NS}(A')$ has rank 4, see [**BL04**, Exercise 5.6 (10) p.142], hence we only need to show that the classes of the curves E'_k generate a primitive sublattice of maximal rank in the Néron-Severi group. By Proposition 3.4, the corresponding Gram matrix has determinant

$$\det\,(E'_i \cdot E'_j) = \det(\delta_{ij}) = -3$$

so the claim follows because -3 is a non-zero, square-free integer.

3.3 Double covers of the equianharmonic product

In order to construct a surface of type *II*, we must find an abelian surface A and a divisor D_A on it such that

- D_A is 2-divisible in $\mathrm{Pic}(A)$;
- $D_A^2 = 24$ and D_A has precisely two ordinary quadruple points as singularities.

We will construct the pair $(A,\, D_A)$ as an étale double cover of the pair $(A',\, D_{A'})$, where $A' = V/\Lambda_{A'}$ is the equianharmonic product and $D_{A'} = E'_1 + E'_2 + E'_3 + E'_4$ is the sum of four elliptic curves considered in Proposition 3.4.

By the Appell-Humbert theorem, the sixteen 2-torsion divisors on A', i.e. the elements of order 2 in $\mathrm{Pic}^0(A')$, correspond to the sixteen characters

$$\chi\colon \Lambda_{A'} \longrightarrow \{\pm 1\}. \tag{19}$$

Any such character is specified by its values at the elements of the ordered basis $\{\lambda_1,\, \lambda_2,\, e_1,\, e_2\}$ of $\Lambda_{A'}$ given in (13), so it can be denoted by

$$\chi = (\chi(\lambda_1),\, \chi(\lambda_2),\, \chi(e_1),\, \chi(e_2)).$$

For instance, $\chi_0 := (1,\, 1,\, 1,\, 1)$ is the trivial character, corresponding to the trivial divisor $\mathcal{O}_{A'}$. We will write

$$\begin{array}{lll} \chi_1 := (-1,\, -1,\, 1, -1), & \chi_2 := (1,\, -1,\, -1,\, 1), & \chi_3 := (-1,\, 1,\, -1,\, -1), \\ \chi_4 := (1,\, 1,\, -1,\, 1), & \chi_5 := (-1,\, 1,\, 1,\, 1), & \chi_6 := (-1,\, 1,\, -1,\, 1), \\ \chi_7 := (1,\, 1,\, 1,\, -1), & \chi_8 := (1,\, -1,\, 1,\, -1), & \chi_9 := (-1,\, 1,\, 1,\, -1), \\ \chi_{10} := (1,\, 1,\, -1,\, -1), & \chi_{11} := (-1,\, -1,\, -1,\, 1), & \chi_{12} := (1,\, -1,\, 1,\, 1), \\ \chi_{13} := (1,\, -1,\, -1,\, -1), & \chi_{14} := (-1,\, -1,\, 1,\, 1), & \chi_{15} := (-1,\, -1,\, -1,\, -1) \end{array} \tag{20}$$

for the fifteen non-trivial characters. To any non-trivial 2-torsion divisor on A', and so to any non-trivial character χ as in (19), it corresponds an isogeny of degree two $f_{\chi}\colon A_{\chi} \longrightarrow A'$; in fact, $\ker \chi \subset \Lambda_{A'}$ is a sublattice of index 2 and A_{χ} is the abelian surface

$$A_{\chi} = V/\ker\chi. \tag{21}$$

Let us set

$$E_i := f_{\chi}^*(E'_i), \quad D_{A_{\chi}} := f_{\chi}^*(D_{A'}) = E_1 + E_2 + E_3 + E_4$$

and write Σ for the subgroup of $\mathrm{Pic}^0(A')$ generated by χ_1 and χ_2, namely

$$\Sigma := \{\chi_0,\, \chi_1,\, \chi_2,\, \chi_3\}. \tag{22}$$

We are now ready to prove the key result of this subsection.

Proposition 3.8. *The following are equivalent*:

- ***(a)*** *the divisor* $D_{A_{\chi}}$ *is* 2*-divisible in* $\mathrm{Pic}(A_{\chi})$;
- ***(a′)*** *the divisor* $D_{A_{\chi}}$ *is* 2*-divisible in* $\mathrm{NS}(A_{\chi})$;
- ***(b)*** *every* E_i *is an irreducible elliptic curve in* A_{χ};
- ***(c)*** *the character* χ *is a non-trivial element of* Σ.

Proof. We first observe that $\mathrm{Pic}^0(A_{\chi}) = \mathrm{Pic}(A_{\chi})/\mathrm{NS}(A_{\chi})$ is a divisible group, so ***(a)*** is equivalent to ***(a′)***.

Next, the curve $E_i \subset A_{\chi}$ is irreducible if and only if the 2-torsion divisor corresponding to the character $\chi\colon \Lambda_{A'} \longrightarrow \{\pm 1\}$ restricts non-trivially to E'_i. This in turn means that χ restricts non-trivially to the sublattice Λ_i, and so ***(b)*** occurs if and only if χ restricts non-trivially to all $\Lambda_1,\, \Lambda_2,\, \Lambda_3,\, \Lambda_4$. By using the generators given in (15), a long but elementary computation (or a quick computer calculation) shows that this happens if and only if ***(c)*** holds.

It remains to prove that ***(a′)*** and ***(c)*** are equivalent. The isogeny $f_{\chi}\colon A_{\chi} \longrightarrow A$ lifts to the identity $1_V\colon V \longrightarrow V$ so, if $h\colon V \times V \longrightarrow \mathbb{C}$ is the hermitian form that represents the class of $D_{A'}$ in $\mathrm{NS}(A')$, then the same form also represents the class of $D_{A_{\chi}}$ in $\mathrm{NS}(A_{\chi})$. By the Appell-Humbert theorem the group $\mathrm{NS}(A_{\chi})$ can be identified with the group of hermitian forms on V whose imaginary part takes integral values on the lattice $\ker\chi$, so (21) implies that condition ***(a′)*** is equivalent to

$$\mathrm{Im}\, h(\ker\chi,\, \ker\chi) \subseteq 2\mathbb{Z}. \tag{23}$$

The non-zero values assumed by the alternating form $\operatorname{Im} h$ on the generators $\lambda_1,\ \lambda_2,\ e_1,\ e_2$ of $\Lambda_{A'}$ can be computed by using the hermitian matrix H given in Proposition 3.5, obtaining Table 1 below:

$(\cdot,\ \cdot)$	$(\lambda_1,\ \lambda_2)$	$(\lambda_1,\ e_1)$	$(\lambda_1,\ e_2)$	$(\lambda_2,\ e_1)$	$(\lambda_2,\ e_2)$	$(e_1,\ e_2)$
$\operatorname{Im} h(\cdot,\ \cdot)$	-1	3	-2	-1	3	-1

Table 1: Non-zero values of $\operatorname{Im} h$ at the generators of $\Lambda_{A'}$

Now we show that (23) holds if and only if χ is a non-trivial element of Σ. In fact we have seen that, if $\chi \notin \{\chi_1,\ \chi_2,\ \chi_3\}$, then one of the effective divisors $E_i = f_\chi^*(E_i')$ is a disjoint union of two elliptic curves, say $E_i = E_{i1} + E_{i2}$. But then, using the projection formula, we find

$$D_{A_\chi} \cdot E_{i1} = f_\chi^*(D_{A'}) \cdot E_{i1} = D_{A'} \cdot f_{\chi *}(E_{i1}) = D_{A'} \cdot E_i' = 3$$

which is not an even integer, so D_{A_χ} is not 2-divisible in this situation.

