On an open problem of M.Cohen concerning smash products

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The Question

Question (Miriam Cohen, 1985)

Is the smash product $A\#H$ semiprime in case H is a semisimple Hopf algebra acting on a semiprime algebra A ?

Semiprime rings

Definition

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R is semiprime iff it has no nilpotent $\neq 0$ ideals.

Preliminaries

For this talk $\mathrm{char}(k) = 0$.

Definition

A coalgebra (C, Δ, ϵ) is a k-vector space with comultiplication

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes 1} \\ C \otimes C & \xrightarrow[1\otimes\Delta]{} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} k \otimes C & \xleftarrow{\epsilon\otimes 1} & C \otimes C & \xrightarrow{1\otimes\epsilon} & C \otimes k \\ & \nwarrow & \uparrow{\scriptstyle\Delta} & \nearrow & \\ & & C & & \end{array}$$

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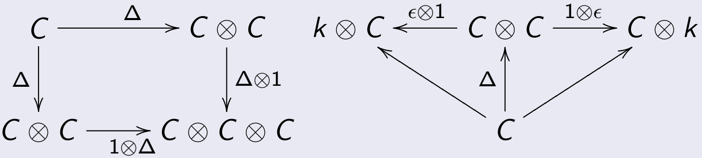

Definition

Given a coalgebra (C, Δ, ϵ) and an algebra $(A, m, 1)$ makes $\mathrm{Hom}(C, A)$ into an algebra with convolution product:

$$(f * g)(c) = m \circ (f \otimes g)\Delta(c) :$$

Hopf algebras

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Example

The dual $H^* = \mathrm{Hom}(H, k)$ of a finite dimensional Hopf algebra H is a Hopf algebra.

Examples

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2. If G is finite, then the dual group ring $H^* = \mathrm{Hom}(k[G], k)$ with dual basis $\{p_g\}_{g \in G}$ is a Hopf algebra with

$$\Delta(p_g) = \sum_{h \in G} p_{gh^{-1}} \otimes p_h, \qquad \epsilon(p_g) = \delta_{e,g}, \qquad S(p_g) = p_{g^{-1}}.$$

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Definition

A Hopf algebra isomorphic to one of these is called trivial.

Semisimple Hopf algebras

Theorem (Larson-Radford, 1988, char(k)=0)

The following are equivalent:

1. H *is a semisimple Hopf algebra;*
2. H^* *is a semisimple Hopf algebra;*
3. $S^2 = id$.

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Let H *be any semisimple Hopf algebra.*

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trivial = semisimple and (commutative or cocommutative)

A non-trivial semisimple Hopf algebra

Example (Fukuda, 1997)

$$H_8 = \mathbb{C}[G][z;\sigma]/\langle z^2 - \frac{1}{2}(1+x+y-xy)\rangle$$

is an 8-dimensional non-trivial semisimple Hopf algebra where $G = C_2 \times C_2$ with generators x, y and $\sigma(x) = y$, $\sigma(y) = x$ and

$$\Delta(z) = \frac{1}{2}(1\otimes 1 + 1\otimes x + y\otimes 1 - y\otimes x)(z\otimes z).$$

Hopf module algebras

The category H-Mod of left H-modules is a tensor category

$$h \cdot (v \otimes w) = \Delta(h)(V \otimes W) = \sum_{(h)} (h_1 \cdot v) \otimes (h_2 \cdot w),$$

for all $h \in H, v \in V, w \in W$ *for* $V, W \in H$*-Mod.*

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Definition

An algebra A in the category of left H-modules is called a (left) H-module algebra.

That means H acts on A such that

$$h \cdot (ab) = \sum_{(h)} (h_1 \cdot a)(h_2 \cdot b) \qquad \text{and} \qquad h \cdot 1_A = \epsilon(h) 1_A.$$

smash product

Definition (Smash product)

The smash product of a Hopf algebra H and a module algebra A is defined on $A\#H := A \otimes H$ with multiplication:

$$(a\#h)(b\#g) = \sum_{(h)} a(h_1 \cdot b)\#h_2 g,$$

with identity $1_A\#1_H$.

