Integral Calculus On Quantum Exterior Algebras

Serkan Karaçuha

MATHEMATICS DEPARTMENT, UNIVERSITY OF PORTO

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Differential Graded Algebra

DGA over A

\[ \Omega(A) := \bigoplus_{n \geq 0} \Omega^n(A) \text{ such that } \Omega^0(A) = A, \]
**Differential Graded Algebra**

**DGA over A**

- \( \Omega(A) := \bigoplus_{n \geq 0} \Omega^n(A) \) such that \( \Omega^0(A) = A \),
- \( d : \Omega^k(A) \to \Omega^{k+1}(A) \) linear map of degree one, which satisfies
Differential Graded Algebra

**DGA over A**

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  (i) $d^2 = 0$,
Differential Graded Algebra

DGA over $A$

- $\Omega(A) := \bigoplus_{n \geq 0} \Omega^n(A)$ such that $\Omega^0(A) = A$,
- $d : \Omega^k(A) \rightarrow \Omega^{k+1}(A)$ linear map of degree one, which satisfies
  (i) $d^2 = 0$,
  (ii) $d(\omega \nu) = d(\omega) \nu + (-1)^k \omega d(\nu)$, $\forall \omega \in \Omega^k(A), \nu \in \Omega(A)$.

FODC

The pair $(\Omega^1(A), d)$ is referred to as a first order differential calculus on $A$. 
Non-commutative Connection

Connection

Given an \( FODC (\Omega^1(A), d) \) over \( A \) and a right \( A \)-module \( M \), a linear map \( \nabla^0 : M \rightarrow M \otimes_A \Omega^1(A) \) satisfying

\[
\nabla^0(ma) = \nabla^0(m)a + m \otimes_A d(a)
\]

is called a \textit{connection} in \( M \).
Non-commutative Hom-connection

**Hom-connection (T. Brzezinski)**

A right *hom-connection* w.r.t. a *dga* $(\Omega(A), d)$ over $A$, is a pair $(M, \nabla_0)$, where $M$ is a right $A$-module and

$$\nabla_0 : \text{Hom}_A(\Omega^1(A), M) \rightarrow M$$

is a linear mapping s.t.

$$\nabla_0(fa) = \nabla_0(f)a + f(d(a)) \quad \forall a \in A, f \in \text{Hom}_A(\Omega^1(A), M)$$
Non-commutative Hom-connection

Hom-connection

Any hom-connection $(M, \nabla_0)$ can be extended to maps $\nabla_m : \text{Hom}_A(\Omega^{m+1}(A), M) \rightarrow \text{Hom}_A(\Omega^m(A), M)$ by

$$\nabla_m(f)(\omega) = \nabla_0(f \omega) + (-1)^{m+1} f(d\omega),$$

$\forall f \in \text{Hom}_A(\Omega^{m+1}(A), M), \omega \in \Omega^m(A)$.

The vector space $\bigoplus_{m \geq 0} \text{Hom}_A(\Omega^m(A), M)$ is a right $\Omega(A)$-module by the action

$$f \omega(\nu) := f(\omega \nu)$$

where $\omega \in \Omega^m(A), f \in \text{Hom}_A(\Omega^{m+n}(A), M), \nu \in \Omega^n(A)$. 

Non-commutative Hom-connection

Curvature

The right $A$-module homomorphism $F := \nabla_0 \circ \nabla_1$ is called the \textit{curvature} of the hom-connection $(M, \nabla_0)$. 
Curvature

- The right $A$-module homomorphism $F := \nabla_0 \circ \nabla_1$ is called the *curvature* of the hom-connection $(M, \nabla_0)$.
- $(M, \nabla_0)$ is said to be *flat* provided that $F = 0$. We can associate a chain complex $(\bigoplus_{m \geq 0} \text{Hom}_A(\Omega^m(A), M), \nabla)$ to a flat hom-connection $(M, \nabla_0)$. 
Curvature

