Poisson and Hochschild cohomology and the semiclassical limit

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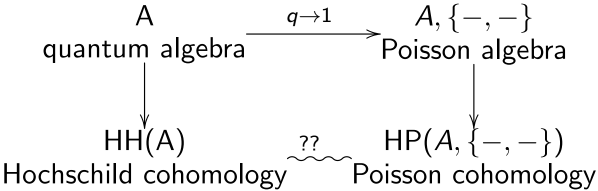
http://arxiv.org/abs/1304.6003

Motivation


A
quantum algebra
$q \to 1$
$A, \{-,-\}$
Poisson algebra
HH(A)
Hochschild cohomology
??
$\mathrm{HP}(A, \{-,-\})$
Poisson cohomology

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Example

A is a Poisson module over itself.

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The **Poisson enveloping algebra** $P(A)$ is an associative algebra such that

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Example

$A = k[x,y], \{x,y\} = xy$. *Then* $P(A)$ *is generated by* $x, y, \Omega(x), \Omega(y)$ *subject to*

$$\begin{array}{ll} xy = yx & \Omega(y)x - x\Omega(y) = -xy \\ x\Omega(x) = \Omega(x)x & \Omega(x)y - y\Omega(x) = xy \\ y\Omega(y) = \Omega(y)y & \Omega(x)\Omega(y) - \Omega(y)\Omega(x) = x\Omega(y) + y\Omega(x) \end{array}$$

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If A is a polynomial Poisson algebra, we define the **Poisson cohomology** by

$$\mathrm{HP}^*(A) := \mathrm{Ext}^*_{P(A)}(A, A)$$

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Otherwise, this may not be equivalent to the standard definition.

Graded Poisson algebras

Definition

A graded Poisson algebra is a graded commutative algebra B with a bilinear bracket $\{-,-\}$ such that

$$\{a,b\} = (-1)^{|a||b|+1}\{b,a\}$$
$$\{a,bc\} = \{a,b\}c + (-1)^{|a||b|}b\{a,c\}$$
$$(-1)^{|a||c|}\{a,\{b,c\}\} + (-1)^{|a||b|}\{b,\{c,a\}\} + (-1)^{|b||c|}\{c,\{a,b\}\} = 0$$

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A **quadratic** Poisson bracket on $A = k[x_1,\ldots,x_n]$ is one such that $\{x_i,x_j\}$ is a quadratic polynomial for all i,j. It is determined by a map

$$\beta : \Lambda^2(\langle x_1,\ldots,x_n\rangle) \to S^2(\langle x_1,\ldots,x_n\rangle).$$

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If $B = \Lambda(U)$ is an exterior algebra, a quadratic graded Poisson bracket is determined by a map $\gamma : S^2(U) \to \Lambda^2(U)$.

Homological aspects

Theorem

If $A = k[V]$ is a quadratic Poisson algebra with Poisson bracket determined by

$$\beta : S^2(V) \to \Lambda^2(V)$$

then $P(A)$ is Koszul and $P(A)^! = \mathrm{Ext}^*_{P(A)}(k, k)$ is isomorphic to the graded Poisson enveloping algebra $P_{gr}(B)$ of the exterior algebra B on V^* with graded Poisson bracket determined by

$$\beta^* : \Lambda^2(V^*) \to S^2(V^*).$$

Semiclassical limits

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Write $\bar{x}$ for the image of x in A. Then

$$\{\bar{x}, \bar{y}\} := \overline{\beta(x, y)}$$

is a Poisson bracket on A. This Poisson algebra is called the **semiclassical limit** of $\mathcal{A}$.

Example semiclassical limit

Example

$$\mathcal{A} = \frac{\mathscr{R}\langle x, y\rangle}{(xy - qyx)}$$

Then $xy - yx = (q-1)yx$, *so the semiclassical limit is* $A = k[\bar{x}, \bar{y}]$ *with Poisson bracket*

$$\{\bar{x}, \bar{y}\} = \bar{x}\bar{y}$$

Observation

Let $\mathrm{B} = k_q[\mathrm{x}, \mathrm{y}] = \frac{k(q)\langle \mathrm{x},\mathrm{y}\rangle}{\mathrm{xy}-q\mathrm{yx}}$. Then the enveloping algebra $\mathrm{B}^e = \mathrm{B} \otimes \mathrm{B}^{\mathrm{op}}$ is a deformation of $P(\bar{A})$.

