On an open problem of M. Cohen concerning smash products

Christian Lomp

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The problem

Conditions on $A$

Conditions on $H$

Conclusion

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The Question

Question (Miriam Cohen, 1985)

*Is the smash product* \( A \# H \) *semiprime in case* \( H \) *is a semisimple Hopf algebra acting on a semiprime algebra* \( A \) ?
Semiprime rings

Definition

A ring $R$ is semiprime if its prime radical $P(R)$ is zero.
Semiprime rings

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\[ P(R) = \bigcap \{ P \trianglelefteq R \mid P \text{ is prime} \}. \]
Semiprime rings

**Definition**

A ring $R$ is semiprime if its prime radical $P(R)$ is zero.

$$P(R) = \bigcap \left\{ P \trianglelefteq R \mid P \text{ is prime} \right\}.$$ 

$R$ is semiprime iff it has no nilpotent $\neq 0$ ideals.
Preliminaries

For this talk $\text{char}(k) = 0$.

**Definition**

A coalgebra $(C, \Delta, \epsilon)$ is a $k$-vector space with comultiplication:

$$
\xymatrix{ 
C \ar[r]^-\Delta & C \otimes C \\
C \otimes C \ar[u]^-\Delta & C \otimes C \otimes C \ar[l]_-{1 \otimes \Delta} \\
& C \otimes C \otimes k \\
& C \otimes k}
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**Definition**

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\[
\begin{align*}
C & \xrightarrow{\Delta} C \otimes C \\
C & \otimes C & k & \otimes C & C & \otimes C & \otimes C & k & \otimes k
\end{align*}
\]

**Definition**

Given a coalgebra $(C, \Delta, \epsilon)$ and an algebra $(A, m, 1)$ makes $\text{Hom}(C, A)$ into an algebra with convolution product:

\[
(f \ast g)(c) = m \circ (f \otimes g)\Delta(c)
\]
Hopf algebras

Definition
A Hopf algebra $H$ over $k$ is an algebra and a coalgebra $(H, \Delta, \epsilon)$ such that $\Delta$ and $\epsilon$ are algebra maps and $id_H$ has an inverse in $(\text{End}(H), \ast)$. 

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**Example**

The dual $H^* = \text{Hom}(H, k)$ of a finite dimensional Hopf algebra $H$ is a Hopf algebra.
Examples

Let $G$ be a group.
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**Example**

1. The group ring $H = k[G]$ is a Hopf algebra with

\[ \Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}. \]
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1. The group ring $H = k[G]$ is a Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}.$$

2. If $G$ is finite, then the dual group ring $H^* = \text{Hom}(k[G], k)$ with dual basis $\{p_g\}_{g \in G}$ is a Hopf algebra with

$$\Delta(p_g) = \sum_{h \in G} p_{gh^{-1}} \otimes p_h, \quad \epsilon(p_g) = \delta_{e,g}, \quad S(p_g) = p_{g^{-1}}.$$
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$$

**Definition**

A Hopf algebra isomorphic to one of these is called **trivial**.
Semisimple Hopf algebras

Theorem (Larson-Radford, 1988, char(k)=0)

The following are equivalent:

1. $H$ is a semisimple Hopf algebra;
2. $H^*$ is a semisimple Hopf algebra;
3. $S^2 = id$. 

Corollary

Let $H$ be any semisimple Hopf algebra.

1. $H$ commutative $\iff H \cong k\left[G\right]$.
2. $H^*$ commutative $\iff H \cong k\left[G\right]$. 

trivial = semisimple and (commutative or cocommutative)

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The problem

Semisimple Hopf algebras

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trivial = semisimple and (commutative or cocommutative)
A non-trivial semisimple Hopf algebra

Example (Fukuda, 1997)

\[ H_8 = \mathbb{C}[G][z; \sigma]/\langle z^2 - \frac{1}{2}(1 + x + y - xy) \rangle \]

is an 8-dimensional non-trivial semisimple Hopf algebra where \( G = C_2 \times C_2 \) with generators \( x, y \) and \( \sigma(x) = y, \sigma(y) = x \) and

\[ \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z). \]
Hopf module algebras

The category $H$-Mod of left $H$-modules is a tensor category

$$h \cdot (v \otimes w) = \Delta(h)(V \otimes W) = \sum_{(h)} (h_1 \cdot v) \otimes (h_2 \cdot w),$$

for all $h \in H, v \in V, w \in W$ for $V, W \in H$-Mod.
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Definition

An algebra $A$ in the category of left $H$-modules is called a (left) $H$-module algebra.
Hopf module algebras

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for all $h \in H$, $v \in V$, $w \in W$ for $V$, $W \in H$-Mod.