- The right $A$-module homomorphism $F := \nabla_0 \circ \nabla_1$ is called the *curvature* of the hom-connection $(M, \nabla_0)$.
- $(M, \nabla_0)$ is said to be *flat* provided that $F = 0$. We can associate a chain complex $(\bigoplus_{m \geq 0} \text{Hom}_A(\Omega^m(A), M), \nabla)$ to a flat hom-connection $(M, \nabla_0)$.
- We set $M = A$ and $\Omega^*_m = \text{Hom}_A(\Omega^m(A), A)$ to get the following *complex of integral forms* on $A$

\[ \cdots \xrightarrow{\nabla_3} \Omega^*_3 \xrightarrow{\nabla_2} \Omega^*_2 \xrightarrow{\nabla_1} \Omega^*_1 \xrightarrow{\nabla_0} A \]
Twisted Multi-Derivations and Hom-Connections
Twisted Multi-Derivation

Right Twisted Multi-Derivation

- By a *right twisted multi-derivation* in an algebra $A$ we mean a pair $(\partial, \sigma)$, where $\sigma : A \to M_n(A)$ is an algebra homomorphism and $\partial : A \to A^n$ is a $k$-linear map such that, for all $a, b \in A$,

  $$\partial(ab) = \partial(a)\sigma(b) + a\partial(b).$$

- $A^n$ is understood as an $(A-M_n(A))$-bimodule. If we write $\sigma(a) = (\sigma_{ij}(a))_{i,j=1}^n$ and $\partial(a) = (\partial_i(a))_{i=1}^n$ for an element $a \in A$, then we obtain the following $n$ equations

  $$\partial_i(ab) = \sum_j \partial_j(a)\sigma_{ji}(b) + a\partial_i(b), \text{ for } i = 1, 2, \ldots, n.$$
Right Twisted Multi-Derivation

Given a right twisted multi-derivation \((\partial, \sigma)\) on \(A\) we construct a FODC on the free left \(A\)-module

\[
\Omega^1 = A^n = \bigoplus_{i=1}^{n} A\omega_i
\]

with basis \(\omega_1, \ldots, \omega_n\) which becomes an \(A\)-bimodule by

\[
\omega_i a = \sum_{j=1}^{n} \sigma_{ij}(a)\omega_j \quad \text{for all } 1 \leq i \leq n.
\]

The map

\[
d : A \rightarrow \Omega^1, \quad a \mapsto \sum_{i=1}^{n} \partial_i(a)\omega_i
\]

is a derivation and makes \((\Omega^1, d)\) a FODC on \(A\).
Free Right Twisted Multi-Derivation

- A map $\sigma : A \rightarrow M_n(A)$ can be equivalently understood as an element of $M_n(\text{End}_k(A))$. We write $\bullet$ for the product in $M_n(\text{End}_k(A))$, $\mathbb{I}$ for the unit in $M_n(\text{End}_k(A))$ and $\sigma^T$ for the transpose of $\sigma$. 

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Integral Calculus On Quantum Exterior Algebras
Twisted Multi-Derivation

Free Right Twisted Multi-Derivation

- A map $\sigma : A \rightarrow M_n(A)$ can be equivalently understood as an element of $M_n(\text{End}_k(A))$. We write $\bullet$ for the product in $M_n(\text{End}_k(A))$, $\mathbb{I}$ for the unit in $M_n(\text{End}_k(A))$ and $\sigma^T$ for the transpose of $\sigma$.

- We call a right twisted multi-derivation $(\partial, \sigma)$ free, provided there exist algebra maps $\bar{\sigma} : A \rightarrow M_n(A)$ and $\hat{\sigma} : A \rightarrow M_n(A)$ such that

$$
\bar{\sigma} \bullet \sigma^T = \mathbb{I}, \quad \sigma^T \bullet \bar{\sigma} = \mathbb{I},
$$

$$
\hat{\sigma} \bullet \bar{\sigma}^T = \mathbb{I}, \quad \bar{\sigma}^T \bullet \hat{\sigma} = \mathbb{I}.
$$

We denote it by $(\partial, \sigma; \bar{\sigma}, \hat{\sigma})$. 
Twisted Multi-Derivation

**Proposition (Brzezinski, El Kaoutit, Lomp)**

An upper-triangular right twisted multi-derivation \((\partial, \sigma)\) is free if and only if \(\sigma_{11}, \ldots, \sigma_{nn}\) are automorphisms of \(A\).