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$$A^H = \{a \in A \mid h \cdot a = \epsilon(h)a \quad \forall h \in H\}.$$

$$\mathrm{End}_{A\#H}(A) \simeq A^H \qquad f \mapsto f(1).$$

Examples

Example

An action $k[G] \otimes A \to A$ corresponds to the group homomorphism $\varphi : G \to \mathrm{Aut}(A)$ given by

$$\varphi(g)(a) = g \cdot a.$$

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Theorem (Fisher-Montgomery, 1978; Lorenz-Passman, 1980)

$A\#k[G]$ is semiprime for a finite group G and A semiprime.

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Example

An action $k[G]^* \otimes A \to A$ corresponds to the grading $A = \bigoplus_{g\in G} A_g$ given by

$$A_g = \{p_g \cdot a \in A \mid a \in A\}.$$

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Moreover $A\#k[G]^*$ has been used in the study of group gradings.

Theorem (Cohen-Montgomery, 1984)

$A\#k[G]^*$ is semiprime for a finite group G and A semiprime.

The Question

Question (Miriam Cohen, 1985)

Is $A\#H$ semiprime if H is a semisimple and A semiprime ?

A non-trivial action

$$H_8 = \mathbb{C}[C_2 \times C_2][z;\sigma]/\langle z^2 - \frac{1}{2}(1 + x + y - xy)\rangle$$

Example (Kirkman-Kuzmanovich-Zhang, 2009)

H_8 acts on the quantum plane $A = \mathbb{C}_q[u,v]$ with $q^2 = -1$ by

$$\begin{array}{lclclclclcl} x \cdot u & = & -u, & & y \cdot u & = & u, & & z \cdot u & = & v \\ x \cdot v & = & v, & & y \cdot v & = & -v, & & z \cdot v & = & u. \end{array}$$

Note that $z \cdot (uv) = -vu \neq vu = (z \cdot u)(z \cdot v)$.

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Semisimple Hopf algebras

Theorem (Sweedler)

The following are equivalent:

1. H *is a semisimple Hopf algebra;*
2. H *is a separable* k*-algebra;*
3. k *is a projective left* H*-module;*
4. $\exists t \in H : \forall h \in H : ht = \epsilon(h)t$ *and* $\epsilon(t) = 1$.

Separable extensions

Definition (Hirata-Sugano, 1966)

A ring extension $R \subseteq S$ is separable if mult : $S \otimes_R S \to S$ splits as S-bimodule, i.e.

$$\exists \gamma = \sum_{i=1}^{n} e_i \otimes f_i \in S \otimes_R S, \qquad \mathrm{mult}(\gamma) = 1 \text{ and } s\gamma = \gamma s \ \forall s \in S.$$

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Corollary

H is semisimple iff $A \subseteq A\#H$ is a separable extension for all A.

Choose $\gamma = \sum_{(t)} 1\#t_1 \otimes_A 1\#S(t_2) \in (A\#H) \otimes_A (A\#H)$.

Von Neumann regular algebras

Transfer of homological properties via separable extensions:

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Let H be a semisimple Hopf algebra acting on A.

1. *If A is von Neumann regular, i.e. all A-modules are flat, then $A\#H$ is von Neumann regular.*
2. *If A is semisimple Artinian, then $A\#H$ is semisimple Artinian.*

Finiteness conditions

Question

Can one embed A into another H-module algebra Q such that

$$A \text{ semiprime} \Longrightarrow Q\#H \text{ semiprime} \Longrightarrow A\#H \text{ semiprime ?}$$

Classical ring of quotient

Theorem (Skryabin-VanOystayen, 2006)

If A has a right Artinian classical ring of quotient Q, then any left H-action on A extends to Q.

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Example

$\mathbb{C}_q[u, v]\#H_8$ is semiprime for $q^2 = -1$.