Let us consider now the case $\chi \in \{\chi_1,\ \chi_2,\ \chi_3\}$. We can easily see that the integral bases of $\ker \chi_1$, $\ker \chi_2$, $\ker \chi_3$ are given by

$$\begin{aligned} \mathscr{B}_1 &:= \{e_1,\ \lambda_1 + e_2,\ \lambda_2 + e_2,\ 2e_2\}, \\ \mathscr{B}_2 &:= \{\lambda_2 + e_1,\ \lambda_1,\ e_2,\ 2e_1\}, \\ \mathscr{B}_3 &:= \{\lambda_1 + e_2,\ \lambda_2,\ 2e_2,\ e_1 + e_2\}, \end{aligned} \tag{24}$$

respectively. Then, by using Table 1, it is straightforward to check that $\operatorname{Im} h(b_1,\ b_2) \in 2\mathbb{Z}$ for all $b_1,\ b_2 \in \mathscr{B}_1$; for instance, we have

$$\begin{aligned} \operatorname{Im} h(\lambda_1 + e_2,\ \lambda_2 + e_2) &= \operatorname{Im} h(\lambda_1, \lambda_2) + \operatorname{Im} h(\lambda_1, e_2) + \operatorname{Im} h(e_2, \lambda_2) + \operatorname{Im} h(e_2, e_2) \\ &= -1 - 2 - 3 + 0 = -6 \in 2\mathbb{Z}. \end{aligned}$$

This shows that the inclusion (23) holds for χ_1. The proof that it also holds for χ_2 and χ_3 is analogous. □

Remark 3.9. Writing the details in the proof of Proposition 3.8, one sees that every non-trivial character χ in (20) restricts trivially to *at most one* curve E_i'. Identifying A' with $\operatorname{Pic}^0(A')$ via the principal polarization Θ described in Remark 3.6, this corresponds to the fact that every 2-torsion point of A' is contained in at most one of the E_i'. More precisely, every E_i' contains exactly three non-zero 2-torsion points of A', so it remain $16 - (4 \times 3 + 1) = 3$ of them that are not contained in any of the E_i'. Via the identification above, they clearly correspond to the three 2-torsion divisors restricting non-trivially to all the E_i', namely to the three non-trivial characters in the group Σ.

Summing up, we have the following existence result for surfaces of type *II*.

Proposition 3.10. *Let χ be any non-trivial element in the group Σ, and write $f\colon A \longrightarrow A'$ instead of $f_\chi\colon A_\chi \longrightarrow A'$. Then there exists a double cover $\alpha_X\colon X \longrightarrow A$ branched precisely over the 2-divisible effective divisor $D_A \subset A$. The smooth minimal resolution S of X is a surface with $p_g = q = 2$, $K^2 = 8$ and Albanese map of degree 2, belonging to type II.*

Proof. It only remains to compute the invariants of S. From the double cover formulas (see [**BHPVdV04**, Chapter V.22]) we see that, if we impose an ordinary quadruple point to the branch locus, then χ decreases by 1 and K^2 decreases by 2, hence we get

$$\chi(\mathcal{O}_S) = \frac{1}{8}D_A^2 - 2 = 1 \quad \text{and} \quad K_S^2 = \frac{1}{2}D_A^2 - 4 = 8.$$

Since $q(S) \geq q(A) = 2$, we have $p_g(S) = q(S) \geq 2$. Assume that $p_g(S) = q(S) \geq 3$. By [**HP02**], [**Pir02**] and [**Deb82**, Beauville appendix], we have two possibilities:

- $p_g(S) = q(S) = 4$ and S is the product of two curves of genus 2;

- $p_g(S) = q(S) = 3$ and $S = (C_2 \times C_3)/\mathbb{Z}_2$, where C_2 is a smooth curve of genus 2 with an elliptic involution τ_2, C_3 is a smooth curve of genus 3 with a free involution τ_3, and the cyclic group $\mathbb{Z}_2$ acts freely on the product $C_2 \times C_3$ via the involution $\tau_2 \times \tau_3$.

In both cases above, S contains no elliptic curves. On the other hand, all our surfaces of type II contain four elliptic curves, coming from the strict transform of D_A. Therefore the only possibility is $p_g(S) = q(S) = 2$. $\square$

4 Surfaces of type II : classification

4.1 Holomorphic and anti-holomorphic diffeomorphisms of cyclic covers

Let $n \geq 2$ be an integer and let D be an effective divisor on a smooth projective variety Y, such that

$$\mathcal{O}_Y(D) = \mathcal{L}_1^{\otimes n} = \mathcal{L}_2^{\otimes n}$$

for some line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(Y)$. Canonically associated to such data, there exists two simple n-cyclic covers

$$\pi_1\colon X_1 \longrightarrow Y \quad \text{and} \quad \pi_2\colon X_2 \longrightarrow Y,$$

both branched over D. We want to provide conditions ensuring that the two compact complex manifolds underlying X_1 and X_2 are biholomorphic or anti-biholomorphic.

Following [**KK02**, Section3], let us denote by $\mathrm{Kl}(Y)$ the group of holomorphic and anti-holomorphic diffeomorphisms of Y. There is a short exact sequence

$$1 \longrightarrow \mathrm{Aut}(Y) \longrightarrow \mathrm{Kl}(Y) \longrightarrow H \longrightarrow 1,$$

where $H = \mathbb{Z}/2\mathbb{Z}$ or $H = 0$. To any anti-holomorphic element $\sigma \in \mathrm{Kl}(Y)$ we can associate a $\mathbb{C}$-antilinear map

$$\sigma^*\colon \mathbb{C}(Y) \longrightarrow \mathbb{C}(Y)$$

on the function field $\mathbb{C}(Y)$ by defining

$$(\sigma^* f)(x) := \overline{f(\sigma(x))}$$

for all $f \in \mathbb{C}(Y)$. That action extends the usual action of $\mathrm{Aut}(Y)$ on $\mathbb{C}(Y)$ in a natural way (note that in [**KK02**] the notation σ^* is used only for holomorphic maps, whereas for anti-holomorphic maps the corresponding notation is $\sigma^!$). We have $\sigma^{-1}(\mathrm{div}\,(f)) = \mathrm{div}\,(\sigma^* f)$, hence the action $\sigma^*\colon \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(Y)$ induces an action $\sigma^*\colon \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(Y)$, such that $\sigma^* K_Y = K_Y$. Moreover, the intersection numbers are also preserved by the action of any $\sigma \in \mathrm{Kl}(Y)$.