**Theorem (Brzezinski, El Kaoutit, Lomp)**

For any free right twisted multi-derivation \((\partial, \sigma; \bar{\sigma}, \hat{\sigma})\) on \(A\) with the induced FODC \((\Omega^1(A), d)\) with generators \(\omega_i\), the map

\[
\nabla : \text{Hom}_A(\Omega^1(A), A) \to A, \quad f \mapsto \sum_i \partial^\sigma_i (f (\omega_i))
\]

is a hom-connection, where \(\partial^\sigma_i := \sum_{j, k} \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}\), for each \(i = 1, 2, \ldots, n\).
Differential Calculi on Quantum Exterior Algebras
DC On Quantum Exterior Algebras

Quantum Exterior Algebras

- We call an $n \times n$-matrix $Q = (q_{ij})$ over $K$ a *multiplicatively antisymmetric matrix* if $q_{ij}q_{ji} = q_{ii} = 1$ for all $i, j$. 
We call an $n \times n$-matrix $Q = (q_{ij})$ over $K$ a \textit{multiplicatively antisymmetric matrix} if $q_{ij}q_{ji} = q_{ii} = 1$ for all $i, j$.

Let $M$ be an $A$-bimodule which is free as left and right $A$-module with basis $\{\omega_1, \ldots, \omega_n\}$. The \textit{quantum exterior algebra} of $M$ over $A$ w.r.t. a multiplicatively antisymmetric matrix $Q$ is defined as

$$\bigwedge^Q(M) := T_A(M)/\langle \omega_i \otimes \omega_j + q_{ij}\omega_j \otimes \omega_i, \omega_i \otimes \omega_i \mid i, j = 1, \ldots, n \rangle.$$
The quantum exterior algebra is a free left and right $A$-module of rank $2^n$ with basis
\[ \{1\} \cup \{\omega_{i_1} \wedge \omega_{i_2} \cdots \wedge \omega_{i_k} \mid i_1 < i_2 < \cdots < i_k, 1 \leq k \leq n\}. \]

Question
When a bimodule derivation $d : A \rightarrow M$ can be extended to an exterior derivation $d : \bigwedge^Q(M) \rightarrow \bigwedge^Q(M)$ of the quantum exterior algebra?
Proposition

Let \((\partial, \sigma)\) be a right twisted multi-derivation of rank \(n\) on a \(k\)-algebra \(A\) with associated FODC \((\Omega^1(A), d)\). Let \(Q\) be an \(n \times n\) multiplicatively antisymmetric matrix over \(k\). Then \(d : A \to \Omega^1(A)\) can be extended to make \(\Omega = \bigwedge^Q(\Omega^1(A))\) an \(n\)-dimensional differential calculus on \(A\) with \(d(\omega_i) = 0\) for all \(i = 1, \ldots, n\) if and only if

\[
\partial_i \partial_j = q_{ji} \partial_j \partial_i \quad \text{and} \quad \partial_i \sigma_{kj} - q_{ji} \partial_j \sigma_{ki} = q_{ji} \sigma_{kj} \partial_i - \sigma_{ki} \partial_j, \quad \forall i < j, \forall k.
\]
Theorem (Karaçuha, Lomp)

Let \((\partial, \sigma)\) be a free upper triangular twisted multi-derivation on \(A\) with associated FODC \((\Omega^1, d)\). Suppose that \(d : A \to \Omega^1\) can be extended to an \(n\)-dimensional differential calculus \((\Omega, d)\) where \(\Omega = \wedge^{Q}(\Omega^1)\) is the quantum exterior algebra of \(\Omega^1\) for some matrix \(Q\). Then the following hold:

1. \(\bar{\omega}a = \det(\sigma)\bar{\omega}\), for all \(a \in A\), where \(\det \sigma = \sigma_{11} \circ \cdots \circ \sigma_{nn}\).
2. The maps \(\Theta_m : \Omega^m \to Hom_A(\Omega^{n-m}(A), A)\) given by \(\Theta_m(\nu) = (-1)^{m(n-1)} \beta \nu\) for all \(\nu \in \Omega^m\) are isomorphisms of right \(A\)-modules.
3. Moreover if
Twisted Multi-Derivations and Hom-Connections
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Coordinate Ring of Quantum n-space

Theorem (Karačuha, Lomp)

$$\partial_i^\sigma = \left( \prod_j q_{ij} \right) \det(\sigma)^{-1} \partial_i \det(\sigma) \quad \forall i = 1, \ldots, n$$

holds, then $\Theta = (\Theta_m)_{m=0}^n$ is a chain map, that is, the following diagram commutes:

$$\begin{array}{cccccccc}
A & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1} & \xrightarrow{d} & \Omega^n \\
\Theta_0 & \downarrow & \Theta_1 & \downarrow & & \Theta_{n-1} & \downarrow & \Theta_n & \\
\Omega^*_n & \xrightarrow{\nabla_{n-1}} & \Omega^*_{n-1} & \xrightarrow{\nabla_{n-2}} & \cdots & \xrightarrow{\nabla_1} & \Omega^*_1 & \xrightarrow{\nabla_0} & A
\end{array}$$
Multivariate Quantum Polynomials
Skew Derivations

We have a diagonal bimodule structure on $\Omega^1 = A^n$ if $\sigma_{ij} = \delta_{ij}\sigma_i$ for all $i, j$ where $\sigma_1, \ldots, \sigma_n$ are endomorphisms of $A$. Moreover if $\sigma$ is diagonal and $(\partial, \sigma)$ is a right twisted multi-derivation on $A$, then the maps $\partial_i$, for all $a, b \in A$ and $i$, satisfy

$$\partial_i(ab) = \partial_i(a)\sigma_i(b) + a\partial_i(b)$$

which are then called right $\sigma_i$-skew derivations.
Conversely, given any right $\sigma_i$-derivations $\partial_i$ on $A$, for $i = 1, \ldots, n$ one can form a corresponding diagonal twisted multi-derivation $(\partial, \sigma)$ on $A$. Such diagonal twisted multi-derivation $(\partial, \sigma)$ is free if and only if the maps $\sigma_1, \ldots, \sigma_n$ are automorphisms. The associated $A$-bimodule structure on $\Omega^1 = A^n$ with left $A$-basis $\omega_1, \ldots, \omega_n$ is given by

$$\omega_i a = \sigma_i(a)\omega_i$$

for all $i$ and $a \in A$. 
Corollary

Let $A$ be an algebra over a field $K$, $\sigma_i$ automorphisms and $\partial_i$ right $\sigma_i$-skew derivations on $A$, for $i = 1, \ldots, n$ and let $(\Omega^1, d)$ be the associated FODC on $A$.

1. The derivation $d : A \rightarrow \Omega^1$ extends to an $n$-dimensional differential calculus $(\Omega, d)$ where $\Omega = \bigwedge^Q(\Omega^1)$ is the quantum exterior algebra with respect to some $Q$ such that $d(\omega_i) = 0$ for all $i = 1, \ldots, n$ if and only if

$$\partial_i \sigma_j = q_{ji} \sigma_j \partial_i \quad \text{and} \quad \partial_i \partial_j = q_{ji} \partial_j \partial_i \quad \forall i < j.$$  

2. If $\partial_i \sigma_j = q_{ji} \sigma_j \partial_i$ for all $i, j$ and $\partial_i \partial_j = q_{ji} \partial_j \partial_i$ for all $i < j$, then the de Rham and the integral complexes on $A$ are isomorphic relative to $(\Omega, d)$. 