Idea of Skryabin-VanOystaeyen's proof

1. A is a left module algebra if and only if there exists an algebra homomorphism
$$\rho : A \longrightarrow \mathrm{Hom}(H, A).$$

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$$H = \sum \{C_i \mid C_i \text{ is f.d. subcoalgebra of } H\}.$$

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3. For any C the following diagram can be completed:
$$\begin{array}{ccc} A & \xrightarrow{\rho_{|C}} & \mathrm{Hom}(C, A) \\ \downarrow & & \downarrow \\ Q & \overset{\exists \rho'_{|C}}{\dashrightarrow} & \mathrm{Hom}(C, Q) \end{array}$$

Gabriel localization

Question

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Theorem (Montgomery, 1992; Selvan, 1994)

A sufficient condition for this to happen is that for any right ideal $I \in \mathcal{F}$ there exists an H-stable right ideal $I_H \in \mathcal{F}$ with $I_H \subseteq I$.
Equivalently for any $h \in H$ the action

$$\rho_h : A \longrightarrow A \qquad a \longmapsto h \cdot a$$

is continuous with respect to the Gabriel topology induced by $\mathcal{F}$.

(see also Rumynin, 1993; Sidorov 1996; L. 2002)

Martindale ring of quotient

Theorem (Cohen, 1985)

Let A be any semiprime left H-module algebra. The H-action extends to

$$Q_0 = \lim\{\mathrm{Hom}({}_AI, {}_AA) \mid I \trianglelefteq A \textit{ is } H\textit{-stable and } \mathrm{Ann}(I) = 0\}.$$

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J.Matczuk, 1991, used Q_0 to define the H-central closure of A as the subalgebra $\langle A, Z(Q_0)^H \rangle$.

Extended Centroid

Definition

$B = A \otimes A^{op} \otimes H$ becomes an algebra with

$$(a \otimes b \otimes h)(a' \otimes b' \otimes g) = \sum_{(h)} a(h_1 \cdot a') \otimes (h_3 \cdot b')b \otimes h_2 g.$$

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The self-injective hull $\widehat{A}$ of A as left B-module is a left H-module algebra with subalgebra A.

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1. $\widehat{A} \simeq \langle A, Z(Q_0)^H \rangle$;
2. $\mathrm{End}_B(\widehat{A}) \simeq Z(Q_0)^H$.
3. $Z(Q_0)^H$ *is von Neumann regular and self-injective.*

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A commutative semprime and H semisimple $\Rightarrow$ $A\#H$ semiprime.

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5. $A\#H$ is semiprime (central extension).

□

Actions on integral domains

Theorem (Etingof-Walton, 2013)

Let A be an integral domain. For any action of a semisimple Hopf algebra H on A exists a Hopf ideal I of H and a group G such that

$$I \cdot A = 0 \qquad \text{and} \qquad H/I \simeq k[G].$$

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This means that H acts virtually as a group algebra.

PI algebras

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3. (Blattner-Montgomery duality):

$$A\#H\#H^* \simeq M_n(A) \qquad \text{with } n = \dim(H).$$

Hence Cohen's question is equivalent to:
$S\#H^*$ semiprime $\Rightarrow S$ semiprime for $S = A\#H$?

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(Borges-L., 2011): proof works for weak Hopf algebras

Large invariants

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Bergman-Isaacs, 1973 proved this for group actions.

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If $A\#H$ is semiprime and H semisimple, then any non-zero H-stable left ideal of A contains non-zero H-invariant elements.

Question

Is it always true, that $I \cap A^H \neq 0$ for I an H-stable left ideal of A, H semisimple and A semiprime ?

Bergman-Isaacs, 1973 proved this for group actions.
Bathurin-Linchenko, 1998 gave criteria for Hopf actions.

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Trivial Hopf algebras

Recall that a semisimple Hopf algebra H is trivial if it is commutative or cocommutative, i.e. if $H = k[G]$ or $H = k[G]^*$.

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Theorem (Zhu, 1994)

A Hopf algebra of prime dimension is a group ring.

Theorem (Etingof-Gelaki, 1998)

Any Hopf algebra whose dimension is a product pq of two prime numbers is trivial.