Example 4.1. Let $A_1 = V_1/\Lambda_1$ and $A_2 = V_2/\Lambda_2$ be two abelian varieties, and let $\sigma\colon A_2 \longrightarrow A_1$ be an anti-holomorphic homomorphism with analytic representation $\mathfrak{S}\colon V_2 \longrightarrow V_1$ and rational representation $\mathfrak{S}_\Lambda\colon \Lambda_2 \longrightarrow \Lambda_1$ (note that $\mathfrak{S}$ is a $\mathbb{C}$-antilinear map). Then, for any $\mathcal{L}(h, \chi) \in \mathrm{Pic}(A_1)$, we have the following analog of (11):

$$\sigma^*\mathcal{L}(h, \chi) = \mathcal{L}(\overline{\mathfrak{S}^* h}, \overline{\mathfrak{S}_\Lambda^* \chi}), \tag{25}$$

In fact, looking at the transition function of the anti-holomorphic line bundle $\mathcal{L}(\mathfrak{S}^* h, \mathfrak{S}_\Lambda^* \chi)$ we see that, in order to obtain a holomorphic one, we must take the conjugated hermitian form $\overline{\mathfrak{S}^* h}$ and the conjugated semicharacter $\overline{\mathfrak{S}_\Lambda^* \chi}$.

Let us now denote by $\mathrm{Kl}(Y, D)$ and $\mathrm{Aut}(Y, D)$ the subgroups of $\mathrm{Kl}(Y)$ and $\mathrm{Aut}(Y)$ given by diffeomorphisms such that $\sigma^* D = D$. Again, $\mathrm{Aut}(Y, D)$ is a normal subgroup of $\mathrm{Kl}(Y, D)$, of index 1 or 2.

Proposition 4.2.

(i) *Let $\sigma \in \mathrm{Kl}(Y, D)$ be such that $\sigma^*\mathcal{L}_2 \simeq \mathcal{L}_1$. Then there exists a diffeomorphism $\tilde{\sigma}\colon X_1 \longrightarrow X_2$ such that $\sigma \circ \pi_1 = \pi_2 \circ \tilde{\sigma}$. Moreover, $\tilde{\sigma}$ is holomorphic (respectively, anti-holomorphic) if and only if σ is so.*

(ii) *A diffeomorphism $\sigma \in \mathrm{Kl}(Y)$ lifts to X_i if and only if $\sigma \in \mathrm{Kl}(Y,\, D)$ and $\sigma^*\mathcal{L}_i \simeq \mathcal{L}_i$. Moreover, if σ lifts than it lifts in n different ways.*

Proof. Let us prove ***(i)***. Let $\mathbb{L}_1, \mathbb{L}_2$ be the total spaces of $\mathcal{L}_1, \mathcal{L}_2$ and let $p_1 \colon \mathbb{L}_1 \longrightarrow Y$, $p_2 \colon \mathbb{L}_2 \longrightarrow Y$ be the corresponding projections. Let $s \in H^0(Y,\, \mathcal{O}_Y(D))$ be a section vanishing exactly along D (if $D = 0$, we take for s the constant function 1). If $t_i \in H^0(\mathbb{L}_i,\, p_i^*\mathcal{L}_i)$ denotes the tautological section, by [**BHPVdV04**, I.17] it follows that global equations for X_1 and X_2, as analytic subvarieties of $\mathbb{L}_1$ and $\mathbb{L}_2$, are provided by

$$t_1^n - p_1^*s = 0 \quad \text{and} \quad t_2^n - p_2^*s = 0,$$

the covering maps π_1 and π_2 being induced by the restrictions of p_1 and p_2, respectively. Since $\sigma \in \mathrm{Kl}(Y,\, D)$, we have $\sigma^*s = \lambda s$ with $\lambda \in \mathbb{C}^*$. Moreover, $\sigma^*\mathcal{L}_2 \simeq \mathcal{L}_1$ implies that there exists $\tilde{\sigma} \colon \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ such that $p_2 \circ \tilde{\sigma} = \sigma \circ p_1$, hence

$$\tilde{\sigma}^*(p_2^*s) = p_1^*(\sigma^*s) = \lambda\, p_1^*s.$$

Moreover, we have $\tilde{\sigma}^*t_2 = \mu t_1$, with $\mu \in \mathbb{C}^*$. Up to rescaling t_2 by a constant factor we can assume $\mu = \sqrt[n]{\lambda}$, so that

$$\tilde{\sigma}^*(t_2^n - p_2^*s) = \lambda(t_1^n - p_1^*s).$$

This means that $\tilde{\sigma} \colon \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ restricts to a diffeomorphism $\tilde{\sigma} \colon X_1 \longrightarrow X_2$, which is compatible with the two covering maps π_1 and π_2. By construction, such a diffeomorphism is holomorphic (respectively, anti-holomorphic) if and only if σ is so.
The part ***(ii)*** follows from part ***(i)***, setting $\mathcal{L}_1 = \mathcal{L}_2$, so that $X_1 = X_2$. The existence of n different choices for the lifting of σ is a consequence of the fact that there are n different choices for $\sqrt[n]{\lambda}$. □

In the case of double covers induced by the Albanese map, we have the following converse of Proposition 4.2, ***(i)***.

Proposition 4.3. *Set $n = 2$, let $Y = A$ be an abelian variety and assume that the double cover $\pi_i \colon X_i \longrightarrow A$ is the Albanese map of X_i, for $i = 1,\, 2$. If there is a holomorphic (respectively, anti-holomorphic) diffeomorphism $\tilde{\sigma} \colon X_1 \longrightarrow X_2$, then there exists a holomorphic (respectively, anti-holomorphic) diffeomorphism $\sigma \in \mathrm{Kl}(A,\, D)$ such that $\sigma^*\mathcal{L}_2 \simeq \mathcal{L}_1$.*

Proof. We first assume that $\tilde{\sigma}$ is holomorphic. By the universal property of the Albanese map the morphism $\pi_2 \circ \tilde{\sigma} \colon X_1 \longrightarrow A$ factors through π_1, in other words there exists $\sigma \colon A \longrightarrow A$ such that $\sigma \circ \pi_1 = \pi_2 \circ \tilde{\sigma}$. The map σ is an isomorphism because $\tilde{\sigma}$ is an isomorphism, then it sends the branch locus of π_1 to the branch locus of π_2, or equivalently $\sigma^*D = D$. Finally, looking at the direct image of the structural sheaf $\mathcal{O}_{X_1}$ we get

$$\begin{aligned} (\sigma \circ \pi_1)_*\mathcal{O}_{X_1} &= (\pi_2 \circ \tilde{\sigma})_*\mathcal{O}_{X_1}, \quad &&\text{that is} \\ \sigma_*(\mathcal{O}_A \oplus \mathcal{L}_1^{-1}) &= \pi_{2*}(\tilde{\sigma}_*\mathcal{O}_{X_1}), \quad &&\text{that is} \\ \mathcal{O}_A \oplus (\sigma_*\mathcal{L}_1^{-1}) &= \mathcal{O}_A \oplus \mathcal{L}_2^{-1}. \end{aligned}$$