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Integral Calculus On Quantum Exterior Algebras
Multivariate Quantum Polynomials

Quantum Polynomial Algebra

- $Q = (q_{ij})$ is a $n \times n$ multiplicatively antisymmetric matrix over a field $k$. The multivariate quantum polynomial algebra with respect to $Q$ is defined as:

$$A = O_Q(k^n) := k\langle x_1, \ldots, x_n \rangle / \langle x_i x_j - q_{ij} x_j x_i \mid 1 \leq i, j \leq n \rangle.$$

- For two generic monomials $x^\alpha$ and $x^\beta$ with $\alpha, \beta \in \mathbb{N}^n$ one has

$$x^\alpha x^\beta = \left( \prod_{1 \leq j < i \leq n} q_{ij}^{\alpha_i \beta_j} \right) x^{\alpha + \beta} = \mu(\alpha, \beta) x^{\alpha + \beta},$$

where $\mu(\alpha, \beta) = \prod_{1 \leq j < i \leq n} q_{ij}^{\alpha_i \beta_j}$. 
Multivariate Quantum Polynomials

Quantum Polynomial Algebra

- We define automorphisms $\sigma_1, \ldots, \sigma_n$ and right $\sigma_i$-derivations of $A$ as follows: For a generic monomial $x^\alpha$ with $\alpha \in \mathbb{N}^n$ one sets

$$\sigma_i(x^\alpha) := \lambda_i(\alpha)x^\alpha \quad \text{and} \quad \partial_i(x^\alpha) := \alpha_i \delta_i(\alpha)x^{\alpha - \epsilon^i}$$

where $\lambda_i(\alpha) = \prod_{j=1}^{n} q_{ij}^{\alpha_j}$, $\delta_i(\alpha) = \prod_{i<j} q_{ij}^{\alpha_j}$ and $\epsilon^i \in \mathbb{N}^n$ such that $\epsilon_i^j = \delta_{ij}$.

Then by the previous Corollary we get
Corollary

Let $A = \mathcal{O}_Q(K^n)$ be the multivariate quantum polynomial algebra and let $\Omega = \wedge^Q(\Omega^1)$ be the associated quantum exterior algebra. Then the derivation $d : A \rightarrow \Omega^1$ with $d(x^\alpha) = \sum_{i=1}^n \partial_i(x^\alpha)\omega_i$ makes $\Omega$ into a differential calculus such that the de Rham complex and the integral complex are isomorphic.
Manin’s Quantum n-space
Manin’s Quantum n-space

Let $q \in k \setminus \{0\}$. For the matrix $Q = (q_{ij})$ with $q_{ij} = q$ and $q_{ji} = q^{-1}$ for all $i < j$ and $q_{ii} = 1$, the algebra $\mathcal{O}_Q(k^n)$ is called the coordinate ring of quantum n-space or Manin’s quantum n-space and will be denoted by $A = k_q[x_1, \ldots, x_n]$. We have the following defining relations of the algebra $A$

$$x_ix_j = qx_jx_i, \quad i < j.$$
Manin’s Quantum n-space

For $\alpha \in \mathbb{N}^n$ and $1 \leq i \leq n$ we have:

$$\lambda_i(\alpha)x^\alpha x_i = x^{\alpha + \epsilon_i} = \overline{\lambda}_i(\alpha)x_i x^\alpha,$$

where

$$\lambda_i(\alpha) = \prod_{i<j} q^{\alpha_j} \quad \text{and} \quad \overline{\lambda}_i(\alpha) = \prod_{j<i} q^{-\alpha_j}.$$

More generally

$$x^{\alpha + \beta} = \left( \prod_{j=1}^{n-1} \lambda_j(\alpha) \beta_j \right) x^\alpha x^\beta = \prod_{1 \leq s < j \leq n} q^{\alpha_s \beta_j} x^\alpha x^\beta.$$
An FODC On Manin’s Quantum n-space