Normal sub-Hopfalgebras

Definition

A Hopf subalgebra U of H is called normal if it is stable under the adjoint action, i.e.

$$\forall h \in H : \mathrm{ad}_h(U) = \sum_{(h)} h_1 U S(h_2) \subseteq U.$$

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If U is normal in H, then $\overline{H} = H/U^+$ becomes a Hopf algebra with $U^+ = U \cap \mathrm{Ker}(\epsilon)$. Moreover H can be recovered from U and $\overline{H}$ as a crossed product

$$H \simeq U \#_\sigma \overline{H}.$$

Semisolvable Hopf algebras

Definition (Montgomery-Whiterspoon, 1998)

A Hopf algebra H is called semisolvable if it has a normal series

$$k = H_0 \trianglelefteq H_1 \trianglelefteq \cdots H_{m-1} \trianglelefteq H_m = H$$

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Example

For $H_8 = \mathbb{C}[C_2 \times C_2][z;\sigma]/\langle z^2 - \frac{1}{2}(1 + x + y - xy)\rangle$ one has

$$U = \mathbb{C}[C_2 \times C_2] \trianglelefteq H_8 \qquad \text{and} \qquad H_8/U^+ \simeq \mathbb{C}[C_2].$$

Positive answer for semisolvable Hopf algebras

Theorem (Montgomery-Schneider, 1999)

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Every Hopf algebra of dimension p^n, p prime, is semisolvable.

Theorem (Natale)

The only semisimple Hopf algebra of dimension less than 60 that is not semisolvable is a "twist" of $k[S_3 \times S_3]$ in dimension 36.

Twists

Definition

A twist for a Hopf algebra H is an invertible element $J \in H \otimes H$, such that

$$(J \otimes 1)(\Delta \otimes 1)(J) = (1 \otimes J)(1 \otimes \Delta)(J),$$

$$(\epsilon \otimes 1)(J) = 1 = (1 \otimes \epsilon)(J)$$

holds.

Twisting a Hopf algebra

Definition

Let $J \in H^{\otimes 2}$ be a twist. Then $(H, m, \Delta^J, \epsilon, S^J)$ is also a Hopf algebra with

$$\Delta^J(h) := J\Delta(h)J^{-1}, \qquad S^J(h) := US(h)U^{-1}$$

for all $h \in H$ with $U := m(1 \otimes S)(J) = \sum J^1 S(J^2)$.

Twisted module algebras

Definition

Let $(A, \mu, 1)$ be a left H-module algebra and J a twist for H. The new multiplication on A defined by

$$a \cdot_J b := \mu^J(a \otimes b) := \sum (Q^1 \cdot a)(Q^2 \cdot b) \quad \text{for all } a, b \in A.$$

makes A a left H^J-module algebra.

Here $J^{-1} = \sum Q^1 \otimes Q^2$.

Cohen's question for twists

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$A\#H \simeq A^J\#H^J$ as algebras.

Theorem

If $A\#H$ is semiprime for all semiprime H-module algebras A, then the same is true for any H^J-module algebra over a twist H^J.

Etingof-Gelaki 1998

Definition

A Hopf algebra is called *triangular*, if there exists an invertible element $\mathcal{R} \in H \otimes H$ with

$$(\Delta \otimes 1)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23},$$

$$(1 \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

$$\Delta^{cop} = \mathcal{R}\Delta\mathcal{R}^{-1} \text{ und } \mathcal{R}^{-1} = \tau(\mathcal{R}).$$

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Theorem (Etingof-Gelaki, 1998)

Any semisimple triangular Hopf algebra is isomorphic to a twist of a group algebra.

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1. *all semisimple Hopf algebras H that are twists of semisolvable Hopf algebras (in particular for $\dim(H) \leq 60$).*
2. *all semiprime module algebras A that either satisfy a PI or have an Artinian quotient ring.*

Future directions

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1. *Find a semisimple Hopf algebra H that is not a twist of a semisolvable Hopf algebra and a suitable H-action on a semiprime algebra A.*

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4. *Recovered known results for module categories over fusion categories.*