By [**Ati56**] a direct sum decomposition of a vector bundle into irreducible subbundles is unique up to isomorphisms, so we obtain $\sigma_*\mathcal{L}_1^{-1} \simeq \mathcal{L}_2^{-1}$. By using the projection formula and dualizing we infer $\mathcal{L}_1 = \sigma^*\mathcal{L}_2$, as desired. If $\tilde{\sigma}$ is anti-holomorphic, it suffices to apply the same proof to the holomorphic diffeomorphism which is complex-conjugated to it. □

Summing up, Propositions 4.2 and 4.3 together imply

Corollary 4.4. *With the same assumption as in Proposition* (4.3)*, there exists a holomorphic (respectively, anti-holomorphic) diffeomorphism $\tilde{\sigma} \colon X_1 \longrightarrow X_2$ if and only if there exists a holomorphic (respectively, anti-holomorphic) element $\sigma \in \mathrm{Kl}(A,\, D)$ such that $\sigma^*\mathcal{L}_2 = \mathcal{L}_1$.*

4.2 The uniqueness of the abelian surface A

We follow the notation of Section 3.3. If χ_i is any non-trivial element of the group $\varSigma = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ for the sake of brevity we will write A_i, D_{A_i} and $f_i \colon A_i \longrightarrow A'$ instead of A_{χ_i}, $D_{A_{\chi_i}}$ and $f_{\chi_i} \colon A_{\chi_i} \longrightarrow A'$, respectively. We will also denote by $\mathcal{L}_i \in \operatorname{Pic}^0(A')$ the 2-torsion line bundle corresponding to χ_i, so that $f_{i*}\mathcal{O}_{A_i} = \mathcal{O}_{A'} \oplus \mathcal{L}_i^{-1}$.

Proposition 4.5. *The abelian surfaces* A_1, A_2, A_3 *are pairwise isomorphic. More precisely, for all* $i, j \in \{1, 2, 3\}$ *there exists an isomorphism* $\tilde{\gamma}_{ij} \colon A_j \longrightarrow A_i$ *such that* $\tilde{\gamma}_{ij}^* D_{A_i} = D_{A_j}$.

Proof. By Proposition 4.2 it suffices to prove that there exists an automorphism $\gamma_{ij} \in \operatorname{Aut}(A', D_{A'})$ such that $\gamma_{ij}^* \mathcal{L}_i = \mathcal{L}_j$. Consider the linear automorphism $\gamma \colon V \longrightarrow V$ whose action on the standard basis is $\gamma(e_1) = -\zeta e_1$, $\gamma(e_2) = e_1 + e_2$. It preserves the lattice $\Lambda_{A'}$, in fact we have

$$\gamma(\lambda_1) = e_1 - \lambda_1, \quad \gamma(\lambda_2) = \lambda_1 + \lambda_2, \quad \gamma(e_1) = -\lambda_1, \quad \gamma(e_2) = e_1 + e_2, \tag{26}$$

so it descends to an automorphism of A' that we still denote by $\gamma \colon A' \longrightarrow A'$. An easy calculation shows that γ is an element of order 3 in $\operatorname{Aut}(A', D_{A'})$, so it induces by pull-back an action of $\langle \gamma \rangle \simeq \mathbb{Z}/3\mathbb{Z}$ on $\operatorname{NS}(A')$. Such an action is obtained by composing a character $\chi \colon \Lambda_{A'} \longrightarrow \{\pm 1\}$ with (26), and it is straightforward to check that it restricts to an action on the subgroup $\varSigma$ (defined in (22)), namely the one generated by the cyclic permutation $(\chi_1 \; \chi_3 \; \chi_2)$. This shows that $\langle \gamma \rangle$ acts transitively on the non-trivial characters of $\varSigma$. Since the action of γ on the characters χ_i corresponds to the pullback action on the corresponding 2-torsion divisors $\mathcal{L}_i$, by setting $\gamma_{13} = \gamma_{21} = \gamma_{32} = \gamma$ and $\gamma_{31} = \gamma_{12} = \gamma_{23} = \gamma^2$ we obtain $\gamma_{ij}^* \mathcal{L}_i = \mathcal{L}_j$, as desired. ☐

4.3 A rigidity result for surfaces of type II

Let us first recall the notions of deformation equivalence and global rigidity, ([**Cat13**, Section 1]).

Definition 4.6.

- Two complex surfaces S_1, S_2 are said to be *direct deformation equivalent* if there is a proper holomorphic submersion with connected fibres $f \colon \mathcal{Y} \longrightarrow \mathbb{D}$, where $\mathcal{Y}$ is a complex manifold and $\mathbb{D} \subset \mathbb{C}$ is the unit disk, and moreover there are two fibres of f biholomorphic to S_1 and S_2, respectively;
- two complex surfaces S_1, S_2 are said to be *deformation equivalent* if they belong to the same deformation equivalence class, where by deformation equivalence we mean the equivalence relation generated by direct deformation equivalence;
- a complex surface S is called *globally rigid* if its deformation equivalence class consists of S only, i.e. if every surface which is deformation equivalent to S is actually isomorphic to S.

The following result is a characterization of the equianharmonic product, that can be found in [**KH05**, Proposition 5].

Proposition 4.7. *Let* A' *be an abelian surface containing four elliptic curves, that intersect pairwise at the origin* o' *and not elsewhere. Then* A' *is isomorphic to the equianharmonic product* $E' \times E'$ *and, up to the action of* $\operatorname{Aut}(A')$*, the four curves are* E_1', E_2', E_3', E_4'.

A more conceptual proof of Proposition 4.7, exploiting some results of Shioda and Mitani on abelian surfaces with maximal Picard number, can be found in [**Aid**]. Using Proposition 4.7 we obtain:

Theorem 4.8. *Let* S *be a surface with* $p_g(S) = q(S) = 2$, $K_S^2 = 8$ *and Albanese map* $\alpha \colon S \longrightarrow A$ *of degree* 2*. If* S *belongs to type* II*, then the pair* (A, D_A) *is isomorphic to an étale double cover of the pair* $(A', D_{A'})$*, where* A' *is the equianharmonic product and* $D_{A'} = E_1' + E_2' + E_3' + E_4'$*. In particular, all surfaces of type* II *arise as in Proposition* 3.10*. Finally, all surfaces of type* II *are globally rigid.*

Proof. Let us consider the Stein factorization $\alpha_X \colon X \longrightarrow A$ of the Albanese map $\alpha \colon S \longrightarrow A$; then α_X is a finite double cover branched over D_A.

By Proposition 1.6 we have $D_A = E_1 + E_2 + E_3 + E_4$, where the E_i are four elliptic curves intersecting pairwise transversally at two points p_1, p_2 and not elsewhere. Up to a translation, we may assume that p_1 coincides with the origin of $o \in A$. Then $p_2 = a$, where a is a non-zero, 2-torsion point of A (in fact the E_i are subgroups of A, so the same is true for their intersection $\{o, a\}$).

If we consider the abelian surface $A' := A/\langle a \rangle$, then the projection $f \colon A \longrightarrow A'$ is an isogeny of degree 2. Moreover, setting $E'_i := f(E_i)$, we see that $E'_i, \ldots, E'_4$ are four elliptic curves intersecting pairwise transversally at the origin $o' \in A'$ and not elsewhere. Then the claim about A' and $D_{A'}$ follows from Proposition 4.7.