**Definition**

An algebra $A$ in the category of left $H$-modules is called a (left) $H$-module algebra.

That means $H$ acts on $A$ such that

$$h \cdot (ab) = \sum_{(h)} (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \epsilon(h)1_A.$$
Definition (Smash product)

The smash product of a Hopf algebra $H$ and a module algebra $A$ is defined on $A \# H := A \otimes H$ with multiplication:

$$(a \# h)(b \# g) = \sum_{(h)} a(h_1 \cdot b) \# h_2 g,$$

with identity $1_A \# 1_H$. 

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Definition (Invariants)

$A^H = \{ a \in A \mid h \cdot a = \epsilon(h)a \forall h \in H \}$. 
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Definition (Invariants)

$$A^H = \{a \in A \mid h \cdot a = \epsilon(h)a \ \forall h \in H\}.$$
Examples

**Example**

An action \( k[G] \otimes A \to A \) corresponds to the group homomorphism \( \varphi : G \to \text{Aut}(A) \) given by

\[
\varphi(g)(a) = g \cdot a.
\]

Moreover, \( A \# k[G] \) is the ordinary **skew group ring**.
Example

An action $k[G] \otimes A \to A$ corresponds to the group homomorphism $\varphi : G \to \text{Aut}(A)$ given by

$$\varphi(g)(a) = g \cdot a.$$ 

Moreover $A \# k[G]$ is the ordinary skew group ring.

Theorem (Fisher-Montgomery, 1978; Lorenz-Passman, 1980)

$A \# k[G]$ is semiprime for a finite group $G$ and $A$ semiprime.
Example

An action $k[G]^* \otimes A \to A$ corresponds to the grading $A = \bigoplus_{g \in G} A_g$ given by

$$A_g = \{ p_g \cdot a \in A \mid a \in A \}.$$ 

Moreover $A \# k[G]^*$ has been used in the study of group gradings.
Examples

Example

An action $k[G]^* \otimes A \to A$ corresponds to the grading $A = \bigoplus_{g \in G} A_g$ given by

$$A_g = \{ p_g \cdot a \in A \mid a \in A \}.$$

Moreover $A\# k[G]^*$ has been used in the study of group gradings.

Theorem (Cohen-Montgomery, 1984)

$A\# k[G]^*$ is semiprime for a finite group $G$ and $A$ semiprime.
The Question

Question (Miriam Cohen, 1985)

Is $A \# H$ semiprime if $H$ is a semisimple and $A$ semiprime?
A non-trivial action

\[ H_8 = \mathbb{C}[C_2 \times C_2][z; \sigma]/\langle z^2 - \frac{1}{2}(1 + x + y - xy) \rangle \]

**Example (Kirkman-Kuzmanovich-Zhang, 2009)**

\( H_8 \) acts on the quantum plane \( A = \mathbb{C}_q[u, v] \) with \( q^2 = -1 \) by

\[
\begin{align*}
x \cdot u &= -u, & y \cdot u &= u, & z \cdot u &= v \\
x \cdot v &= v, & y \cdot v &= -v, & z \cdot v &= u.
\end{align*}
\]

Note that \( z \cdot (uv) = -vu \neq vu = (z \cdot u)(z \cdot v) \).
The problem

1 Preliminaries
2 Hopf algebra actions

Conditions on A

2 Separable extensions
3 Finiteness conditions on A
4 Extending the Hopf-action
5 Commutativity
6 Necessary condition

Conditions on H

3 Semisolvable Hopf algebras
4 Drinfeld twists

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Semisimple Hopf algebras

**Theorem (Sweedler)**

The following are equivalent:

1. $H$ is a semisimple Hopf algebra;
2. $H$ is a separable $k$-algebra;
3. $k$ is a projective left $H$-module;
4. $\exists t \in H : \forall h \in H : ht = \epsilon(h)t \text{ and } \epsilon(t) = 1.$
Separable extensions

Definition (Hirata-Sugano, 1966)

A ring extension \( R \subseteq S \) is separable if \( \text{mult} : S \otimes_R S \to S \) splits as \( S \)-bimodule, i.e.

\[
\exists \gamma = \sum_{i=1}^{n} e_i \otimes f_i \in S \otimes_R S, \quad \text{mult}(\gamma) = 1 \text{ and } s\gamma = \gamma s \ \forall s \in S.
\]
Separable extensions

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\]

Corollary

\( H \) is semisimple iff \( A \subseteq A \# H \) is a separable extension for all \( A \).