We take the following two-parameter first order differential calculus $\Omega^1$ which is freely generated by $\{\omega_1, \ldots, \omega_n\}$ over $A$ subject to the relations

$$\omega_i x_j = q x_j \omega_i + (p - 1) x_i \omega_j, \quad i < j,$$

$$\omega_i x_i = p x_i \omega_i,$$

$$\omega_j x_i = pq^{-1} x_i \omega_j, \quad i < j.$$

Set $\pi_i(\alpha) = \prod_{s < i} p^{\alpha_s}, \ i = 1, \ldots, n$ for the following lemma.
Lemma

For $\alpha \in \mathbb{N}^n$ the entries of the matrix $\sigma(x^\alpha)$ are as follows:

$\sigma_{ij}(x^\alpha) = 0$ for $i > j$ and

$$\sigma_{ij}(x^\alpha) = \eta_{ij}(\alpha)x^{\alpha+\epsilon^i-\epsilon^j};$$

$$\eta_{ij}(\alpha) = \begin{cases} 
\pi_j(\alpha)\bar{\lambda}_i(\alpha)\lambda_j(\alpha)(p^{\alpha_j} - 1) & \text{for } i < j, \\
\pi_i(\alpha)\bar{\lambda}_i(\alpha)\lambda_i(\alpha)p^{\alpha_i} & \text{for } i = j.
\end{cases}$$
The Derivation

We have a derivation \( d : K_q[x_1, \ldots, x_n] \to \Omega^1 \) such that \( d(x_i) = \omega_i \) for all \( i \). For any \( \alpha \in \mathbb{N}^n \) we set

\[
d(x^\alpha) = \sum_{i=1}^n \partial_i(x^\alpha)\omega_i
\]

where

\[
\partial_i(x^\alpha) = \delta_i(\alpha)x^{\alpha - \varepsilon^i} ; \quad \delta_i(\alpha) = \pi_i(\alpha)\lambda_i(\alpha)\frac{p^{\alpha_i} - 1}{p - 1}.
\]

for all \( i = 1, \ldots, n \). Also for \( i, k \) we have:

\[
\delta_i(\alpha) = q^{\mp 1}\delta_i(\alpha \pm \varepsilon^k), \quad \text{if } i < k ; \quad \delta_i(\alpha) = p^{\mp 1}\delta_i(\alpha \pm \varepsilon^k), \quad \text{if } i > k.
\]
Lemma

The pair \((\partial, \sigma)\) is a right twisted multi-derivation of \(K_q[x_1, \ldots, x_n]\) satisfying the equations ensuring the extension of the FODC to make \(\Omega = \bigwedge^Q(\Omega^1)\) an \(n\)-dimensional DC with respect to the multiplicatively antisymmetric matrix \(Q'\) whose entries are \(Q'_{ij} = p^{-1}q\) for \(i < j\).
### Lemma cont.

In particular

\[ \partial_i \partial_j = pq^{-1} \partial_j \partial_i, \quad \forall i < j \]

holds as well as for all \( i, k, j \):

\begin{align*}
\partial_i \sigma_{kj} &= pq^{-1} \sigma_{kj} \partial_i, \quad i < k \leq j \\
\partial_i \sigma_{kj} &= pq^{-1} \partial_j \sigma_{ki}, \quad k < i < j \\
\sigma_{ki} \partial_j &= pq^{-1} \sigma_{kj} \partial_i, \quad k < i < j \\
\partial_i \sigma_{ij} - pq^{-1} \partial_j \sigma_{ii} &= pq^{-1} \sigma_{ij} \partial_i - \sigma_{ii} \partial_j, \quad i < j
\end{align*}
Theorem (Karaçuha, Lomp)

The derivation \( d : K_q[x_1, \ldots, x_n] \to \Omega^1 \) extends to a differential calculus \( \bigwedge^{p-1}q(\Omega^1) \) on \( K_q[x_1, \ldots, x_n] \). Furthermore the de Rham and the integral complex associated to the differential calculus \( (\bigwedge^{p-1}q(\Omega^1), d) \) are isomorphic.
Thanks!