Since there are finitely many possibilities for both the double covers $f \colon A \longrightarrow A'$ and $a_X \colon X \longrightarrow A$, it follows that S, being the minimal desingularization of X, belongs to only finitely many isomorphism classes. This implies that S is globally rigid, because by [**Gie77**] the moduli space of surfaces of general type is separated. □

Remark 4.9. It is straightforward to check that the class of the point $\zeta e_1 = (\zeta, 0)$ in $A = V/\Lambda_A$ is contained in all the curves $E_1, \ldots, E_4$, so we obtain $a = \zeta e_1 + \Lambda_A$.

4.4 The groups $\mathrm{Aut}(A,\, D_A)$ and $\mathrm{Kl}(A,\, D_A)$

In the sequel we will write $A := V/\Lambda_A$ in order to denote any of the pairwise isomorphic abelian surfaces A_1, A_2, A_3, see Proposition 4.5. Choosing for instance $\Lambda_A = \ker \chi_1$, by (24) we have

$$\Lambda_A = \mathbb{Z}\mathbf{e}_1 \oplus \mathbb{Z}\mathbf{e}_2 \oplus \mathbb{Z}\mathbf{e}_3 \oplus \mathbb{Z}\mathbf{e}_4, \tag{27}$$

where

$$\mathbf{e}_1 := e_1, \quad \mathbf{e}_2 := \lambda_1 + e_2, \quad \mathbf{e}_3 := \lambda_2 + e_2, \quad \mathbf{e}_4 := 2e_2. \tag{28}$$

Note that $\mathbf{e}_1 = (1,\, 0)$ and $\mathbf{e}_2 = (\zeta,\, 1)$ form a basis for V. Let Γ_ζ and E' be as in (12) and set

$$E'' := \mathbb{C}/\Gamma_{2\zeta}, \quad \Gamma_{2\zeta} := \mathbb{Z}[2\zeta].$$

The next result implies that A is actually isomorphic to the product $E'' \times E'$.

Lemma 4.10. *We have* $\Lambda_A = \Gamma_{2\zeta}\, \mathbf{e}_1 \oplus \Gamma_\zeta\, \mathbf{e}_2$.

Proof. We check that the base-change matrix between the $\mathbb{Q}$-bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ and $\mathbf{e}_1, \mathbf{e}_2, 2\zeta\mathbf{e}_1, \zeta\mathbf{e}_2$ of $H_1(A, \mathbb{Q})$ is in $\mathsf{GL}(4, \mathbb{Z})$. □

We will use Lemma 4.10 in order to describe the groups $\mathrm{Aut}(A,\, D_A)$ and $\mathrm{Kl}(A,\, D_A)$. In what follows, we will identify an automorphism $A \longrightarrow A$ with the matrix of its analytic representation $V \longrightarrow V$ with respect to the standard basis $\{e_1,\, e_2\}$. Moreover, we will write $\tau = \tau_a \colon A \longrightarrow A$ for the translation by the 2-torsion point $a = \zeta e_1 + \Lambda_A$ defined in the proof of Theorem 4.8; see also Remark 4.9.

Proposition 4.11. *The following holds.*

- **(*a*)** *We have*

$$\mathrm{Aut}(A,\, D_A) = \mathrm{Aut}_0(A,\, D_A) \times \mathbb{Z}/2\mathbb{Z}, \tag{29}$$

where $\mathbb{Z}/2\mathbb{Z}$ *is generated by the translation* τ*, whereas* $\mathrm{Aut}_0(A,\, D_A)$ *is the subgroup of group automorphisms of* A *generated by the elements*

$$g_2 = \begin{pmatrix} \zeta & -1 \\ \zeta & -\zeta \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & \zeta - 1 \\ 1 - \zeta & \zeta - 1 \end{pmatrix}. \tag{30}$$

As an abstract group, $\mathrm{Aut}_0(A,\, D_A)$ *is isomorphic to* $\mathsf{SL}(2, \mathbb{F}_3)$ *; it has order* 24.

(b) *The group* $\mathrm{Kl}(A,\, D_A)$ *is generated by* $\mathrm{Aut}(A,\, D_A)$ *together with the anti-holomorphic involution* $\sigma\colon A \longrightarrow A$ *induced by the* $\mathbb{C}$*-antilinear involution of* V *given by*

$$(z_1,\, z_2) \mapsto ((\zeta - 1)\bar{z}_2,\, (\zeta - 1)\bar{z}_1). \tag{31}$$

Furthermore, the two involutions τ *and* σ *commute, so that we can write*

$$\mathrm{Kl}(A,\, D_A) = \mathrm{Kl}_0(A,\, D_A) \times \mathbb{Z}/2\mathbb{Z}, \tag{32}$$

where $\mathrm{Kl}_0(A,\, D_A)$ *contains* $\mathrm{Aut}_0(A,\, D_A)$ *as a subgroup of index* 2.

Proof. ***(a)*** Let us work using the basis $\{\mathbf{e}_1,\, \mathbf{e}_2\}$ of V defined in (28). With respect to this basis, using (14) we see that the four elliptic curves $E_1, \ldots, E_4$ have tangent spaces

$$\begin{aligned} V_1 &= \mathrm{span}(\mathbf{e}_1), & V_2 &= \mathrm{span}(-\zeta\mathbf{e}_1 + \mathbf{e}_2), \\ V_3 &= \mathrm{span}((1-\zeta)\mathbf{e}_1 + \mathbf{e}_2), & V_4 &= \mathrm{span}((1-2\zeta)\mathbf{e}_1 + \mathbf{e}_2). \end{aligned}$$

Then, up to the translation τ, we are looking at the subgroup $\mathrm{Aut}_0(A,\, D_A)$ of the group automorphisms of A whose elements have matrix representation preserving the set of four points $\mathscr{P} = \{P_1,\, P_2,\, P_3,\, P_4\} \subset \mathbb{P}^1$, where

$$P_1 = [1 : 0], \quad P_2 = [-\zeta : 1], \quad P_3 = [1-\zeta : 1], \quad P_4 = [1-2\zeta : 1].$$

The cross ratio $(P_1,\, P_2,\, P_3,\, P_4)$ equals ζ^{-1}, hence $\mathscr{P}$ is an equianharmonic quadruple and so the group $\mathrm{PGL}(2,\, \mathbb{C})$ acts on it as the alternating group A_4, see [**Mai07**, p. 7]. Such a group can be presented as

$$\mathsf{A}_4 = \langle \alpha,\, \beta \mid \alpha^2 = \beta^3 = (\alpha\beta)^3 = 1 \rangle, \tag{33}$$

where $\alpha = (12)(34)$ and $\beta = (123)$, so we need to find matrices $\tilde{g}_2,\, \tilde{g}_3 \in \mathrm{GL}(2,\, \mathbb{C})$, acting as an isomorphism on the lattice Λ_A and inducing the permutations $(P_1\, P_2)(P_3\, P_4)$ and $(P_1\, P_2\, P_3)$ on $\mathscr{P}$, respectively. Using Lemma 4.10, we see that $\tilde{g} \in \mathrm{GL}(2,\, \mathbb{C})$ preserves Λ_A if and only if it has the form

$$\tilde{g} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \text{with } a_{11} \in \Gamma_{2\zeta},\ \ a_{12} \in 2\Gamma_\zeta,\ \ a_{21},\, a_{22} \in \Gamma_\zeta,$$

and its determinant belongs to the group of units of Γ_ζ, namely $\{\pm 1,\, \pm\zeta,\, \pm\zeta^2\}$.