Choose \( \gamma = \sum_{(t)} 1 \# t_1 \otimes_A 1 \# S(t_2) \in (A \# H) \otimes_A (A \# H) \).
Von Neumann regular algebras

Transfer of homological properties via separable extensions:

Theorem

Let $H$ be a semisimple Hopf algebra acting on $A$. 

Transfer of homological properties via separable extensions:

**Theorem**

*Let $H$ be a semisimple Hopf algebra acting on $A$.*

1. *If $A$ is von Neumann regular, i.e. all $A$-modules are flat, then $A\# H$ is von Neumann regular.*
Von Neumann regular algebras

Transfer of homological properties via separable extensions:

**Theorem**

Let $H$ be a semisimple Hopf algebra acting on $A$.

1. If $A$ is von Neumann regular, i.e. all $A$-modules are flat, then $A \# H$ is von Neumann regular.

2. If $A$ is semisimple Artinian, then $A \# H$ is semisimple Artinian.
Finiteness conditions

Question

Can one embed $A$ into another $H$-module algebra $Q$ such that

$A$ semiprime $\implies Q \# H$ semiprime $\implies A \# H$ semiprime?
Classical ring of quotient

Theorem (Skryabin-VanOystayen, 2006)

If A has a right Artinian classical ring of quotient Q, then any left H-action on A extends to Q.
The problem

Classical ring of quotient

Theorem (Skryabin-VanOystayen, 2006)

If $A$ has a right Artinian classical ring of quotient $Q$, then any left $H$-action on $A$ extends to $Q$.

Corollary

If $A$ is semiprime right Noetherian and $H$ semisimple, then $A\#H$ is semiprime.
The problem

Conditions on $A$

Conditions on $H$

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Classical ring of quotient

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If $A$ is semiprime right Noetherian and $H$ semisimple, then $A\# H$ is semiprime.

Example

$\mathbb{C}_q[u, v] \# H_8$ is semiprime for $q^2 = -1$. 
A is a left module algebra if and only if there exists an algebra homomorphism

\[ \rho : A \rightarrow \text{Hom}(H, A). \]
Idea of Skryabin-VanOystaeyen’s proof

1. A is a left module algebra if and only if there exists an algebra homomorphism

   \[ \rho : A \rightarrow \text{Hom}(H, A). \]

2. Reduction to finite dimensional coalgebras measuring A, i.e.

   \[ H = \sum \{ C_i \mid C_i \text{ is f.d. subcoalgebra of } H \}. \]
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1. $A$ is a left module algebra if and only if there exists an algebra homomorphism

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   $$H = \sum \{ C_i \mid C_i \text{ is f.d. subcoalgebra of } H \}.$$ 

3. For any $C$ the following diagram can be completed:

   $\begin{array}{ccc}
   A & \xrightarrow{\rho|_C} & \text{Hom}(C, A) \\
   \downarrow & & \downarrow \\
   Q \xrightarrow{\exists \rho'|_C} & \text{Hom}(C, Q)
   \end{array}$
Gabriel localization

**Question**

*When does the $H$-action extends to a localization $Q = A_\mathcal{F}$ with respect to a Gabriel filter?*
Gabriel localization

Question

When does the $H$-action extends to a localization $Q = A_{\mathcal{F}}$ with respect to a Gabriel filter?

Theorem (Montgomery, 1992; Selvan, 1994)

A sufficient condition for this to happen is that for any right ideal $I \in \mathcal{F}$ there exists an $H$-stable right ideal $I_H \in \mathcal{F}$ with $I_H \subseteq I$.

Equivalently for any $h \in H$ the action

$$\rho_h : A \rightarrow A \hspace{1cm} a \mapsto h \cdot a$$

is continuous with respect to the Gabriel topology induced by $\mathcal{F}$.