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Write

$$\mathrm{X} = \mathrm{x} \otimes 1 \qquad \mathrm{Y} = \mathrm{y} \otimes 1$$
$$O(\mathrm{x}) = (\mathrm{x} \otimes 1 - 1 \otimes \mathrm{x})/(q-1) \qquad O(\mathrm{y}) = (\mathrm{y} \otimes 1 - 1 \otimes \mathrm{y})/(q-1)$$

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(elements of B^e). Then:

$$\begin{aligned} \mathrm{XY} &= q\mathrm{YX} & O(\mathrm{y})\mathrm{X} - \mathrm{X}O(\mathrm{y}) &= -\mathrm{XY} \\ O(\mathrm{x})\mathrm{X} &= \mathrm{X}O(\mathrm{x}) & O(\mathrm{x})\mathrm{Y} - \mathrm{Y}O(\mathrm{x}) &= \mathrm{XY} \\ O(\mathrm{y})\mathrm{Y} &= \mathrm{Y}O(\mathrm{y}) & qO(\mathrm{x})O(\mathrm{y}) - O(\mathrm{y})O(\mathrm{x}) &= \mathrm{X}O(\mathrm{y}) + \mathrm{Y}O(\mathrm{x}) \end{aligned}$$

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Compare with the relations for $P(A)$ given earlier:

$$\begin{array}{ll} xy = yx & \Omega(y)x - x\Omega(y) = -xy \\ \Omega(x)x = x\Omega(x) & \Omega(x)y - y\Omega(x) = xy \\ \Omega(y)y = y\Omega(y) & \Omega(x)\Omega(y) - \Omega(y)\Omega(x) = x\Omega(y) + y\Omega(x) \end{array}$$

Deformation theorem

Theorem

Let B *be a* $\mathbb{Z}_{\geq 0}$*-graded* $k(q)$*-algebra generated by elements* $\mathrm{x}_1, \ldots, \mathrm{x}_n$ *of degree one, with a PBW basis of polynomial type, having a* $\mathscr{R}$*-form* $\mathcal{B}$ *with a semiclassical limit* A *polynomial on the images of the* x_i*. Then there is an* $\mathscr{R}$*-subalgebra* $\mathcal{S}$ *of* B^e *such that*

$$\mathcal{S} \otimes_{\mathscr{R}} k(q) \cong \mathrm{B}^e \qquad \mathcal{S} \otimes_{\mathscr{R}} k \cong P(A)$$

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In this situation we say "B^e is a q-deformation of $P(A)$", and "$\mathcal{S}$ is an $\mathscr{R}$-form of B^e".

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Algebras B satisfying these conditions will be called $\mathcal{P}$-algebras. They include quantum affine spaces and coordinate rings of quantum matrices.

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$\mathrm{HP}(A)$ is the special case $\Lambda = P(A)$, $M = A$.

Koszul algebras

A **Koszul module** over a $\mathbb{Z}_{\geq 0}$-graded connected algebra Λ is one admitting a graded free resolution

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Such an algebra has the form $\Lambda = T(V)/(R)$, $R \subseteq V \otimes V$, and $\mathrm{Ext}^*_\Lambda(k,k) \cong T(V^*)/(R^\perp) = \Lambda^!$.

Cohomology: Poisson vs. Hochschild

Recall that our goal was to compare

$$\mathrm{HH}(\mathrm{B}) = \mathrm{Ext}_{\mathrm{B}^e}(\mathrm{B}, \mathrm{B}) \quad \text{and} \quad \mathrm{HP}(A) = \mathrm{Ext}_{P(A)}(A, A)$$

where B is a $\mathcal{P}$-algebra with semiclassical limit A.

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Equally, $P(A)$ is a Koszul algebra and A is a Koszul $P(A)$-module, and $A \otimes A^!$ is a DGA whose homology is $\mathrm{HP}(A)$.

Deformation theorem II

Theorem

If B *is a* $\mathcal{P}$*-algebra with semiclassical limit* A *then* $\mathrm{B} \otimes \mathrm{B}^!$ *has a* $\mathscr{R}$*-sub-DGA* $\mathcal{C}$ *such that*

$$\mathcal{C} \otimes_{\mathscr{R}} k(q) \cong \mathrm{B} \otimes \mathrm{B}^! \quad \mathcal{C} \otimes_{\mathscr{R}} k \cong A \otimes A^!$$

as DGAs.

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Corollary

If $H^*(\mathcal{C})$ *has no* $(q-1)$*-torsion then* $\mathrm{HH}^*(\mathrm{B})$ *is a q-deformation of* $\mathrm{HP}^*(A)$.