Now an elementary computation yields the matrices $\pm\tilde{g}_2,\, \pm\tilde{g}_3$, where

$$\tilde{g}_2 = \begin{pmatrix} 1 & 2\zeta - 2 \\ \zeta & -1 \end{pmatrix}, \quad \tilde{g}_3 = \begin{pmatrix} -1 & 0 \\ 1-\zeta & \zeta \end{pmatrix},$$

and from this we can obtain the matrix representations $g_2,\, g_3$ of our automorphisms in the basis $\{e_1,\, e_2\}$ of V by taking

$$g_i = N\tilde{g}_i N^{-1}, \quad \text{with } N = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}.$$

This gives (30). Setting $h = -I_2$ and lifting the presentation (33) we get the presentation

$$\mathrm{Aut}_0(A,\, D_A) = \langle g_2,\, g_3,\, h \mid h^2 = 1,\ g_2^2 = g_3^3 = (g_2 g_3)^3 = h \rangle, \tag{34}$$

showing the isomorphism $\mathrm{Aut}_0(A,\, D_A) \simeq \mathsf{SL}(2,\, \mathbb{F}_3)$.

Finally, a standard computation shows that τ commutes with both g_2 and g_3. Since $\mathrm{Aut}(A)$ is the semidirect product of the translation group of A by the group automorphisms, it follows that $\mathrm{Aut}(A,\, D_A)$ is the direct product of $\langle\tau\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ by $\mathrm{Aut}_0(A,\, D_A)$, hence we obtain (29).

(b) Let us consider the $\mathbb{C}$-antilinear map $V \longrightarrow V$

$$(z_1,\, z_2) \mapsto (\bar{z}_1,\, (\zeta - 1)\bar{z}_1 + \zeta\bar{z}_2),$$

expressed with respect to the basis $\{\mathbf{e}_1,\, \mathbf{e}_2\}$. It preserves the lattice Λ_A and so it defines an anti-holomorphic involution $\sigma\colon A \longrightarrow A$, inducing the transposition $(P_1\, P_2)$ on the set $\mathscr{P} = \{P_1,\, P_2,\, P_3,\, P_4\}$. Since $\mathrm{Aut}(A, D)$ has index at most 2 in $\mathrm{Kl}(A,\, D_A)$, it follows that $\mathrm{Aut}(A,\, D_A)$ and σ generate $\mathrm{Kl}(A,\, D_A)$. Moreover, a change of coordinates allows us to come back to the basis $\{e_1,\, e_2\}$ and to obtain the expression of σ given in (31). The subgroup $\mathrm{Kl}_0(A,\, D_A)$ generated by $g_2,\, g_3$ and the involution σ contains $\mathrm{Aut}_0(A,\, D_A)$ as a subgroup of index 2; a straightforward computation now shows that $[\tau,\, \sigma] = 0$, so (32) follows from (29) and the proof is complete. □

Remark 4.12. The proof of Proposition 4.11 also shows that there are central extensions

$$\begin{aligned} 1 \longrightarrow \langle -I_2 \rangle \longrightarrow \mathrm{Aut}_0(A,\, D_A) \longrightarrow \mathsf{A}_4 \longrightarrow 0, \\ 1 \longrightarrow \langle -I_2 \rangle \longrightarrow \mathrm{Kl}_0(A,\, D_A) \longrightarrow \mathsf{S}_4 \longrightarrow 0 \end{aligned}$$

such that $-I_2 = [g_2,\, g_3]^2$. In fact, $\mathrm{KL}_0(A,\, D_A)$ is isomorphic as an abstract group to $\mathsf{GL}(2,\, \mathbb{F}_3)$, see the proof of Theorem 4.13.

4.5 The action of $\mathrm{Kl}(A,\, D_A)$ on the square roots of $\mathcal{O}_A(D_A)$

The main result of this subsection is the following

Theorem 4.13. *Up to isomorphism, there exist exactly two surfaces of type II. These surfaces* S_1, S_2 *have conjugated complex structures, in other words there exists an anti-holomorphic diffeomorphism* $S_1 \longrightarrow S_2$.

In order to prove this result, we must study the action of the groups $\mathrm{Aut}(A,\, D_A)$ and $\mathrm{Kl}(A,\, D_A)$ on the sixteen square roots $\mathcal{L}_1, \ldots, \mathcal{L}_{16}$ of the line bundle $\mathcal{O}_A(D_A) \in \mathrm{Pic}(A)$. The Appell-Humbert data of such square roots are described in the following

Proposition 4.14. *For* $k \in \{1, \ldots, 16\}$, *we have* $\mathcal{L}_k = \mathcal{L}\left(\frac{1}{2} h_A,\, \psi_k\right)$ *where*

- $h_A \colon V \times V \longrightarrow \mathbb{C}$ *is the hermitian form on* V *whose associated alternating form* $\mathrm{Im}\, h_A$ *assumes the following values at the generators* $\mathbf{e}_1, \ldots, \mathbf{e}_4$ *of* Λ_A :

$(\cdot,\, \cdot)$	$(\mathbf{e}_1,\, \mathbf{e}_2)$	$(\mathbf{e}_1,\, \mathbf{e}_3)$	$(\mathbf{e}_1,\, \mathbf{e}_4)$	$(\mathbf{e}_2,\, \mathbf{e}_3)$	$(\mathbf{e}_2,\, \mathbf{e}_4)$	$(\mathbf{e}_3,\, \mathbf{e}_4)$
$\mathrm{Im}\, h_A(\cdot,\, \cdot)$	-4	0	-2	-6	-4	6

Table 2: The values of $\mathrm{Im}\, h_A$ at the generators of Λ_A

- *Using the notation* $\psi_k := (\psi_k(\mathbf{e}_1),\, \psi_k(\mathbf{e}_2),\, \psi_k(\mathbf{e}_3),\, \psi_k(\mathbf{e}_4))$, *the semicharacters* $\psi_k \colon \Lambda_A \longrightarrow \mathbb{C}^*$ *are as follows*:

$$\begin{array}{lll} \psi_1 := (i,\, 1,\, i,\, 1), & & \\ \psi_2 := (-i,\, -1,\, i, -1), & \psi_3 := (i,\, -1,\, -i,\, 1), & \psi_4 := (-i,\, 1,\, -i,\, -1), \\ \psi_5 := (i,\, 1,\, -i,\, 1), & \psi_6 := (-i,\, 1,\, i,\, 1), & \psi_7 := (-i,\, 1,\, -i,\, 1), \\ \psi_8 := (i,\, 1,\, i,\, -1), & \psi_9 := (i,\, -1,\, i,\, -1), & \psi_{10} := (-i,\, 1,\, i,\, -1), \\ \psi_{11} := (i,\, 1,\, -i,\, -1), & \psi_{12} := (-i,\, -1,\, -i,\, 1), & \psi_{13} := (i,\, -1,\, i,\, 1), \\ \psi_{14} := (i,\, -1,\, -i,\, -1), & \psi_{15} := (-i,\, -1,\, i,\, 1), & \psi_{16} := (-i,\, -1,\, -i,\, -1). \end{array} \tag{35}$$