(see also Rumynin, 1993; Sidorov 1996; L. 2002)
Martindale ring of quotient

Theorem (Cohen, 1985)

Let $A$ be any semiprime left $H$-module algebra. The $H$-action extends to

$$Q_0 = \lim\{\text{Hom}(\_I, AA) \mid I \subseteq A \text{ is } H\text{-stable and } \text{Ann}(I) = 0\}.$$

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The problem

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Conditions on $A$

Theorem (Cohen, 1985)

Let $A$ be any semiprime left $H$-module algebra. The $H$-action extends to

$$Q_0 = \lim \{ \text{Hom}(A I, A A) \mid I \unlhd A \text{ is } H\text{-stable and } \text{Ann}(I) = 0 \}.$$ 

J. Matczuk, 1991, used $Q_0$ to define the $H$-central closure of $A$ as the subalgebra $\langle A, Z(Q_0)^H \rangle$.

Conditions on $H$

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On an open problem of M. Cohen concerning smash products
Extended Centroid

**Definition**

\[ B = A \otimes A^{op} \otimes H \] becomes an algebra with

\[
(a \otimes b \otimes h)(a' \otimes b' \otimes g) = \sum_{(h)} a(h_1 \cdot a') \otimes (h_3 \cdot b') b \otimes h_2 g.
\]
Extended Centroid

### Definition

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\[ (a \otimes b \otimes h)(a' \otimes b' \otimes g) = \sum_{(h)} a(h_1 \cdot a') \otimes (h_3 \cdot b')b \otimes h_2g. \]

### Theorem (L.2002)

*The self-injective hull \( \hat{A} \) of \( A \) as left \( B \)-module is a left \( H \)-module algebra with subalgebra \( A \).*

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**Extended Centroid**

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**Theorem (L.2002)**

*The self-injective hull \( \hat{A} \) of \( A \) as left \( B \)-module is a left \( H \)-module algebra with subalgebra \( A \).*

1. \( \hat{A} \simeq \langle A, Z(Q_0)^H \rangle \);
Extended Centroid

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\[ (a \otimes b \otimes h)(a' \otimes b' \otimes g) = \sum_{(h)} a(h_1 \cdot a') \otimes (h_3 \cdot b') b \otimes h_2 g. \]

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The self-injective hull \( \hat{A} \) of \( A \) as left \( B \)-module is a left \( H \)-module algebra with subalgebra \( A \).

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2. \( \text{End}_B(\hat{A}) \simeq Z(Q_0)^H \).
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\[ B = A \otimes A^{\text{op}} \otimes H \text{ becomes an algebra with} \]

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2. \( \text{End}_B(\hat{A}) \simeq Z(Q_0)^H \).
3. \( Z(Q_0)^H \) is von Neumann regular and self-injective.
Commutative module algebras

Theorem

A commutative semiprime and $H$ semisimple $\Rightarrow A \# H$ semiprime.
The problem

Commutative module algebras

Theorem

A commutative semiprime and $H$ semisimple $\Rightarrow A\#H$ semiprime.

Proof.

1. $A$ commutative $\Rightarrow \hat{A}$ commutative;

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4. $\hat{A}\#H$ is von Neumann regular (separability).
5. $A\#H$ is semiprime (central extension).
The problem

Conditions on $A$

Conditions on $H$

Conclusion

**Actions on integral domains**

---

**Theorem (Etingof-Walton, 2013)**

*Let $A$ be an integral domain. For any action of a semisimple Hopf algebra $H$ on $A$ exists a Hopf ideal $I$ of $H$ and a group $G$ such that*

\[ I \cdot A = 0 \quad \text{and} \quad H/I \cong k[G]. \]
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Let $A$ be an integral domain. For any action of a semisimple Hopf algebra $H$ on $A$ exists a Hopf ideal $I$ of $H$ and a group $G$ such that

$$I \cdot A = 0 \quad \text{and} \quad H/I \cong k[G].$$

This means that $H$ acts virtually as a group algebra.
The problem

Conditions on $A$

Conditions on $H$

Conclusion

PI algebras

Theorem (Linchenko-Montgomery, 2007)

A P.I. semprime and $H$ semisimple $\Rightarrow$ $A\#H$ semiprime.

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PI algebras

Theorem (Linchenko-Montgomery, 2007)

A P.I. semprime and $H$ semisimple $\Rightarrow A\#H$ semiprime.