Proof. Let us consider the double cover $f \colon A \longrightarrow A'$. If the hermitian form $h \colon V \times V \longrightarrow \mathbb{C}$ and the semicharacter $\chi_{D_{A'}} \colon \Lambda_{A'} \longrightarrow \{\pm 1\}$ are as in Proposition 3.5 and Table 1, then we have $\mathcal{O}_A(D_A) = \mathcal{L}(h_A,\, \chi_{D_A})$, where $h_A = f^* h, \quad \chi_{D_A} = f^* \chi_{D_{A'}}$. From this, using (28) we can compute the values of the alternating form $\mathrm{Im}\, h_A$ and of the semicharacter χ_{D_A} at $\mathbf{e}_1, \ldots,\, \mathbf{e}_4$, obtaining Table 2 and

$$\chi_{D_A} = (\chi_{D_A}(\mathbf{e}_1),\, \chi_{D_A}(\mathbf{e}_2),\, \chi_{D_A}(\mathbf{e}_3),\, \chi_{D_A}(\mathbf{e}_4)) = (-1,\, 1,\, -1,\, 1).$$

Then, setting $\mathcal{L}_k = \mathcal{L}(h_k,\, \psi_k)$, the equality $\mathcal{L}_k^{\otimes 2} = \mathcal{O}_A(D_A)$ implies

$$2h_k = h_A, \quad \psi_k^2 = \chi_{D_A},$$

hence $h_k = \frac{1}{2} h_A$ for all k. Moreover we can set $\psi_1 = (i,\, 1,\, i,\, 1)$, whereas the remaining 15 semicharacters ψ_k are obtained by multiplying ψ_1 by the 15 non-trivial characters $\Lambda_A \longrightarrow \{\pm 1\}$. $\square$

The hermitian form h_A is $\mathrm{Kl}(A,\, D_A)$-invariant (according with the fact that the divisor D_A is so), hence the action of $\mathrm{Kl}(A,\, D_A)$ on the set $\{\mathcal{L}_1, \ldots, \mathcal{L}_{16}\}$ is completely determined by its permutation action on the set $\{\psi_1, \ldots, \psi_{16}\}$, namely

$$\varrho\colon \mathrm{Kl}(A, D_A) \longrightarrow \mathsf{Perm}(\psi_1, \ldots, \psi_{16}), \quad \varrho(g)(\psi_k) := g^* \psi_k.$$

After identifying the group $\mathsf{Perm}(\psi_1, \ldots, \psi_{16})$ with the symmetric group S_{16} on the symbols $\{1, \ldots, 16\}$, we get the following

Proposition 4.15. *With the notation of* Proposition 4.11*, we have*

$$\begin{aligned} \varrho(g_2) &= (1\ 3\ 7\ 12)(2\ 9\ 14\ 16)(3\ 5\ 15\ 6)(4\ 11\ 8\ 10), \\ \varrho(g_3) &= (1\ 13\ 5\ 7\ 12\ 6)(2\ 4\ 11\ 14\ 8\ 10)(3\ 15)(9\ 16), \\ \varrho(-I) = \varrho(g_2^2) = \varrho(g_3^3) = \varrho(\tau) &= (1\ 7)(2\ 14)(3\ 15)(4\ 8)(5\ 6)(9\ 16)(10\ 11)(12\ 13), \\ \varrho(\sigma) &= (1\ 14)(2\ 7)(3\ 16)(4\ 5)(6\ 8)(9\ 15)(10\ 12)(11\ 13). \end{aligned}$$

Proof. Using the explicit expressions given in Proposition 4.11, by a standard computation we can check that g_2, g_3, τ and σ send the ordered basis $\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4\}$ of Λ_A to the bases

$$\begin{aligned} &\{\mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4,\ -2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4,\ -\mathbf{e}_1 - \mathbf{e}_2 - 2\mathbf{e}_3 + 2\mathbf{e}_4,\ -2\mathbf{e}_1 - 2\mathbf{e}_3 + \mathbf{e}_4\}, \\ &\{-\mathbf{e}_3 + \mathbf{e}_4,\ -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4,\ -2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - 2\mathbf{e}_4,\ -2\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4\}, \\ &\{-\mathbf{e}_3\ - \mathbf{e}_2,\ -\mathbf{e}_3,\ -\mathbf{e}_4\}, \\ &\{\mathbf{e}_3 - \mathbf{e}_4,\ -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4,\ -\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_4,\ -2\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_4\}, \end{aligned} \tag{36}$$

respectively. For any $g \in \mathrm{Aut}(A,\, D_A)$, calling $G_\Lambda \colon \Lambda_A \longrightarrow \Lambda_A$ the corresponding rational representation we have $\varrho(g)(\psi_k) = \psi_k \circ G_\Lambda$, whereas $\varrho(\sigma)(\psi_k) = \overline{\psi_k \circ \mathfrak{S}_\Lambda}$ (see (25)). Then another long but straightforward calculation using (35) and (36) concludes the proof. $\square$

We are now ready to give the

Proof of Theorem 4.13. Since any surface S of type II is a double cover $f\colon S \longrightarrow A$, branched over D_A, by Proposition 4.3 it follows that the number of surfaces of type II up to isomorphisms (respectively, up to holomorphic and anti-holomorphic diffeomorphisms) equals the number of orbits for the permutation action of $\mathrm{Aut}(A,\, D_A)$ (respectively, of $\mathrm{Kl}(A,\, D_A)$) on the set $\{\mathcal{L}_1, \ldots, \mathcal{L}_{16}\}$ of the sixteen square roots of $\mathcal{O}_A(D_A)$. We have seen that such an action is determined by the permutation action on the set of sixteen semicharacters $\{\psi_1, \ldots, \psi_{16}\}$, so we only have to compute the number of orbits for the subgroup of S_{16} whose generators are described in Proposition 4.15. This can be done by hand, but it is easier to write a short script using the Computer Algebra System `GAP4` ([**GAP16**]):

```
g2:=(1, 13, 7, 12)(2, 9, 14, 16)(3, 5, 15, 6)(4, 11, 8, 10);;
g3:=(1, 13, 5, 7, 12, 6)(2, 4, 11, 14, 8, 10)(3, 15)(9, 16);;
sigma:=(1, 14)(2, 7)(3, 16)(4, 5)(6, 8)(9, 15)(10, 12)(11, 13);;
Aut:=Group(g2, g3);;
Kl:=Group(g2, g3, sigma);;
StructureDescription(Aut) = "SL(2,3)";
OrbitsPerms(Aut, [ 1 .. 16 ] ) =
[ [ 1, 7, 12, 13, 3, 15, 5, 6 ], [ 2, 14, 16, 9, 10, 11, 4, 8 ] ];
StructureDescription(Kl) = "GL(2,3)";
OrbitsPerms(Kl, [1..16] ) =
[ [ 1, 7, 12, 13, 15, 3, 6, 5, 14, 2, 10, 11, 9, 16, 8, 4 ] ];
```

The output shows that ϱ induces an embedding of $\mathrm{Aut}_0(A,\, D_A)$ in S_{16}, and that the corresponding permutation subgroup has precisely two orbits. Therefore there are exactly two surfaces S_1, S_2 of type II, up to isomorphisms. Furthermore, ϱ induces an embedding of $\mathrm{Kl}_0(A,\, D_A)$ in S_{16}, and the corresponding permutation subgroup has only one orbit. This means that there exists an anti-holomorphic diffeomorphism $S_1 \longrightarrow S_2$, hence these surfaces are not isomorphic, but they have conjugated complex structure. $\square$

Let us finally show that surfaces of type II are not uniformized by the bidisk (unlike surfaces of type I, see Corollary 2.7).

Proposition 4.16. *Let S be a surface of type II and $\widetilde{S} \longrightarrow S$ its universal cover. Then $\widetilde{S}$ is* not *biholomorphic to $\mathbb{H} \times \mathbb{H}$.*

Proof. Looking at diagram (2) in Section 1, we see that, in case II, the map $\varphi \colon B \longrightarrow A$ is the blow-up of A at the two quadruple points p_1, p_2 of the curve D_A and that $\bar{S} = S$. Moreover, considering $\beta \colon S \longrightarrow B$ we have

$$\beta^* D_B = C_1 + C_2 + C_3 + C_4,$$

where the C_i are (pairwise disjoint) elliptic curves with $C_i^2 = -1$. The embedding $C_i \longrightarrow S$, composed with the universal cover $\mathbb{C} \longrightarrow C_i$, gives a non-constant holomorphic map $\mathbb{C} \longrightarrow S$, that in turn lifts to a non-constant holomorphic map $\mathbb{C} \longrightarrow \widetilde{S}$. If $\widetilde{S}$ were isomorphic to $\mathbb{H} \times \mathbb{H}$, projecting onto one of the two factors we would obtain a non-constant holomorphic map $\mathbb{C} \longrightarrow \mathbb{H}$, whose existence would contradict Liouville's theorem because $\mathbb{H}$ is biholomorphic to the bounded domain $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$. □

4.6 Concluding remarks

Remark 4.17. In the argument in the proof of Proposition 4.16, we could have used one of the elliptic curves Z_1, Z_2 instead of the C_i (see Remark 1.4).

Remark 4.18. Denoting by χ_{top} the topological Euler number, we have

$$\left(K_S + \sum_{i=1}^{4} C_i\right)^2 = 12 = 3\,\chi_{\text{top}}\left(S - \sum_{i=1}^{4} C_i\right)$$

and

$$\left(K_S + \sum_{i=1}^{2} Z_i\right)^2 = 12 = 3\,\chi_{\text{top}}\left(S - \sum_{i=1}^{2} Z_i\right).$$

This implies that the open surfaces $S - \sum_{i=1}^{4} C_i$ and $S - \sum_{i=1}^{2} Z_i$ both have the structure of a complex ball-quotient, see [**Rou14**] for references and further details.

Remark 4.19. The two non-isomorphic surfaces of type II exhibit a new occurrence of the so-called `Diff`$\not\Rightarrow$`Def` phenomenon, meaning that their diffeomorphism type does not determine their deformation class. In fact, they are (anti-holomorphically) diffeomorphic, but not deformation equivalent since they are rigid. See [**Man01**], [**KK02**], [**Cat03**], [**CW07**] for further examples of this situation.

Remark 4.20. It is possible to give a different geometric construction of the abelian surfaces A', A and of the divisor D_A as follows. Unfortunately, at present we do not know how to recover the 2-divisibility of the curve D_A in $\mathrm{Pic}(A)$ by using this alternative approach.

Let F_1, F_2, F_3 and G_1, G_2, G_3 be general fibres of the two rulings $f, g \colon \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, respectively; then the two reducible divisors $F_1 + F_2 + F_3$ and $G_1 + G_2 + G_3$ meet at nine distinct points. Consider three of these points, say p_1, p_2, p_3, with the property that each F_i and each G_i contain exactly one of them. Then there exists precisely one (smooth) curve C_1 of bidegree (1, 1) passing through p_1, p_2, p_3. Similarly, if we choose three other points $q_1, q_2, q_3 \notin \{p_1, p_2, p_3\}$ with the same property, there exist a unique curve C_2 of bidegree (1, 1) passing through q_1, q_2, q_3. The curves C_1 and C_2 meet at two points, say r_1, r_2, different from the points p_i and q_i.

Let us call F_4 and G_4 the fibres of f and g passing through one of these two points, say r_1. Then the reducible curve B of bidegree (4, 4) defined as

$$B = F_1 + \cdots + F_4 + G_1 + \cdots + G_4$$

has sixteen ordinary double points as only singularities, and the double cover $\phi \colon Q' \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ branched over B gives a singular Kummer surface Q'; let us write A' for the associate abelian surface. We can easily show that

$$\phi^* C_1 = C_{11} + C_{12}, \quad \phi^* C_2 = C_{21} + C_{22},$$

where all the C_{ij} are smooth and irreducible. Moreover we see that C_{11} and C_{22} intersect at exactly one point, which is a node of Q'. Writing

$$\phi^* F_4 = 2\widehat{F}_4, \quad \phi^* G_4 = 2\widehat{G}_4,$$

we see that the rational curves C_{11}, C_{22}, $\widehat{F}_4$, $\widehat{G}_4$ meet at one node of Q' and that each of them contains precisely four nodes of Q'. Hence the pullback of these curves via the double cover $A' \longrightarrow Q'$ yields four elliptic curves in A' intersecting pairwise and transversally at a single point.

Let us choose now i, j, h, $k \in \{1, 2, 3, 4\}$, with $i \neq j$ and $h \neq k$, and consider the eight nodes of B different from the nodes of the curve $H = F_i + F_j + G_h + G_k$. The 2-divisibility of H in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ implies that the corresponding set Ξ of eight nodes in the Kummer surface Q' is 2-divisible, so we can consider the double cover $Q \longrightarrow Q'$ branched over Ξ. The surface Q is again a singular Kummer surface and, calling A the abelian surface associate with Q, we obtain a degree 2 isogeny $A \longrightarrow A'$. We can choose (in three different ways) i, j, h, k so that each of the four curves C_{11}, C_{22}, $\widehat{F}_4$ and $\widehat{G}_4$ contains exactly two nodes of Ξ. Therefore we obtain four rational curves in Q, all passing through two of the nodes of Q and containing four nodes each. This in turn gives four elliptic curves in A meeting at two common points and not elsewhere, and the union of these curves is the desired divisor D_A.

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