Proof.

1. Reduction to finite dimensional factors $A/P$ of $A$
The problem

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Theorem (Linchenko-Montgomery, 2007)

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Proof.

1. Reduction to finite dimensional factors $A/P$ of $A$
2. (Linchenko, 2003): $\text{Jac}(A/P)$ is $H$-stable for $A/P$ f.d.
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**Theorem (Linchenko-Montgomery, 2007)**

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**Proof.**

1. Reduction to finite dimensional factors $A/P$ of $A$
2. (Linchenko, 2003): $\text{Jac}(A/P)$ is $H$-stable for $A/P$ f.d.
3. (Blattner-Montgomery duality):

$$A\#H\#H^* \simeq M_n(A) \quad \text{with} \quad n = \dim(H).$$

Hence Cohen’s question is equivalent to:
$S\#H^*$ semiprime $\Rightarrow$ $S$ semiprime for $S = A\#H$?
### PI algebras

#### Theorem (Linchenko-Montgomery, 2007)

A P.I. semiprime and $H$ semisimple $\Rightarrow A\#H$ semiprime.

#### Proof.

1. **Reduction to finite dimensional factors $A/P$ of $A$**
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3. (Blattner-Montgomery duality):

   $$A\#H\#H^* \cong M_n(A) \quad \text{with } n = \dim(H).$$

Hence Cohen's question is equivalent to:

$S\#H^*$ semiprime $\Rightarrow S$ semiprime for $S = A\#H$?

(Borges-L., 2011): proof works for weak Hopf algebras
Let \( t \in H \) be with \( ht = \epsilon(h)t, \forall h \in H \) and \( \epsilon(t) = 1 \).
Large invariants

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*If $A\#H$ is semiprime and $H$ semisimple, then any non-zero $H$-stable left ideal of $A$ contains non-zero $H$-invariant elements.*
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**Question**

Is it always true, that $I \cap A^H \neq 0$ for $I$ an $H$-stable left ideal of $A$, $H$ semisimple and $A$ semiprime?
Large invariants

Let $t \in H$ be with $ht = \epsilon(h)t, \forall h \in H$ and $\epsilon(t) = 1$. If $A \neq H$ is semiprime, then $(I \neq t)^2 \neq 0$ for any $H$-stable left ideal $I$ of $A$, i.e. $0 \neq (t \cdot I) \subseteq I \cap A^H$.

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Bergman-Isaacs, 1973 proved this for group actions.
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Bathurin-Linchenko, 1998 gave criteria for Hopf actions.
Contents

1. The problem
   - Preliminaries
   - Hopf algebra actions

2. Conditions on A
   - Separable extensions
   - Finiteness conditions on A
   - Extending the Hopf-action
   - Commutativity
   - Necessary condition

3. Conditions on H
   - Semisolvable Hopf algebras
   - Drinfeld twists

4. Conclusion
Recall that a semisimple Hopf algebra $H$ is trivial if it is commutative or cocommutative, i.e. if $H = k[G]$ or $H = k[G]^*$. 
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**Theorem (Zhu, 1994)**

*A Hopf algebra of prime dimension is a group ring.*
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**Theorem (Zhu, 1994)**

A Hopf algebra of prime dimension is a group ring.

**Theorem (Etingof-Gelaki, 1998)**

Any Hopf algebra whose dimension is a product $pq$ of two prime numbers is trivial.
Normal sub-Hopf algebras

Definition

A Hopf subalgebra $U$ of $H$ is called normal if it is stable under the adjoint action, i.e.

$$\forall h \in H : \text{ad}_h(U) = \sum_{(h)} h_1 U S(h_2) \subseteq U.$$
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If $U$ is normal in $H$, then $\overline{H} = H/U^+$ becomes a Hopf algebra with $U^+ = U \cap \text{Ker}(\epsilon)$. Moreover $H$ can be recovered from $U$ and $\overline{H}$ as a crossed product

$$H \simeq U \#_\sigma \overline{H}.$$
Semisolvable Hopf algebras

Definition (Montgomery-Whiterspoon, 1998)

A Hopf algebra $H$ is called **semisolvable** if it has a normal series

$$k = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{m-1} \trianglelefteq H_m = H$$

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Example

For $H_8 = \mathbb{C}[C_2 \times C_2][z;\sigma]/\langle z^2 - \frac{1}{2}(1 + x + y - xy) \rangle$ one has

$$U = \mathbb{C}[C_2 \times C_2] \unlhd H_8 \quad \text{and} \quad H_8/U^+ \cong \mathbb{C}[C_2].$$
Positive answer for semisolvable Hopf algebras

Theorem (Montgomery-Schneider, 1999)

*If* $H$ *is a semisolvable and semisimple Hopf algebra and* $A$ *a semiprime, then* $A\#H$ *is semiprime.*
The problem

Conditions on $A$

Conditions on $H$

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*Every Hopf algebra of dimension $p^n$, $p$ prime, is semisolvable.*
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Theorem (Natale)

The only semisimple Hopf algebra of dimension less than 60 that is not semisolvable is a “twist” of $k[S_3 \times S_3]$ in dimension 36.
Twists

Definition

A twist for a Hopf algebra $H$ is an invertible element $J \in H \otimes H$, such that

$$(J \otimes 1)(\Delta \otimes 1)(J) = (1 \otimes J)(1 \otimes \Delta)(J),$$

$$(\epsilon \otimes 1)(J) = 1 = (1 \otimes \epsilon)(J)$$

holds.
Twisting a Hopf algebra

Definition

Let $J \in H \otimes^2$ be a twist. Then $(H, m, \Delta^J, \epsilon, S^J)$ is also a Hopf algebra with

$$\Delta^J(h) := J\Delta(h)J^{-1}, \quad S^J(h) := US(h)U^{-1}$$

for all $h \in H$ with $U := m(1 \otimes S)(J) = \sum J^1 S(J^2)$. 

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On an open problem of M. Cohen concerning smash products
Twisted module algebras

Definition

Let \((A, \mu, 1)\) be a left \(H\)-module algebra and \(J\) a twist for \(H\). The new multiplication on \(A\) defined by

\[
a \cdot_J b := \mu^J(a \otimes b) := \sum (Q^1 \cdot a)(Q^2 \cdot b) \quad \text{for all } a, b \in A.
\]

makes \(A\) a left \(H^J\)-module algebra.

Here \(J^{-1} = \sum Q^1 \otimes Q^2\).
Cohen’s question for twists

Theorem (Majid, 1997)

\[ A\#H \simeq A^J \# H^J \text{ as algebras.} \]
The problem

Cohen’s question for twists

Theorem (Majid, 1997)

\[ A \# H \cong A^J \# H^J \] as algebras.

Theorem

If \( A \# H \) is semiprime for all semiprime \( H \)-module algebras \( A \), then the same is true for any \( H^J \)-module algebra over a twist \( H^J \).
Definition

A Hopf algebra is called *triangular*, if there exists an invertible element $R \in H \otimes H$ with

$$(\Delta \otimes 1)(R) = R_{13}R_{23},$$

$$(1 \otimes \Delta)(R) = R_{13}R_{12},$$

$${\Delta}^{cop} = R\Delta R^{-1} \text{ und } R^{-1} = \tau(R).$$
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$\Delta^{\text{cop}} = \mathcal{R}\Delta\mathcal{R}^{-1}$ und $\mathcal{R}^{-1} = \tau(\mathcal{R})$.

Any semisimple triangular Hopf algebra is isomorphic to a twist of a group algebra.
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Cohen’s question has a positive solution for

1. all semisimple Hopf algebras $H$ that are twists of semisolvable Hopf algebras (in particular for $\dim(H) \leq 60$).
## Corollary

Cohen’s question has a positive solution for

1. all semisimple Hopf algebras $H$ that are twists of semisolvable Hopf algebras (in particular for $\dim(H) \leq 60$).

2. all semiprime module algebras $A$ that either satisfy a PI or have an Artinian quotient ring.
Future directions

1. Find a semisimple Hopf algebra $H$ that is not a twist of a semisolvable Hopf algebra and a suitable $H$-action on a semiprime algebra $A$.

2. Extend Etingof-Walton's result on integral domains. Study Hopf algebra actions on domains (simple domains or free algebras).

3. Look at more general actions than Hopf algebra actions like actions of weak Hopf algebras, Hopfish algebras, bialgebroids to find counterexamples.

4. Recover known results for module categories over fusion categories.
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